Scattering Amplitudes in QFT WS 2023/24

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https://www.ph.nat.tum.de/ttpmath/teaching/ws-2023-2024/

Sheet 12: Box contributions in Generalised Unitarity

In the lectures you have seen that one-loop n-point amplitudes up to $\mathcal{O}(\epsilon^0)$ can be reduced to a linear combination of basis integrals, which involve scalar boxes, triangles, bubbles and tadpoles. The coefficients of these *master integrals* can be computed from tree-level on-shell amplitudes that result after performing "generalised unitarity cuts" on the original amplitude.

Exercise 1 - Box contributions to one-loop n-point amplitudes

Consider the decomposition of a tensorial 4-point integral,

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\prod_{j=1}^r (l \cdot u_j)}{D_0 D_1 D_2 D_3} = d_0 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{D_0 D_1 D_2 D_3} + \sum_{n=1}^{4} d_n \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{(l \cdot n_4)^n}{D_0 D_1 D_2 D_3} + \text{ lower-point integrals,}$$

where $r \leq 4$ and the inverse propagators are

$$D_0 = l^2 - m_0^2\,, \quad D_1 = (l+q_1)^2 - m_1^2\,, \quad D_2 = (l+q_2)^2 - m_2^2\,, \quad D_3 = (l+q_3)^2 - m_3^2\,, \tag{1}$$

with region momenta q_i. The unit vector n_4 is orthogonal to all region momenta, $n_4 \cdot q_i = 0$.

1. Show that at the *integrand* level in *strictly 4 space-time dimensions*, we actually have

$$\frac{d(l)}{D_0 D_1 D_2 D_3} \equiv \frac{d_0 + \sum_{n=1}^4 d_n (l \cdot n_4)^n}{D_0 D_1 D_2 D_3} = \frac{d + \tilde{d}(l \cdot n_4)}{D_0 D_1 D_2 D_3} + \text{ lower-point integrands}\,, \tag{2}$$

   so we do not need to keep any higher powers of $(l \cdot n_4)$.

2. The coefficient $d(l)$ can be isolated via a quadruple unitarity cut in $D = 4$ space-time dimensions. For simplicity, set all internal masses to zero, $m_i^2 = 0$. Use the Van Neerven-Vermaseren decomposition for the loop momentum, using its three region momenta q_1, q_2, q_3 and the extra momentum n_4, to show that the quadruple cut "freezes" all components of the loop momentum l^μ to the two solutions

$$\bar{l}_\pm^\mu = -\frac{1}{2} \sum_{i=1}^{3} q_i^2 v^\mu \pm \frac{1}{2} \sqrt{-\left(q_1^2 v_1^\mu + q_2^2 v_2^\mu + q_3^2 v_3^\mu\right)^2}\, n_4^\mu. \tag{3}$$

Consider now an n-point one-loop amplitude with integrand A_n. One-loop integrand reduction in four space-time dimensions allows us to write

$$A_n = \frac{d(l)}{D_0 D_1 D_2 D_3} + \text{ other boxes } + \text{ lower point contributions}\,. \tag{4}$$

On the quadruple cut, the left-hand side can be written as the product of four tree-level amplitudes A_i^{tree}, $i = 1, \ldots, 4$, while on the right-hand side the cut isolates the coefficient $d(l) = d + \tilde{d}(l \cdot n_4)$.

3. Show that the scalar box coefficient d can be written as,

$$d = \frac{D_+ + D_-}{2} \tag{5}$$

with

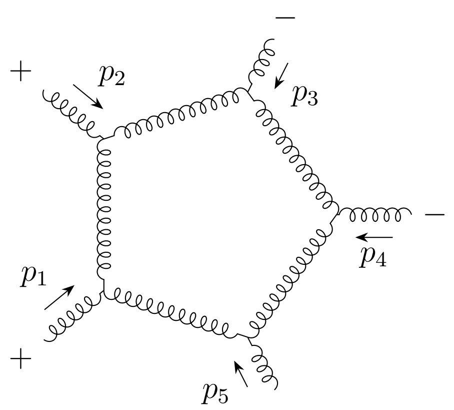
$$D_\pm = A_1^{\text{tree}}(\bar{l}_\pm) A_2^{\text{tree}}(\bar{l}_\pm) A_3^{\text{tree}}(\bar{l}_\pm) A_4^{\text{tree}}(\bar{l}_\pm). \tag{6}$$

What is the corresponding formula for $\tilde{d}$? Do we need to compute it and if yes, for what?

Exercise 2 - Box coefficients of a five-gluon amplitude

In this exercise we will consider the computation of the coefficients of the boxes that appear in the reduction of a one-loop five-point diagram contributing to the one-loop colour-ordered five-gluon amplitude $\mathcal{A}^{(1)}[1^+,\ 2^+,\ 3^-,\ 4^-,\ 5^-]$,

$$A^{(1)}(1^+,\ 2^+,\ 3^-,\ 4^-,\ 5^-) \ = \tag{7}$$



1. Explain that the contributing boxes are

$$I(s_{12}),\ I(s_{23}),\ I(s_{34}),\ I(s_{45}),\ I(s_{51}),$$

where $s_{ij} = (p_i + p_j)^2$ and $I(s_{ij})$ represents the box integral with momenta p_i and p_j entering in the same corner.

Focus now on the computation of the coefficient of $I(s_{12})$, which we denote as d_{12}. Concretely, the $I(s_{12})$ scalar box integral is defined as

$$I(s_{12}) = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2(l+q_2)^2(l+q_3)^2(l+q_4)^2} \tag{8}$$

with region momenta $q_1 = p_1$, $q_2 = q_1 + p_2$, $q_3 = q_2 + p_3$, $q_4 = q_3 + p_4$.

2. Consider the helicity amplitudes that result from the quadruple cut of $A^{(1)}(1^+,\ 2^+,\ 3^-,\ 4^-,\ 5^-)$ in such a way as to isolate the contribution of $I(s_{12})$. What choices are allowed for the helicities of the resulting tree amplitudes?

3. Compute the solution to the quadruple cut constraints.

   Hint: For convenience you can define $l_i \equiv l + q_i$, for $i = 1 \ldots 4$ and solve the cut conditions for an appropriate choice of l_i instead of l.

4. Finally, use (6) to compute $D_\pm$ in terms of specific helicity amplitudes and use (5) to obtain the box coefficient d_{12}.