Advanced Methods for Collider Physics

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Ex-sheet 01: Tensor decomposition and Integration-By-Parts Identities

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1 Tensor Decomposition

In this exercise, we get familiar with tensor decomposition as introduced in class.

1.1 't Hooft-Veltman (tHV) versus Conventional Dimensional Regularization (CDR)

Consider the production of a Z boson and a jet in quark-antiquark annihilation, i.e.

$$q(p_1) + \bar{q}(p_2) \to g(p_3) + Z(p_4)$$

. We assume that the quarks are *massless*.

1. Working in QCD and assuming that the Z boson only couples through a vector current (i.e. the theory is CP even), derive the tensor decomposition for this process in CDR. How many tensors do you find?
2. With the same assumptions as above, derive the tensor decomposition for this process in tHV. Explain the difference in the number of tensors.

1.2 Tensor decomposition with massive quarks

1. How does the situation change if the initial state quarks are massive?
2. A situation where a massive quark line appears is the production of a top-antitop pair. How does the tensor basis for $gg \to t\bar{t}$ would look like (always assuming only CP-preserving interactions).

1.3 Tensor decomposition with axial-vector couplings

Consider now a real Z boson, coupling with an axial and a vector current and repeat the exercises of problem 1.1, (i.e. explain what changes in CDR and in tHV).

1.4 Tensor decomposition for n-gluon amplitudes

Considering the scattering of n massless on-shell spin-1 vector bosons (i.e. gluons).

1. Provide a basis of tensor structures for $n = 5$ gluons in tHV. How would your argument change in CDR?
2. Provide a basis of tensor structures for $n = 6$ gluons in tHV. How would your argument change in CDR?

2 IBPs for one-loop three-point functions

Consider the production of a Higgs boson in gluon-gluon annihilation at 1 loop in QCD. We write

$$g(p_1) + g(p_2) \to H(q)$$

with $p_1^2 = p_2^2 = 0$ and $q^2 = 2p_1 \cdot p_2 = m_H^2$. The quark running in the loop has mass m_t.

> **BONUS:** *This is just matter of writing down the Feynman diagrams and computing the traces, but if you feel like, you can try to prove that, working in the cyclic axial gauge*
>
> $$\epsilon_1 \cdot p_1 = \epsilon_2 \cdot p_2 = \epsilon_1 \cdot p_2 = \epsilon_2 \cdot p_1 = 0\,,$$
>
> *the scattering amplitude for this process $\mathcal{S}$ at leading order in QCD (and working modulo prefactors) takes the form*
>
> $$\mathcal{S} \propto \delta_{a_1 a_2} \mathcal{A}$$
>
> *with*
>
> $$\mathcal{A} = 2\left\{\int \frac{d^D k}{(2\pi)^D}\left[\frac{4(\epsilon_1 \cdot k)(\epsilon_2 \cdot k)}{D_1 D_2 D_3}\right] - \epsilon_1 \cdot \epsilon_2 \left[\frac{m_H^2}{2}\int \frac{d^D k}{(2\pi)^D}\frac{1}{D_1 D_2 D_3} + \int \frac{d^D k}{(2\pi)^D}\frac{1}{D_1 D_3}\right]\right\}, \tag{1}$$
>
> *where $D_1 = k^2 - m_t^2$, $D_2 = (k-p_1)^2 - m_t^2$, $D_3 = (k-p_1-p_2)^2 - m_t^2$.*

Based on the general form in eq. (1), we can define the integral family relevant to compute this amplitude as

$$I(a_1, a_2, a_3) = \int \frac{d^D k}{(2\pi)^D}\frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3}} \tag{2}$$

where, again, $D_1 = k^2 - m_t^2$, $D_2 = (k-p_1)^2 - m_t^2$, $D_3 = (k-p_1-p_2)^2 - m_t^2$

2.1 Reducing bubble integrals

Here we will prove that the integral $I(1,-1,1)$ (a so-called rank-1 bubble integral) and the integral $I(2,0,1)$ (a "dotted bubble") can both be reduced to a scalar bubble and a scalar tadpole with rank 0 using either symmetry relations or IBPs.

1. Perform the shift $k \to -k + p_1 + p_2$ and show that

$$I(1,-1,1) = I(1,0,0) - \frac{m_H^2}{2} I(1,0,1)\,.$$

> **BONUS:** *Show how to obtain the same result using a combination of IBPs with all three momenta k, p_1, p_2.*

2. Work out the two IBPs

$$\int \left[\frac{\partial}{\partial k^\mu}\frac{k^\mu}{(k^2 - m_t^2)}\right] = 0\,, \quad \int \left[\frac{\partial}{\partial k^\mu}\frac{k^\mu}{(k^2 - m_t^2)((k-q)^2 - m_t^2)}\right] = 0\,, \quad \text{with} \quad q = p_1 + p_2 \tag{3}$$

and use them to prove that

$$I(2,0,1) = \frac{D-2}{2m_t^2(m_H^2-4m_t^2)} I(1,0,0) - \frac{D-3}{m_H^2-4m_t^2} I(1,0,1)\,.$$

> **BONUS:** *Prove that the other IBP*
>
> $$\int \left[\frac{\partial}{\partial k^\mu} \frac{q^\mu}{(k^2-m_t^2)((k-q)^2-m_t^2)}\right] = 0$$
>
> *does not provide any useful information.*

2.2 Reducing triangle integrals

Consider the same family of integrals as in (2), but assume that $m_t = 0$. Derive the three IBP identities for $a_1 = a_2 = a_3 = 1$

$$\int \frac{d^D k}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \left[\frac{u^\mu}{D_1 D_2 D_3}\right] = 0 \tag{4}$$

for the three choices of momenta $u^\mu = k^\mu, p_1^\mu, p_2^\mu$. You should find

$$\begin{aligned} m_H^2 I(1,1,2) + (D-4) I(1,1,1) &= 0 \\ m_H^2 I(1,1,2) + I(1,0,2) + I(2,0,1) &= 0 \\ m_H^2 I(2,1,1) + I(1,0,2) + I(2,0,1) &= 0\,. \end{aligned} \tag{5}$$

1. Argue that only 2 of the 3 IBPs are linearly independent.
2. Use these identities to prove that the massless triangle can be reduced to a massless bubble.

> **BONUS:** *How does these IBP change if we reinstate the internal mass? Is the triangle still reducible?*