Scattering Amplitudes in QFT WS 2023/24

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https://www.ph.nat.tum.de/ttpmath/teaching/ws-2023-2024/

Sheet 03: Tree-Level QCD four-point Amplitudes

In this exercise sheet, we use the techniques introduced in the lecture to efficiently compute tree-level four-point amplitudes in QCD.

Exercise 1 - Tree-level amplitude for $q\bar{q}gg \to 0$

Consider the process (as usual we take all particles as incoming)

$$q_{\lambda_1,i_1}(p_1) \;+\; q_{\lambda_2,i_2}(p_2) \;+\; g_{\lambda_3,a_3}(p_3) \;+\; g_{\lambda_4,a_4}(p_4) \to 0\,, \tag{1}$$

where λ_j, i_j, and a_j, denote helicity indices, colour indices in the fundamental, and colour indices in the adjoint representation, respectively. Momentum conservation implies $p_1 + p_2 = p_3 + p_4$. We denote the corresponding amplitude as $\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}$.

1. Draw all the relevant Feynman diagrams contributing to the amplitude at $\mathcal{O}(g_s^2)$ (tree-level) and write down an expression for $\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}$ by inserting the QCD Feynman rules (see end of the sheet).

2. Use the $SU(N)$-algebra to write down a decomposition of the amplitude in terms of colour-ordered amplitudes,
$$\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4} = \mathcal{M}^{(1)}_{\lambda_1\lambda_2\lambda_3\lambda_4} T^{a_3}_{i_1k} T^{a_4}_{ki_2} + \mathcal{M}^{(2)}_{\lambda_1\lambda_2\lambda_3\lambda_4} T^{a_4}_{i_1k} T^{a_3}_{ki_2}\,. \tag{2}$$
Find a relation between $\mathcal{M}^{(1)}_{\lambda_1\lambda_2\lambda_3\lambda_4}$ and $\mathcal{M}^{(2)}_{\lambda_1\lambda_2\lambda_3\lambda_4}$.

3. List all helicity configurations allowed for the external particles. Taking the action of parity and charge conjugation into account, how many independent helicity amplitudes are there?

4. Compute all the independent colour-stripped amplitudes $\mathcal{M}^{(j)}_{\lambda_1\lambda_2\lambda_3\lambda_4}$. You should find that all those where the external gluons have the same helicity are zero.

5. Finally, compute the amplitude squared and summed over helicities. You should find the result
$$\sum_{\substack{\lambda_{1,2}=L,R \\ \lambda_{3,4}=\pm}} |\mathcal{M}|^2 = g_s^4 \left[2\frac{(N^2-1)^2}{N} \frac{s_{13}^2 + s_{14}^2}{s_{13}s_{14}} - 4N(N^2-1)\frac{s_{13}^2+s_{14}^2}{s_{12}^2} \right]\,, \tag{3}$$
where $s_{ij} = (p_i + p_j)^2$.

Exercise 2 - Tree-level amplitude for $gggg \to 0$

Consider now the process (again all particles are incoming)

$$g_{\lambda_1,a_1}(p_1) \;+\; g_{\lambda_2,a_2}(p_2) \;+\; g_{\lambda_3,a_3}(p_3) \;+\; g_{\lambda_4,a_4}(p_4) \to 0\,, \tag{4}$$

where λ_i and a_i denote the helicity and colour of the corresponding gauge boson, respectively. We denote the corresponding amplitude as $\mathcal{M}^{\lambda_1,\lambda_2,\lambda_3,\lambda_4}_{a_1,a_2,a_3,a_4}$.

1. Draw again all relevant Feynman diagrams contributing to the process at tree level.

2. Demonstrate that the amplitude can be written in the form

$$\mathcal{M}^{\lambda_1,\lambda_2,\lambda_3,\lambda_4}_{a_1,a_2,a_3,a_4} = 4 \sum_{\sigma\in S_3} \mathrm{tr}\left(T^{a_1} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}\right) \mathcal{A}\left[1, \sigma(2), \sigma(3), \sigma(4)\right] , \tag{5}$$

   where the sum runs over all permutations of $\{2, 3, 4\}$ and T^{a_i} denotes the generator a_i in the fundamental representation. You may find it helpful to write

$$f^{abc} = -2\, i\, \mathrm{tr}\left(\left[T^a, T^b\right] T^c\right) \tag{6}$$

   and to make use of the Fierz completeness relation for $SU(N)$:

$$T^a_{ij}\, T^a_{kl} = \frac{1}{2}\, \delta_{il}\, \delta_{jk} - \frac{1}{2N} \delta_i j\, \delta_{kl} \, . \tag{7}$$

The colour-stripped amplitudes $\mathcal{A}\left[1, \sigma(2), \sigma(3), \sigma(4)\right]$ satisfy several linear relations (see discussion in the lecture). Without going into detail, it can be shown that only one of them is actually linearly independent, say $\mathcal{A}\left[1, 2, 3, 4\right]$. To calculate it explicitly, we will employ spinor-helicity formalism and write $\mathcal{A}\left[1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}\right]$ for the individual helicity amplitudes.

3. Use the QCD Feynman rules to obtain an explicit expression for $\mathcal{A}\left[1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}\right]$.

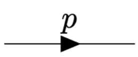
4. Show that $\mathcal{A}\left[1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}\right] = 0$ unless exactly two helicities are + and the other two helicities are −. To do that, use the fact that the reference vector for each external gauge boson can be chosen freely in every individual helicity amplitude[1] but the choice has to be consistent across all diagrams. The non-vanishing helicity amplitudes are referred to as *maximally helicity violating (MHV).*

5. Through crossing symmetry, the action of parity and the relations among the partial amplitudes, it is possible to compute any MHV helicity amplitude through the knowledge of any single other MHV amplitude. So it suffices to explicitly compute a single one of them, say $\mathcal{A}\left[1^+, 2^+, 3^-, 4^-\right]$. By explicit calculation show that

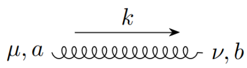
$$\mathcal{A}\left[1^+, 2^+, 3^-, 4^-\right] = g^2\, \frac{\langle 12\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle} \, . \tag{8}$$

[1]This is a consequence of the fact that helicity amplitudes are individually gauge-invariant.

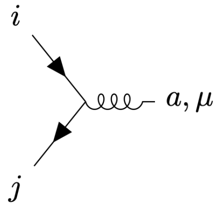
QCD Feynman rules in Feynman gauge



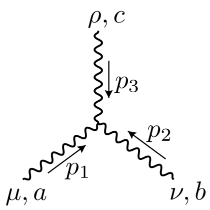
$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i0^+}$$



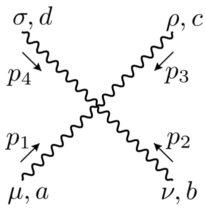
$$= -i\delta_{ab}\left(\frac{g_{\mu\nu}}{k^2 + i0^+}\right)$$



$$= -ig_s\gamma^\mu T^a_{ji}$$



$$= -g_s f^{abc}\left[g^{\mu\nu}(p_1 - p_2)^\rho + g^{\nu\rho}(p_2 - p_3)^\mu + g^{\rho\mu}(p_3 - p_1)^\nu\right]$$



$$= \begin{aligned} -ig_s^2 \big[& f^{abe} f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ & + f^{ace} f^{dbe}(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}) \\ & + f^{ade} f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\big] \end{aligned}$$