Scattering Amplitudes in QFT WS 2023/24

Lecturer: Prof. Lorenzo Tancredi
Assistant: Fabian Wagner
https://www.ph.nat.tum.de/ttpmath/teaching/ws-2023-2024/

Sheet 09:

Exercise 1 - Schouten Identity in $D = 2$ dimensions

Prove the Schouten identity in $D = 2$ dimensions, i.e. show that for a two-dimensional vector l^μ, $\mu \in \{0, 1\}$, it holds that

$$\epsilon^{\mu\nu} l^\rho + \epsilon^{\nu\rho} l^\mu + \epsilon^{\rho\mu} l^\nu = 0 \,, \tag{1}$$

where $\epsilon^{\mu\nu}$ denotes the Levi-Civita tensor in $D = 2$ dimensions.

Exercise 2 - Photon vacuum polarisation in $D = 2$ dimensions

In this exercise, we study the one-loop photon vacuum polarisation in QED in $D = 2$ dimensions, the so-called Schwinger model. The electron is taken as massless.

1. Draw the contributing Feynman diagrams and show that the one-loop contribution to the photon vacuum polarisation reads[1]

$$\Pi^{(1)}(p) = \Pi^{(1)}_{\mu\nu}(p) \varepsilon_1^\mu \varepsilon_2^\nu = -e^2 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\mathrm{Tr}\left(\not{\varepsilon}_1 \not{l} \not{\varepsilon}_2 (\not{l} + \not{p})\right)}{l^2 (l+p)^2} \,, \tag{2}$$

where ε_i denote two-dimensional polarisation vectors of the off-shell photon with momentum p, $p^2 \neq 0$, and we use dimensional regularisation, $D = 2 - 2\epsilon$.

To compute the integral, we perform an integrand reduction using the “van Neerven - Vermaseren” basis. For two-point functions in two dimensions, the physical and transverse spaces are both one-dimensional. They are spanned by unit vectors $v^\mu = p^\mu / \sqrt{p^2}$ and $n_\perp^\mu$, $v^2 = n_\perp^2 = 1$, respectively. Furthermore, we denote the unit-vector spanning the remaining, $(D-2)$-dimensional space by n_ϵ^μ, $n_\epsilon^2 = 1$[2]. By construction, $v \cdot n_\perp = v \cdot n_\epsilon = n_\perp \cdot n_\epsilon = 0$. We can now decompose the loop momentum as

$$l^\mu = (l \cdot v) v^\mu + (l \cdot n_\perp) n_\perp^\mu + (l \cdot n_\epsilon) n_\epsilon^\mu \equiv l_p^\mu + l_\perp^\mu + l_\epsilon^\mu \tag{3}$$

2. Use above decomposition and the trace identity[3]

$$\mathrm{Tr}\left(\gamma^\alpha \gamma^\beta \gamma^\rho \gamma^\sigma\right) = 2 \left[g^{\alpha\beta} g^{\rho\sigma} - g^{\alpha\rho} g^{\beta\sigma} + g^{\alpha\sigma} g^{\beta\rho} \right] \tag{4}$$

to simplify the numerator in eq. (2). Your result should read

$$\Pi^{(1)}(p) = -2e^2 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{l^2 (l+p)^2} \left\{ 2 (l_\perp \cdot \varepsilon_1)(l_\perp \cdot \varepsilon_2) + \frac{p^2}{2} \left[\varepsilon_1 \cdot \varepsilon_2 - \frac{(p \cdot \varepsilon_1)(p \cdot \varepsilon_2)}{p^2} \right] \right\} . \tag{5}$$

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[1] The Feynman prescription is omitted for ease of typing.

[2] Remember that what this notation means, is that you need to consider (-2ϵ) of these extra vectors to span the (-2ϵ)-dimensional space, such that $n_\epsilon^2 \to \sum_i n_{\epsilon,i} \cdot n_{\epsilon,i} \sim -2\epsilon$.

[3] In D dimensions, the identity reads $\mathrm{Tr}\left(\gamma^\alpha \gamma^\beta \gamma^\rho \gamma^\sigma\right) = D \left[g^{\alpha\beta} g^{\rho\sigma} - g^{\alpha\rho} g^{\beta\sigma} + g^{\alpha\sigma} g^{\beta\rho} \right]$.

3. Use Lorentz covariance to demonstrate that

$$\varepsilon_{1,\alpha}\,\varepsilon_{2,\beta}\int\frac{\mathrm{d}^D l}{(2\pi)^D}\,\frac{l_\perp^\alpha l_\perp^\beta}{l^2(l+p)^2}=\left[\varepsilon_1\cdot\varepsilon_2-\frac{(p\cdot\varepsilon_1)(p\cdot\varepsilon_2)}{p^2}\right]\int\frac{\mathrm{d}^D l}{(2\pi)^D}\,\frac{l_\perp^2}{l^2(l+p)^2}\,. \tag{6}$$

<u>Hint</u>: Consider a general, symmetric solution $A\,g_{(2)}^{\alpha\beta}+B\,p^\alpha p^\beta/p^2+C\,g_{(\epsilon)}^{\alpha\beta}$ for the tensorial integral and derive A, B, and C by appropriate contractions. Here, $g_{(2)}^{\alpha\beta}$ is the metric tensor of the physical and transverse space and $g_{(\epsilon)}^{\alpha\beta}$ the one for the extra $D-2$ dimensions.

4. Contract eq. (3) with itself and use eq. (6) to show that

$$\Pi^{(1)}(p)=4e^2\left[\varepsilon_1\cdot\varepsilon_2-\frac{(p\cdot\varepsilon_1)(p\cdot\varepsilon_2)}{p^2}\right]\int\frac{\mathrm{d}^D l}{(2\pi)^D}\,\frac{l_\epsilon^2}{l^2(l+p)^2} \tag{7}$$

5. The result in eq. (7) shows that the only nonzero contribution to $\Pi^{(p)}(p)$ stems from ultraviolet regularisation. Such contributions are known as "rational terms" in generalised unitarity. Compute the remaining loop integral. You should find the final result

$$\Pi^{(1)}(p)=\mathrm{i}\frac{e^2}{\pi}\left[\varepsilon_1\cdot\varepsilon_2-\frac{(p\cdot\varepsilon_1)(p\cdot\varepsilon_2)}{p^2}\right]\,, \tag{8}$$

Resuming the vacuum polarisation contributions, Schwinger showed that the photon propagator in two-dimensional QED becomes

$$\Pi^{\mu\nu}(p)=\frac{-\mathrm{i}}{p^2-m_\gamma^2}\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\,. \tag{9}$$

In other words, through these contributions the photon receives a mass $m_\gamma^2=\frac{e^2}{\pi}$ as already visivle in the result in eq. (8).

Exercise 3 - Reduction of scalar triangles in $D=2$ dimensions

In this exercise, you will show that scalar triangle integrals in $D=2$ dimensions can be reduced to lower-point integrals. We start from the most general one-loop scalar triangle integral,

$$\mathcal{I}_\Delta=\int\frac{\mathrm{d}^D l}{(2\pi)^D}\,\frac{1}{D_0D_1D_2}\,, \tag{10}$$

where the inverse propagators read

$$D_0=l^2-m_0^2\,,\quad D_1=(l+q_1)^2-m_1^2\,,\quad D_0=(l+q_2)^2-m_2^2\,, \tag{11}$$

with region momenta q_1,q_2 and we omitted the Feynman prescription.

1. Write down the van Neerven-Vermaseren basis for this problem and decompose the loop momentum l^μ in this basis plus a vector l_ϵ^μ parameterising the $D-2$ extra dimensions. Express the coefficients of this decomposition purely in terms of kinematic invariants and the inverse propagators.

2. Compute l^2 from the decomposition obtained in the previous point and use $l^2 = D_0 + m_0^2$ to show that

$$l_\epsilon^2 = \mathcal{N} + \mathcal{O}(D_i)\,, \tag{12}$$

   where $\mathcal{N}$ is a function of the kinematics only and independent of the loop momentum.

3. From eq. (12), it follows that

$$\mathcal{I}_\Delta = \frac{1}{\mathcal{N}} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l_\epsilon^2}{D_0 D_1 D_2} + \text{lower point integrals}\,. \tag{13}$$

   Conclude the proof that scalar triangle integrals can be reduced to lower-point integrals in $D = 2$ dimensions by showing that

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l_\epsilon^2}{D_0 D_1 D_2} = \mathcal{O}(\epsilon) \tag{14}$$

   for generic external kinematics.