Classical Field Theory & Noether Theorem

QFT 1

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Our goal in this course is to build a consistent relativistic quantum theory; we have argued that the starting point must be a theory that treats $\vec{x}$ & t on same footing $\Rightarrow$ demote $\vec{x}$ to a label & use "FIELD" or QUANTUM OPERATORS $\phi(x^\mu)$

Now that we have studied and clarified Poincaré group, we finally have all ingredients to build such a theory.

TWO MAIN QUESTIONS

1. How do we build a CLASSICAL RELATIVISTIC field theory

2. How do we quantize it and make contact with Hilbert space & one-particle states as studied in general before?

To answer question 1. we need to learn how to write a theory starting from fields. The idea is very simple: we generalize LAGRANGE & HAMILTON Formulation of Classical Mechanics

$$\left.\begin{array}{l}\text{CLASSICAL}\\ \text{MECHANICS}\end{array}\right\}\quad \begin{array}{l}\text{dynamical variables are } \vec{x}, \dot{\vec{x}}\\ \text{and label is } t\end{array}$$

(or $\vec{x}, \vec{p}$ in Hamiltonian language)

$$\left.\begin{array}{l}\text{(QUANTUM)}\\ \text{FIELD}\\ \text{THEORY}\end{array}\right\}\quad \begin{array}{l}\text{dynamical variables are fields } \phi(x)\\ \text{and labels } x^\mu\end{array}$$

Consider then a mechanical system made of N dofs described by coordinates $q_i(t) = q$ full set

Lagrangian $L(q, \dot{q})$ where q_i $i = 1, \dots, N$ is all $\vec{x}_j \dots$ etc

Usually (but not only) $L(q, \dot{q}) = \underbrace{\frac{1}{2} \sum_i m_i^2 \dot{q}_i^2}_{\text{Kinetic Energy}} - \underbrace{V(q)}_{\text{Potential}}$

We then define the ACTION

$$S = \int dt \, L(q(t), \dot{q}(t))$$

Action principle states that classical trajectory starting at $q_i = q(t_i)$ and ending at $q_f = q(t_f)$ is an $\underbrace{\text{EXTREMUM of the action}}_{\text{stationary point, not necessarily MINIMUM !}}$ with these boundaries

$$\delta \int_{t_i}^{t_f} dt \, L(q, \dot{q}) = 0$$

$$= \int_{t_1}^{t_f} dt \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right]$$

$$= \int_{t_i}^{t_f} dt \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \right] \qquad \text{Int by Parts second term}$$

$$= \int_{t_1}^{t_f} dt \left\{ \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right] \delta q_i + \cancel{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right)} \right\}$$

$\delta q_i = 0$ @ t_i & t_f

" boundary term "

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \qquad \forall i$$

EULER-LAGRANGE EQUATIONS

From here, $p_i = \frac{\partial L}{\partial \dot{q}_i}$ conjugate momenta

HAMILTONIAN $\mathcal{H}(q_i, p_i) = \sum p_i \dot{q}_i - L(q_i, \dot{q}_i(p_i, q_i))$

the power of this formalism is its generality

$\Rightarrow$ in a FIELD THEORY $\quad q_i(t) \longrightarrow \phi_i(x)$

fields $\phi_i(x)$ could be anything $\quad \dot{q}_i(t) \rightarrow \partial_\mu \phi_i(x)$

$\Rightarrow$ we are ultimately interested in relativistic theory, so we will consider $\phi_i(x)$ $\left.\begin{array}{l}\text{SCALAR}\\ \text{SPIN } \frac{1}{2}\\ \text{SPIN } 1\end{array}\right\}$

For Example: $\phi_i(x) = A_\mu(x)$ Electromagnetic Field

but in general $\underline{i = \text{multiindex}}$, $\underline{\underline{\text{not only Lorentz}}}$

$\Rightarrow$ the formalism is general, whatever symmetr<u>ies</u> you might wan<u>t to impose</u> !

LAGRANGIAN (DENSITY) $\mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))$

$\Rightarrow$ Action $\quad S = \int d^4x \, \mathcal{L}(\phi_i, \partial_\mu \phi_i)$

We will limit ourselves to $\mathcal{L}$ depending on FINITE NUMBER OF DERIVATIVES $\Rightarrow$ Local theory

As we want the theory to be Poincaré invariant we will use our knowledge of Lorentz IRREPS. to build invariant combinations of ϕ_i, $\partial_\mu \phi_i$

[remember, all fields transform trivially under translations]

$\Rightarrow$ For now we DO NOT SPECIFY ϕ_i LORENTZ TRANSF. we see what we can say in general!

The integral is usually taken on FULL INFINITE SPACE-TIME and we require the physical boundary condition that all $\phi_i(x)$ go to zero at infinity

NOT NECESSARY, we can also FIX FINITE REGION R with boundary ∂R and allow $\mathcal{L}(\phi_i, \partial_\mu \phi_i, x^\mu)$

explicit dependence on the coordinates

$\Rightarrow$ Would be like explicit time dependence in CLASSICAL MECHANICS

In this set up, let us consider now some infinitesimal transformation of co-ordinates x^μ & fields $\phi_i(x)$

$$\begin{cases} x^\mu \to x'^\mu = x^\mu + \delta x^\mu[x] \\ \phi_i(x) \to \phi'_i(x) = \phi_i(x) + \underbrace{\delta F_i[\phi(x)]}_{\text{in general these might mix the fields in non-trivial way}} \end{cases}$$

$\left[\delta x^\mu, \delta F_i \ll 1 \right.$
infinitesimal of same order $\left. \right]$

δF_i gives only FUNCTIONAL VARIATION since $\phi_i(x)$ and $\phi'_i(x)$ are evaluated at same point!

As we have already seen, the total variation will be

$$\underbrace{\phi'_i(x') - \phi_i(x)}_{\Delta \phi_i} = \phi'_i(x') - \phi_i(x') + \phi_i(x') - \phi_i(x)$$

$$\Delta \phi_i \simeq \delta F_i[\phi(x)] + \partial_\mu \phi_i \cdot \delta x^\mu$$

to first order

the action then transforms as

$$\delta S = \int_R \mathcal{L}(\phi_i', \partial_\mu \phi_i', x'^\mu)\, d^4x' - \int_R \mathcal{L}(\phi_i, \partial_\mu \phi_i, x^\mu)\, d^4x$$

$$d^4x' = J \cdot d^4x \Rightarrow J = \det\left[\frac{\partial x'^\mu}{\partial x^\nu}\right]$$

here used

$\det[\mathbb{1} + \varepsilon A] \simeq$

$\mathbb{1} + \varepsilon \, \mathrm{Tr}[A] + O(\varepsilon^2)$

$$\begin{cases} = \det\left[\delta^\mu_\nu + \partial_\nu \delta X^\mu\right] \\ = 1 + \partial_\mu \delta X^\mu \end{cases}$$

$$\delta S = \int_R \left[\delta \mathcal{L} + \mathcal{L}\, \partial_\mu \delta X^\mu\right] d^4x$$

to first order !

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_i} \delta F_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta[\partial_\mu \phi_i] + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta X^\mu$$

changing ϕ_i at fixed x

what is $\delta[\partial_\mu \phi_i] = \partial_\mu \delta \phi_i = \partial_\mu \delta F_i$

because x kept fixed !

so the two derivatives in $\partial_\mu \cdot \delta x^\mu$ recombine to give

$$\delta S = \int_R \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial \phi_i} \delta F_i}_{\mathrm{I}} + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu (\delta F_i)}_{\mathrm{II}} + \partial_\mu \left(\mathcal{L}\, \delta x^\mu \right) \Big] d^4x$$

total divergence

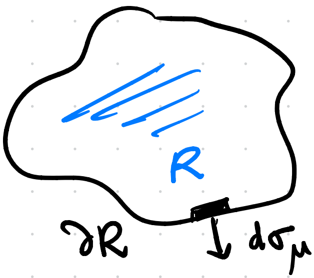
$$\mathrm{II} \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu [\delta F_i] = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta F_i \right]$$

$$- \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \cdot \delta F_i$$

$$\delta S = \int_R \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \right] \delta F_i \, d^4x$$

$$+ \int_R \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta F_i + \mathcal{L}\, \delta x^\mu \right] d^4x$$

Now as we are considering δS on region R, ∂R



use Gauss theorem to rewrite second term as surface integral

$$\delta S = \int_R \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \right] \delta F_i \, d^4x$$

$$+ \int_{\partial R} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta F_i + \mathcal{L} \, \delta X^\mu \right] d\sigma_\mu \qquad (*)$$

$\Rightarrow$ if we assume δX^μ & δF_i ZERO on BOUNDARY ∂R then second term drops; then imposing that ACTION IS STATIONARY $\delta S = 0$ gives

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0}$$

EULER LAGRANGE EQ.

EULER-LAGRANGE EQs. $\Rightarrow$ Equations of Motion of ϕ_i

Note that, if we had considered a DIFFERENT $\mathcal{L}$

$$\mathcal{L}' = \mathcal{L} + \underbrace{\partial_\mu K^\mu}_{\text{Adding a TOTAL DERIVATIVE}}$$

this either does not change the Action if $K^\mu \to 0$ at boundary, or it only changes it by a constant term

$$S' = \int \mathcal{L}' \, d^4x = S + \text{Const}$$

$\Rightarrow \delta S' = \delta S$ so Eq. of Motion are not changed!

We will see examples of field theories and the corresponding equations of motion soon.

Before going there, we can extract more info from $\delta S' = 0$ by changing our point of view.

Interpret transformation

$$\begin{cases} x^\mu \to x'^\mu = x^\mu + \delta X^\mu[x] \\ \phi_i(x) \to \phi'_i(x) = \phi_i(x) + \delta F_i[\phi(x)] \end{cases}$$

<u>not</u> as a "variation" with fixed δX^μ & δF_i

on the boundary ∂R to find stationary point of S

but <u>instead</u> as a SYMMETRY TRANSFORMATION on

our system $\Rightarrow$ reparametization coordinates & fields

which LEAVES THE ACTION INVARIANT

The main formula (*) does not change but

1. R is generic region with $\delta F_i, \delta X^\mu \neq 0$ on ∂R !

2. We <u>IMPOSE</u> $\delta S = 0$, i.e. theory does not change, transformation is SYMMETRY

In this case, let us rewrite the "boundary integral" adding and subtracting $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}[\partial_\nu \phi_i]\, \delta X^\nu$ such that

$$\delta S = \int_R \left\{ \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right] \right\} \delta F_i \, d^4x$$

$$+ \int_{\partial R} \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} [\underbrace{\delta F_i + (\partial_\nu \phi_i) \delta X^\nu}_{\Delta \phi_i \text{ total variation of } \phi_i(x)\,!}] \right.$$

$$\left. - \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \delta^\mu_\nu \mathcal{L} \right]}_{\Theta^\mu_\nu} \delta X^\nu \right\} d\sigma_\mu$$

where we define the ENERGY-MOMENTUM tensor

$$\Theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \delta^\mu_\nu \mathcal{L}$$

now if the transformation is given by a Lie Group

we can write in general

$$\delta x^\mu = \varepsilon^a Y_a^\mu(x)$$

$$\Delta\phi_i = \underbrace{\varepsilon^a G_{i,a}(\phi,\partial\phi)}_{\text{TOTAL VARIATION OF FIELD !}}$$

ε_a set of infinitesimal parameters to parametrize continuous transformation

$a = 1, \dots, N$

Now, let us assume that $\phi_i(x)$ obeys EULER LAGRANGE

then requiring $\delta S = 0$ means

$$\int_{\partial R} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} G_{i,a} - \Theta^\mu_\nu Y_a^\nu \right] \varepsilon^a \, d\sigma_\mu = 0$$

it should be true for arbitrary ε_a !

Define $J^\mu_a = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} G_{i,a} - \Theta^\mu_\nu Y_a^\nu$

$\Rightarrow \int_{\partial R} J^\mu_a \, d\sigma_\mu = 0$ or using Gauss

$$\int_R \left[\partial_\mu J^\mu_a\right] d^4x = 0$$

which should be true for any region R!

$\Rightarrow$ We find a CONSERVED CURRENT J^μ_a

For every infinitesimal $\varepsilon_a \Rightarrow \forall$ generator of the group that is a symmetry of S!

$$\partial_\mu J^\mu_a = 0$$

For each J^μ_a define $Q_a = \int_\Sigma J^\mu_a \, d\sigma_\mu$

where Σ is any SPACE-LIKE surface

typically we choose $t = \text{const} \Rightarrow \Sigma = 3\text{-dim } V$

$$Q_a = \int_V J_a^0 \, d^3\vec{x}$$ $\Rightarrow$ now notice that using $\partial_\mu J_a^\mu = 0$

$$\int_V \partial_\mu J_a^\mu \, d^3\vec{x} = \int_V \partial_0 J_a^0 \, d^3\vec{x} + \int_V \partial_i J_a^i \, d^3\vec{x}$$

$$= \frac{d}{dt} Q_a + \underbrace{\int_{\partial V} J_a^i \cdot d\sigma^i}_{\substack{\text{surface integral} \\ = 0 \text{ if } \partial V \to \infty}}$$

$$\Rightarrow \frac{dQ_a}{dt} = 0$$ we find a CONSERVED CHARGE for every symmetry of the action

this is the statement of NOETHER THEOREM

For every symmetry $\Rightarrow$ conserved quantity !

Let's apply Noether to the special cases

1. TRANSLATIONS

$\delta x^\mu = \varepsilon^\mu \quad (x^\mu \to x^\mu + \varepsilon^\mu)$

total variation! $\Big\} \Rightarrow \Delta\phi_i = \phi'(x+\varepsilon) - \phi(x) = 0$

(all fields invariant!)

then in our notation $Y^\mu_a = \delta^\mu_a$

$G_{ia} = 0$

a = becomes here a Minkowski index

Physics law should of course be invariant, this is part of POINCARE' INVARIANCE

then conserved current is

$$J^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} G_{i,a} - \Theta^\mu{}_\nu Y^\nu_a$$

$\Rightarrow J^\mu = -\Theta^\mu{}_\nu$ ENERGY-MOMENTUM tensor

Conserved Charges $Q_\nu = -\int \theta^0_\nu \, d^3x \equiv -P_\nu$

it turns out to be the 4-momentum of the set of fields $\phi_i(x)$

$$\theta^0_\nu = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \partial_\nu \phi_i - \delta^0_\nu \mathcal{L}$$

$$\Rightarrow \begin{cases} \theta^0_0 = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \dot{\phi}_i - \mathcal{L} \\ \theta^0_j = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \partial_j \phi_i \end{cases}$$

Just as in classical mechanics $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$

$\Rightarrow$ Conjugate momentum to $\phi_i(x)$ $\quad \Pi_i(x) = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i(x))}$

so $\Theta^0_0 = \underbrace{\Pi_i \dot{\phi}_i}_{\text{summed}} - \mathcal{L}$

$= \mathcal{H}$ Hamiltonian DENSITY

and by integrating $H = \int d^3\vec{x}\ \mathcal{H}$ total energy

$\stackrel{!}{=} P_0$ so zeroth component of conserved charge is Energy!

then $P_j = \int \Theta^0_j\, d^3\vec{x} = \int \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \partial_j \phi_i\, d^3\vec{x}$

must be 3-momentum by Lorentz covariance

$\Rightarrow$ only possible choice, given $P_0 = E$!

Of course this means that P^μ always conserved if $\mathcal{L}$ (and then S) don't depend explicitly on x^μ

From its definition, the Energy-momentum tensor does not seem obviously symmetric

$$\Theta^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \delta^{\mu}{}_{\nu} \mathcal{L}$$

IMPORTANT : its definition is not unique and we can always make it symmetric

$$\Theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu} = T^{\mu\nu}$$

with $f^{\lambda\mu\nu} = - f^{\mu\lambda\nu}$ antisymmetric in first two indices

$$\Rightarrow \partial_\mu T^{\mu\nu} = \partial_\mu \Theta^{\mu\nu} + \partial_\mu \partial_\lambda f^{\lambda\mu\nu} = \partial_\mu \Theta^{\mu\nu} = 0$$

still conserved

Also $P^\mu = \int (\Theta^{\mu 0} + \partial_\lambda f^{\lambda 0 \nu}) d^3x =$ does not change because $\int_V \partial_1 f^{1o\nu} d^3x = \int_V \partial_i f^{io\nu} d^3x$

$$\overset{(\text{GAUSS})}{=} \int_\Sigma f^{io\nu} d\sigma^i = 0$$

where $\Sigma \to \infty$ no fields!

$\Rightarrow$ fix $f^{1\mu\nu}$ such that $T^{\mu\nu} = T^{\nu\mu}$

CANONICAL ENERGY MOMENTUM TENSOR

$\Rightarrow P^\mu$ does not change!

A good reason to have a symmetric $T^{\mu\nu}$ is General Relativity, where Einstein Equations read

$$\underset{\substack{\downarrow \\ \text{Ricci tensor} \\ \text{also symmetric!}}}{R_{\mu\nu}} - \frac{1}{2} \underset{\substack{\uparrow \\ \text{metric} \\ = \text{symm!}}}{g_{\mu\nu}} R = - \frac{8\pi G}{c^2} \underset{\substack{\uparrow \\ \text{energy-mom tensor}}}{T_{\mu\nu}}$$

2. LORENTZ TRANSFORMATIONS

Remember for x^μ an infinitesimal Lorentz transf reads

$$x^\mu \to x'^\mu = x^\mu - \frac{i}{2}\omega^{\rho\sigma}\left[S_{\rho\sigma}\right]^\mu{}_\nu\, x^\nu$$

with $\quad [S_{\rho\sigma}]^\mu{}_\nu = i\left(\delta^\mu_\rho\, g_{\nu\sigma} - \delta^\mu_\sigma\, g_{\rho\nu}\right)$

$$x'^\mu = x^\mu + \frac{1}{2}\omega^{\rho\sigma}\left[\delta^\mu_\rho\, x_\sigma - \delta^\mu_\sigma\, x_\rho\right]$$

so going back to our notation

$$\delta x^\mu = \varepsilon^a\, Y^\mu_a(x)$$

$$\Delta\phi_i = \varepsilon^a\, G_{i,a}(\phi, \partial\phi)$$

$$\Rightarrow \quad \varepsilon^a = \omega^{\rho\sigma}\, ; \quad Y^\mu_a = \frac{1}{2}\left[\delta^\mu_\rho\, x_\sigma - \delta^\mu_\sigma\, x_\rho\right]$$

$G_{i,a}$ instead depends on <u>SPIN of FIELDS</u> !

remember, this is total variation and we had

SCALAR: $\Delta\phi = \phi'(x') - \phi(x) = 0$

WEYL L: $\Delta\psi_L = \Lambda_L \psi_L = (-i\vec{\theta} - \vec{\eta})\frac{\vec{\sigma}}{2}\psi_L$

VECTOR: $\Delta V^\mu = \Lambda^\mu{}_\nu V^\nu = -\frac{i}{2}\omega_{\rho\sigma}[S^{\rho\sigma}]^\mu{}_\nu V^\nu$

$\uparrow$ same as for x^μ!

etc etc

Take case of SCALAR FIELD for simplicity, then conserved current is (using $T^{\mu\nu}$ for $\Theta^{\mu\nu}$)

$$j^\mu_a = \cancel{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\mathcal{G}_{i,a}} - \Theta^\mu{}_\nu Y^\nu_a$$

$$j^\mu_{\rho\sigma} = -T^\mu{}_\nu \frac{1}{2}\left[\delta^\nu_\rho x_\sigma - \delta^\nu_\sigma x_\rho\right]$$

$\uparrow$ a = double index in this case

$$j^\mu{}_{\rho\sigma} = -\frac{1}{2}\left[T^\mu{}_\rho x_\sigma - T^\mu{}_\sigma x_\rho \right]$$

from here it's obvious that

$$\partial_\mu j^\mu{}_{\rho\sigma} = -\frac{1}{2}\left[\cancel{\partial_\mu T^\mu{}_\rho x_\sigma} - T^\mu{}_\rho g_{\mu\sigma} - \cancel{\partial_\mu T^\mu{}_\sigma x_\rho} - T^\mu{}_\sigma g_{\rho\mu} \right] \quad \left(\text{using } \partial_\mu T^{\mu\nu} = 0 \right)$$

$$= +\frac{1}{2}\left[T_{\sigma\rho} - T_{\rho\sigma} \right] = 0$$

thanks to symmetry!

Indeed, the zeroth component $j^0{}_{\rho\sigma}$ is

$$j^0{}_{\rho\sigma} = -\frac{1}{2}\left[T^0_\rho x_\sigma - T^0_\sigma x_\rho \right]$$

proportional to angular mom. density of ϕ

$$M_{\mu\nu} = \int \left(T_{0\nu} x_\mu - T_{0\mu} x_\nu \right) d^3x$$

ANGULAR MOM generator of Lorentz!

(notice order $\mu \leftrightarrow \nu$)

$$\frac{d}{dt} M_{\mu\nu} = 0$$

from Noether!