## Advanced Quantum Mechanics SS 2024

Prof. Lorenzo Tancredi (TUM) **Tutors:** Cesare Carlo Mella (cesarecarlo.mella@tum.de) Fabian Johannes Wagner (fabianjohannes.wagner@tum.de) https://www.ph.nat.tum.de/ttpmath/teaching/ss-2024/

Sheet 02: Time dependent problems (to be handed in by 22.05.2024)

## **1** Interaction picture

Prove that operators in the interaction picture evolve in time according to

$$\frac{dO_I}{dt} = \frac{1}{i\hbar} [O_I, H_0], \qquad (1)$$

where the Hamiltonian of the system is defined as

$$H(t) = H_0 + V(t) \,,$$

with  $H_0$  time independent.

# 2 The three level system: dark states and induced transparency

In this exercise we consider a three level system with levels  $\hbar\omega_a$ ,  $\hbar\omega_b$ ,  $\hbar\omega_c$ , associated to the states  $|a\rangle$ ,  $|b\rangle$ ,  $|c\rangle$ . The system is perturbed by a time dependent potential of the form

$$V(t) = \begin{pmatrix} 0 & \frac{1}{2}e^{-i\omega_1 t}W_{ab} & \frac{1}{2}e^{-i\omega_2 t}W_{ac} \\ \frac{1}{2}e^{i\omega_1 t}W_{ab} & 0 & 0 \\ \frac{1}{2}e^{i\omega_2 t}W_{ac} & 0 & 0 \end{pmatrix},$$
(2)

where we can take  $W_{ab}$  and  $W_{ac}$  real. Define  $\delta_1 = (\omega_a - \omega_b) - \omega_1$  and  $\delta_2 = (\omega_a - \omega_c) - \omega_2$ ,

1. Compute the interaction picture potential,  $V_I(t)$ .

At a time t the state vector in interaction picture is written as

$$\psi_I(t) = \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix},\tag{3}$$

- 2. Write down the differential equation for the coefficients a(t), b(t) and c(t).
- 3. We now make an ansatz for the time dependence of the coefficients. Assume  $a(t) = \sum_j a_j(0)e^{i\Omega_j t}$ ,  $b(t) = \sum_j b_j(0)e^{i(\Omega_j \delta_1)t}$ ,  $c(t) = \sum_j c_j(0)e^{i(\Omega_j \delta_2)t}$  and derive an equation do determine the allowed values for  $\Omega$ . You don't have to solve it.
- 4. From now on, assume perfect tuning,  $\delta_1 = \delta_2 = 0$ . Determine the allowed values for  $\Omega$ .
- 5. Write down the general solution and use the above equation to put constrains on the coefficients  $a_j(0), b_j(0)$  and  $c_j(0)$ .

- 6. (Dark states) Consider the case  $W_{ab} = W_{ac}$  and  $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|b\rangle |c\rangle)$ . Compute the probability of finding the system in state  $|a\rangle$  at the time t and discuss your result.
- 7. (Induced transparency) Consider the case  $W_{ab} \gg W_{ac}$  and a(0) = b(0) = 0. Compute the probability of finding the system in state  $|a\rangle$  at the time t and discuss your result.

### **3** Second order perturbation theory

We have seen in class the transition coefficients in the interaction picture, defined as

$$c_f(t) = \langle f | U_I(t, t_0) | i \rangle, \tag{4}$$

where f and i are eigenstates of some free hamiltonian  $H_0$  and  $U_I(t, t_0)$  is the time evolution operator in the interaction picture. In the previous exercise we wrote down a differential equation for the transition coefficients, but this strategy, very often, require the solution of a coupled system of differential equations too difficult to handle. An approximate solution can be obtained in perturbation theory. Starting from the differential equation for the time evolution operation in interaction picture,

$$i\hbar\frac{\partial}{\partial t}U_I(t,t_0) = V_I(t)U_I(t,t_0)$$
(5)

Derive the expressions for the transition coefficients up to second order in the perturbative expansion in the interaction potential.

#### 4 Two level system in perturbation theory

Consider a two level system with free hamiltonian

$$H_0 = E_1 |1\rangle \langle 1| + E_2 |2\rangle \langle 2|, \tag{6}$$

and the perturbation

$$V(t) = \gamma e^{i\omega t} |1\rangle \langle 2| + \gamma e^{-i\omega t} |2\rangle \langle 1|.$$
(7)

Suppose  $E_2 > E_1$ . In interaction picture we set  $|\psi(t)\rangle_I = c_1(t)|1\rangle + c_2(t)|2\rangle$ . We worked out the exact solution of this problem in class. In this exercise we use perturbation theory to solve the same problem. Assume that at t = 0 the state is  $|1\rangle$ . Treating separately the cases

- (a)  $\hbar \omega$  very different from  $E_2 E_1$ (b)  $\hbar \omega \approx E_2 - E_1$
- 1. Compute now  $c_1(t)$  and  $c_2(t)$  to the lowest non vanishing order in perturbation theory. Compute also the probability to find the system in state  $|1\rangle$  and  $|2\rangle$ .
- 2. Compare the result with the exact solution expanded for small  $\gamma$ .