Scattering Amplitudes in QFT WS 2023/24

Lecturer: Prof. Lorenzo Tancredi
Assistant: Fabian Wagner
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Sheet 02: Spinor-Helicity-Formalism for Massive Fermions

Exercise 1 - Spinor-Helicity-Formalism for Massive Fermions

Consider a massive fermion with momentum p such that $p^2 = m^2 > 0$. The associated spinor $u(p)$ satisfies the Dirac equation

$$(\not{p} - m)u(p) = 0. \tag{1}$$

In order to use the spinor helicity formalism, we start by writing the massive momentum as a linear combination of two light-like momenta, i.e. $p = p_1 + p_2$, with $p_1^2 = 0$, $p_2^2 = 0$, $2\,p_1 \cdot p_2 = m^2$.

1. A massive fermion can have two different polarisations, which we will call $u_\pm(p)$. Argue that they can be decomposed as follows:

$$\begin{aligned} u_+(p_t) &= \alpha\; 1\rangle + \beta\; 2]\,, \\ u_-(p_t) &= \gamma\; 1] + \delta\; 2\rangle\,. \end{aligned} \tag{2}$$

2. Use the fact that the spinors in eq. (2) must satisfy the Dirac equation to find relations among $\alpha, \beta, \gamma, \delta$.

3. Argue that we can write the spinors in eq. (2) in the following compact form:

$$u_+(p) = N_+(\not{p} + m)\; 2]\,, \quad u_-(p) = N_-(\not{p} + m)\; 2\rangle\,, \tag{3}$$
$$u_+(p) = \tilde{N}_+(\not{p} + m)\; 1]\,, \quad u_-(p) = \tilde{N}_-(\not{p} + m)\; 1\rangle\,. \tag{4}$$

4. Given that the sum over polarisations should give the correct density matrix,

$$\sum_\lambda u_\lambda(p)\bar{u}_\lambda(p) = (\not{p} + m), \tag{5}$$

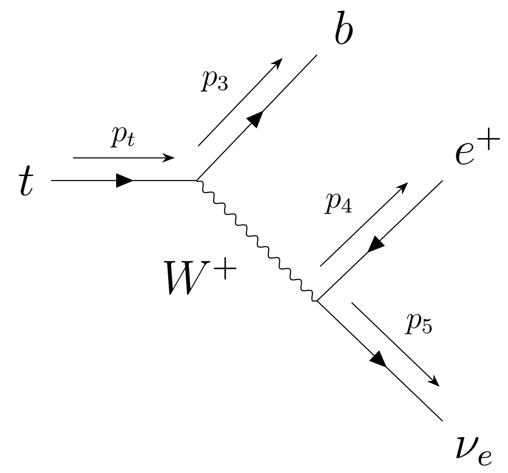
show that

$$u_+(p) = 1\rangle + \frac{m}{[12]}\; 2]\,, \quad u_-(p) = 1] + \frac{m}{\langle 12\rangle}\; 2\rangle\,. \tag{6}$$

<u>Hint</u>: Assume that the normalisation constants are independent of polarisations.

Exercise 2 - Semileptonic Top Quark Decay

Consider the tree-level semileptonic decay of the top quark,



where we assume all fermions to be massless except the top quark.

1. Use the electroweak Feynman rules to write down an expression of the amplitude associated to this Feynman diagram. Which helicity configurations are allowed for the massless fermions?

2. Show that the amplitude can be written in Feynman gauge as

$$\mathcal{M}_\lambda = \frac{g_W^2 V_{tb}^*}{2\left[(p_t - p_3)^2 - m_W^2\right]} \mathcal{A}_\lambda \, \delta_{i_b, i_t}, \quad \text{with} \quad \mathcal{A}_\lambda = \langle 5 \, \gamma^\mu \, 4] \, \langle 3 \, \gamma^\mu \, u_{t,\lambda}(p_t), \tag{7}$$

   where $\lambda = \pm$ labels the top quark polarisation and i_b, i_t are the color indices of the quarks. Applying the result from the previous exercise to the top quark, show that

$$\mathcal{A}_+ = 2 \, \langle 35 \rangle \, \frac{[42]}{[12]} m_t, \quad \mathcal{A}_- = 2 \, \langle 35 \rangle \, [41] \, . \tag{8}$$

3. Calculate the sum over helicities of the squared amplitude $\sum_\lambda |\mathcal{A}_\lambda|^2$ and discuss your result.