Scattering Amplitudes in QFT WS 2023/24

Lecturer: Prof. Lorenzo Tancredi
Assistant: Fabian Wagner
https://www.ph.nat.tum.de/ttpmath/teaching/ws-2023-2024/

Sheet 11: Differential equations for Feynman integrals

Exercise 1 - Differential Equations for the massive bubble integral family

In this exercise, we will (at least in principle) compute all the integrals in the *integral family*

$$\mathcal{I}_{a_1,a_2}(p^2,m^2) = \mathrm{e}^{\epsilon\gamma_E}\int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{1}{[k^2-m^2+\mathrm{i}0^+]^{a_1}[(k-p)^2-m^2+\mathrm{i}0^+]^{a_2}} \equiv \int \mathcal{D}k\, \frac{1}{D_1^{a_1}D_2^{a_2}}\,, \tag{1}$$

where a_1, a_2 are integers, $a_1, a_2 \in \mathbb{Z}$.

1. Show that the integral family in eq. (1) posses the symmetry

$$\mathcal{I}_{a_1,a_2}(p^2,m^2) = \mathcal{I}_{a_2,a_1}(p^2,m^2)\,. \tag{2}$$

The members of an integral family satisfy linear relations coming from integration-by-parts. For the family defined in eq. (1), they can be written as

$$0 = \mathrm{e}^{\epsilon\gamma_E}\int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \left[\frac{\partial}{\partial k^\mu}\, \frac{u^\mu}{[k^2-m^2]^{a_1}[(k-p)^2-m^2]^{a_2}}\right], \tag{3}$$

where $u^\mu \in \{k^\mu, p^\mu\}$ and we hide the Feynman prescription in m^2.

2. Argue why the integration-by-parts identities are satisfied.

3. Derive the two integration-by-parts identities (IBPs) for the bubble family. You should find

$$\begin{aligned} 0 =&(D-2a_1-a_2)\,\mathcal{I}_{a_1,a_2} - 2a_1 m^2\,\mathcal{I}_{a_1+1,a_2} - a_2(2m^2-p^2)\,\mathcal{I}_{a_1,a_2+1} - a_2\,\mathcal{I}_{a_1-1,a_2+1}\,,\\ 0 =&(a_2-a_1)\,\mathcal{I}_{a_1,a_2} + a_1\,\mathcal{I}_{a_1+1,a_2-1} - a_1 p^2\,\mathcal{I}_{a_1+1,a_2} - a_2\,\mathcal{I}_{a_1-1,a_2+1} + a_2 p^2\,\mathcal{I}_{a_1,a_2+1}\,, \end{aligned} \tag{4}$$

where we omitted the dependencies (p^2, m^2) of the integrals.

4. Use the IBPs in eq. (4) to show that

$$\begin{aligned} \mathcal{I}_{a_1,a_2+1}(p^2,m^2) =& \frac{2m^2(a_2-a_1)-p^2(D-2a_1-a_2)}{a_2\,p^2\,(p^2-4m^2)}\,\mathcal{I}_{a_1,a_2}(p^2,m^2) +\\ &+\frac{2a_1\,m^2}{a_2\,p^2\,(p^2-4m^2)}\,\mathcal{I}_{a_1+1,a_2-1}(p^2,m^2) + \frac{p^2-2m^2}{p^2\,(p^2-4m^2)}\mathcal{I}_{a_1-1,a_2+1}(p^2,m^2)\,. \end{aligned} \tag{5}$$

Explain how this relation can be used to write all integrals in the family in terms of a linear combination of the two *master integrals* $\mathcal{I}_{1,0}(p^2,m^2)$ and $\mathcal{I}_{1,1}(p^2,m^2)$.

The IBPs not only allow to write all integrals in the family as a linear combination of two master integrals, they also provide us with a way to compute these master integrals through differential equations.

4. Differentiate $\mathcal{I}_{1,0}(p^2, m^2)$ and $\mathcal{I}_{1,1}(p^2, m^2)$ with respect to the kinematic variables and show that the result can be written in terms of integrals of the same integral family,

$$\partial_{m^2}\mathcal{I}_{1,0}(p^2,m^2) = \mathcal{I}_{2,0}(p^2,m^2)\,, \tag{6}$$

$$\partial_{m^2}\mathcal{I}_{1,1}(p^2,m^2) = \mathcal{I}_{2,1}(p^2,m^2) + \mathcal{I}_{1,2}(p^2,m^2)\,. \tag{7}$$

$$\partial_{p^2}\mathcal{I}_{1,0}(p^2,m^2) = 0\,, \tag{8}$$

$$\partial_{p^2}\mathcal{I}_{1,1}(p^2,m^2) = \frac{1}{2p^2}\,\mathcal{I}_{0,2}(p^2,m^2) - \frac{1}{2p^2}\,\mathcal{I}_{1,1}(p^2,m^2) - \frac{1}{2}\,\mathcal{I}_{1,2}(p^2,m^2) \tag{9}$$

<u>Hint</u>: To act with ∂_{p^2} onto the integrand, show that $\partial_{p^2} = \frac{p^\mu}{2p^2}\frac{\partial}{\partial p^\mu}$.

5. Use IBPs to write the right-hand side of these differential equations in terms of the master integrals and show that you get a closed system of first-order partial differential equations with rational coefficients, which takes the form

$$\partial_{m^2}\begin{pmatrix}\mathcal{I}_{1,0}(p^2,m^2)\\ \mathcal{I}_{1,1}(p^2,m^2)\end{pmatrix} = \begin{pmatrix}\frac{D-2}{2m^2} & 0\\ \frac{D-2}{m^2(p^2-4m^2)} & \frac{2(3-D)}{p^2-4m^2}\end{pmatrix}\begin{pmatrix}\mathcal{I}_{1,0}(p^2,m^2)\\ \mathcal{I}_{1,1}(p^2,m^2)\end{pmatrix} \equiv A_{m^2}\,\vec{\mathcal{I}}\,, \tag{10}$$

$$\partial_{p^2}\begin{pmatrix}\mathcal{I}_{1,0}(p^2,m^2)\\ \mathcal{I}_{1,1}(p^2,m^2)\end{pmatrix} = \begin{pmatrix}0 & 0\\ \frac{2-D}{p^2(p^2-4m^2)} & \frac{(D-4)p^2+4m^2}{2p^2(p^2-4m^2)}\end{pmatrix}\begin{pmatrix}\mathcal{I}_{1,0}(p^2,m^2)\\ \mathcal{I}_{1,1}(p^2,m^2)\end{pmatrix} \equiv A_{p^2}\,\vec{\mathcal{I}}\,. \tag{11}$$

6. Calculate $m^2\,A_{m^2} + p^2\,A_{p^2}$ and explain the result, given that Feynman integrals are homogeneous functions of the kinematic variables.

The result of the previous point clarifies a redundancy contained in the differential equations. Using dimensional analysis, we can pull out an overall scale factor and the remaining integral can then only depend on the ratio $z = m^2/p^2$ of the kinematic variables,

$$\mathcal{I}_{1,0}(p^2,m^2) \equiv (m^2)^{\frac{D-2}{2}}\mathcal{J}_{1,0}(z)\,, \qquad \mathcal{I}_{1,1}(p^2,m^2) \equiv (m^2)^{\frac{D-4}{2}}\mathcal{J}_{1,1}(z)\,. \tag{12}$$

7. Show that the system of differential equations for $\mathcal{J}_{1,0}(z)$ and $\mathcal{J}_{1,1}(z)$ reads

$$\partial_z\begin{pmatrix}\mathcal{J}_{1,0}(z)\\ \mathcal{J}_{1,1}(z)\end{pmatrix} = \begin{pmatrix}0 & 0\\ \frac{2-2\epsilon}{1-4z} & \frac{\epsilon-2z}{z(1-4z)}\end{pmatrix}\begin{pmatrix}\mathcal{J}_{1,0}(z)\\ \mathcal{J}_{1,1}(z)\end{pmatrix} \equiv A(z,\epsilon)\,\vec{\mathcal{J}}(z)\,, \tag{13}$$

where we plugged in $D = 4 - 2\epsilon$.

The differential equation for $\mathcal{J}_{1,1}(z)$ is still not immediate to solve in this form. However, we still have a freedom to change the basis of master integrals we are considering, which we can use to find a simpler differential equation.

8. Consider a new basis $\vec{\mathcal{K}}(z)$ given by

$$\vec{\mathcal{K}}(z) = \begin{pmatrix}\mathcal{K}_1(z)\\ \mathcal{K}_2(z)\end{pmatrix} \equiv \begin{pmatrix}\epsilon\,(m^2)^\epsilon\,\mathcal{I}_{2,0}\\ \epsilon\,p^2(m^2)^\epsilon\sqrt{1-4z}\,\mathcal{I}_{1,2}\end{pmatrix} \equiv \mathcal{T}\,\vec{\mathcal{J}}(z)\,, \tag{14}$$

where $\mathcal{T}$ is a (2×2)-matrix. Determine the matrix $\mathcal{T}$. Then show that the new basis $\vec{\mathcal{K}}(z)$ satisfies the differential equation

$$\partial_z \vec{\mathcal{K}}(z) = B(z,\epsilon)\,\vec{\mathcal{K}}(z)\,, \qquad \text{where} \qquad B(z,\epsilon) = \mathcal{T}\,A(z,\epsilon)\,\mathcal{T}^{-1} + (\partial_z \mathcal{T})\,\mathcal{T}^{-1}\,, \tag{15}$$

and verify that the new differential equations matrix $B(z,\epsilon)$ is ϵ-factorised,

$$B(z,\epsilon) = \epsilon \begin{pmatrix} 0 & 0 \\ -\frac{1}{z\sqrt{1-4z}} & \frac{1}{z(1-4z)} \end{pmatrix}\,. \tag{16}$$

As we are interested only in the first few terms of the Laurent expansion in ϵ of the solution, this differential equation has a very convenient form for us. The coefficient of the next order in ϵ is always just given by an integral over the previous order times the functions in the matrix in eq. (16). However, after the change of basis of master integrals, these functions are now algebraic instead of rational.

9. Change variables to the Landau variable x according to

$$z = -\frac{x}{(1-x)^2} \tag{17}$$

to rationalise the square-root and derive the differential equation of $\vec{\mathcal{K}}$ with respect to x. Thereby, you can take $0 < x < 1$, which corresponds to the Euclidean region $z < 0$. You should find

$$\partial_x \vec{\mathcal{K}}(z(x)) = \begin{pmatrix} 0 & 0 \\ -\frac{\epsilon}{x} & \frac{\epsilon(1-x)}{x(1+x)} \end{pmatrix} \vec{\mathcal{K}}(z(x))\,. \tag{18}$$

We are now ready to integrate the differential equation. For the first integral, this is trivial. The solution is just a constant independent of x, which is determined by the boundary condition. This is expected for integrals that depend on a single kinematic variable. As the dependence on this scale is fixed by dimensional analysis, the differential equation becomes trivial and the boundary condition corresponds to all the non-trivial information on the integral. Therefore, we have no choice but to compute it explicitly, which is easy enough in this example as it is just a tadpole integral. The result reads

$$\mathcal{K}_1 = \epsilon (m^2)^\epsilon\, \mathcal{I}_{2,0} = \mathrm{e}^{\epsilon \gamma_E}\, \Gamma\,(1+\epsilon) = 1 + \frac{\pi^2}{12}\epsilon^2 + \mathcal{O}(\epsilon^3)\,. \tag{19}$$

For the second integral, we can now solve the differential equation to find its kinematic dependence. As $\mathcal{K}_2 \propto \mathcal{I}_{1,2}$, which is UV-finite by power counting and IR-finite as the internal propagators are massive, we can conclude that $\mathcal{K}_2$ is finite for $\epsilon \to 0$ and we can make the ansatz

$$\mathcal{K}_2(z(x)) = \sum_{n=0}^{\infty} \mathcal{K}_2^{(n)}(x)\,\epsilon^n\,. \tag{20}$$

10. Use this ansatz and the differential equation (18) to show that

$$\mathcal{K}_2(x) = B^{(0)} + \epsilon\left(B^{(1)} + (B^{(0)} - 1)\log x - 2B^{(0)}\log(1+x)\right) + \mathcal{O}(\epsilon^2)\,. \tag{21}$$

where $B^{(n)}$ are some constants determined from a boundary condition.

11. Finally, fix the boundary constants by employing the boundary condition

$$\mathcal{I}_{1,1}(p^2 = 0, m^2) = \mathcal{I}_{2,0} = (m^2)^{-\epsilon} \mathrm{e}^{\epsilon\gamma_E}\, \Gamma\left(\epsilon\right) = (m^2)^{-\epsilon} \left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon) \right) . \tag{22}$$

Explain the first equality. Then show that the bubble integral up to the finite piece is given by

$$\mathcal{I}_{1,1}(p^2, m^2) = (m^2)^{-\epsilon} \left[\frac{1}{\epsilon} + 2 + \sqrt{1 - \frac{4m^2}{p^2}} \log \left(\frac{\sqrt{1 - \frac{4m^2}{p^2}} - 1}{\sqrt{1 - \frac{4m^2}{p^2}} + 1} \right) \right] + \mathcal{O}(\epsilon) \,. \tag{23}$$