Lorentz & Poincaré: infinite-dimensional representations

Fields and One-Particle States

QFT 1
WS 2025
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In previous lectures we studied the FINITE DIMENSIONAL representations of Lorentz group. Since the group is non-compact, they are NOT UNITARY $\Rightarrow$ they cannot be used to construct physical states

In this lecture we introduce FIELDS & use them to "uncover" also INFINITE-DIM representations of Lorentz & Full Poincaré groups

We start with LORENTZ and afterwards add translations to get full Poincaré

What is a field $\phi(x)$? It's a function of the coordinates which transforms "covariantly" under Lorentz

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu = x'^\mu$$

then in general $\phi(x) \to \phi'(x') = [D(\Lambda)]^a_b \underbrace{\phi^b(\Lambda x)}$

Finite dim representation of Lorentz group

infinite dim repr.

1. SCALAR FIELDS

Scalar ≡ nothing happens under Lorentz

$$\boxed{\phi'(x') = \phi(x)}$$

$\Delta\phi = \phi'(x') - \phi(x) = 0 \Rightarrow$ "Trivial" representation

What we are considering here is how a SINGLE degree of freedom, the field at x^μ, transformed to point x'^μ; explicitly, write

$$\Delta\phi = \phi'(x') - \phi(x) = \phi'(x + \delta x) - \phi(x)$$

$$= \phi'(x) + \delta x^\mu \partial_\mu \phi'(x) - \phi(x) = 0 \Rightarrow$$

$$\phi'(x) - \phi(x) = \delta x^\mu \partial_\mu \phi'(x)$$

$$= \delta x^\mu \partial_\mu \phi(x) \quad \text{to } \underline{\text{first order}}$$

if we write explicitly $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

then $\delta x^\mu = \omega^\mu{}_\nu x^\nu$ for infinitesimal transformations

which we also wrote as $\delta x^\mu = -\frac{i}{2} \omega_{\rho\sigma} [S^{\rho\sigma}]^\mu{}_\nu x^\nu$

with $[S^{\rho\sigma}]^\mu{}_\nu = i(g^{\rho\mu} \delta^\sigma_\nu - g^{\sigma\mu} \delta^\rho_\nu)$ } USES $S^{\mu\nu}$ FOR "SPIN"

Putting everything together we get

$$\phi'(x) - \phi(x) = \frac{i}{2} \omega_{\rho\sigma} [S^{\rho\sigma}]^\mu{}_\nu x^\nu \partial_\mu \phi(x)$$

$$= \frac{i}{2} \omega_{\rho\sigma}\, i \underbrace{\left[x^\sigma \partial^\rho - x^\rho \partial^\sigma \right]}_{L^{\sigma\rho} = -L^{\rho\sigma}} \phi(x)$$

so we find the following relation

$$\phi'(x) - \phi(x) = -\frac{i}{2}\omega^{\rho\sigma} L_{\rho\sigma}\phi(x)$$

What is $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$?

We can check explicitly that

$$[L_{\mu\nu}, L_{\rho\sigma}] = i\left(g_{\nu\rho} L_{\mu\sigma} - g_{\mu\rho} L_{\nu\sigma} - g_{\nu\sigma} L_{\mu\rho} + g_{\mu\sigma} L_{\nu\rho}\right)$$

$\Rightarrow$ they fulfill Lorentz Algebra !

$L_{\mu\nu}$ are generators in some representation, which one?

$\Rightarrow$ the base space is now the INFINITE DIMENSIONAL SPACE given by $\phi(x)$ at all possible x^μ because we are comparing $\phi'(x) - \phi(x)$

Indeed same (equivalent) result studying

$$\phi(x') - \phi(x) \sim \delta x^\mu \partial_\mu \phi(x)$$

so $L_{\mu\nu} = i(\partial_\mu x_\nu - \partial_\nu x_\mu)$ infinite dim. repr. of Lorentz group acting on "coordinate space"

remembering $i\partial_\mu = p_\mu$ we get

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \quad \& \quad \underbrace{L^i = \frac{1}{2}\varepsilon^{ijk} L^{jk}}_{\substack{\text{orbital angular} \\ \text{momentum} \\ \vec{L} = \vec{r} \times \vec{p}}}$$

2. WEYL FIELDS

remember ψ_L, ψ_R in $(\frac{1}{2}, 0)$ & $(0, \frac{1}{2})$

$$\psi_{L/R}(x) \to \psi'_{L/R}(x') = \Lambda_{L/R} \psi_{L/R}(x)$$

$$= e^{(-i\vec{\theta} \mp \vec{\eta})\frac{\vec{\sigma}}{2}} \psi_{L/R}(x)$$

From here, as before, we we can find the representation of Lorentz generators acting on INFINITE DIMENSIONAL space of $\psi_{L,R}(x)$ $\forall x^\mu$!

take LEFT-HANDED

$$\psi_L'(x') - \psi_L(x) = \psi_L'(x + \delta x) - \psi_L(x)$$

$$= \psi_L'(x) - \psi_L(x) + \delta x^\mu \partial_\mu \psi_L'(x)$$

$$= \psi_L'(x) - \psi_L(x) + \delta x^\mu \partial_\mu \psi_L(x) \quad \text{(to first ord)}$$

$$= \psi_L'(x) - \psi_L(x) + \frac{i}{2}\omega^{\rho\sigma} L_{\rho\sigma}\psi_L \quad \text{(As for scalar)}$$

$$\equiv \delta\Lambda_L \psi_L(x) \quad \text{(transformation of full field!)}$$

so we read off:

$$\psi_L'(x) - \psi_L(x) = -\frac{i}{2}\omega^{\rho\sigma} L_{\rho\sigma}\psi_L + \delta\Lambda_L\psi_L$$

for infinitesimal transformations we write

$$\Lambda_L = e^{[-i\vec{\theta}-\vec{\eta}]\frac{\vec{\sigma}}{2}} \overset{!}{\equiv} e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$

which DEFINES $S^{\mu\nu}$

BOOSTS $\begin{cases} S^{i0} = i\frac{\sigma^i}{2} \\ \omega^{i0} = \eta^i \end{cases}$

(we already encountered definitions η^i, θ^i in terms of $\omega^{\mu\nu}$!)

ROTATIONS $\begin{cases} \frac{1}{2}\varepsilon^{ijk}S^{jk} = \frac{\sigma^i}{2} \\ \frac{1}{2}\varepsilon^{ijk}\omega^{jk} = \theta^i \end{cases}$

$$\Rightarrow \psi_L'(x) - \psi_L(x) = -\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\psi_L$$

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$$

orbital angular momentum infinite dimensional representation

Does not depend on SPIN of field $(L=R)$

spin angular momentum $\begin{cases} L \to R \\ S^{i0} = -i\frac{\sigma^i}{2} \end{cases}$

3. DIRAC SPINOR FIELDS

It's interesting to study what happens to Weyl representation under a PARITY TRANSFORMATION $\mathbb{P}$

$$\vec{x} \to -\vec{x}$$

the reason is that many of the interactions we know (ALL except Electroweak Interaction) are PARITY INVARIANT; if we want to build a field theory, we need fields which are also IRREDUCIBLE REPRESENTATIONS of $\mathbb{P}$

Now, remember

$$\begin{cases} A^i = \dfrac{L^i - i k^i}{2} \\ B^i = \dfrac{L^i + i k^i}{2} \end{cases} \Rightarrow (\ell_A, \ell_B) \text{ Irreps of Lorentz}$$

now under PARITY

$$\vec{k} \to -\vec{k} \quad \text{(vector)}$$

$$\vec{L} \to +\vec{L} \quad \text{(pseudo vector)}$$

which means that under $\mathcal{P}$ $(\ell_A, \ell_B) \to (\ell_B, \ell_A)$

Spinor fields in Weyl representation do not have well-defined transformation properties under $\mathcal{P}$!

How do we construct then a SPINOR which is a basis to build a P-invariant theory? $\Rightarrow$ need $\ell_A = \ell_B$

$\Rightarrow$ DIRAC SPINOR FIELD $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ 4-dim. complex!

Under Lorentz $\psi(x) \to \psi'(x') = \Lambda_D \psi(x)$

$$\Lambda_D = \begin{bmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{bmatrix}$$

this is Dirac Field in a special basis called CHIRAL BASIS, where we nicely see separation into LEFT - RIGHT components!

In this chiral Basis, Parity acts as $x' = \mathbb{P}\, x$

$$x'^{\mu} = (t, -\vec{x}) \leftrightarrow x^{\mu} = (t, \vec{x})$$

$$\mathbb{P}\,\psi(x) = \eta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi(x') = \eta \begin{pmatrix} \psi_R(x') \\ \psi_L(x') \end{pmatrix}$$

where $\eta = \pm 1$ is a possible PHASE, in fact

$$\mathbb{P}^2 \psi = \eta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \psi(x'') = \eta^2 \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}$$

$$\eta^2 = 1 \quad \text{since} \quad \mathbb{P}^2 \psi \equiv \psi \quad !$$

Study CHARGE CONJUGATION

remember $i\sigma^2 \psi_L^* = \psi_R$; $-i\sigma^2 \psi_R^* = \psi_L$ so

Define

$$\psi^c = \begin{bmatrix} -i\sigma^2 \psi_R^* \\ i\sigma^2 \psi_L^* \end{bmatrix} = -i \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} \begin{pmatrix} \psi_L^* \\ \psi_R^* \end{pmatrix}$$

we call $\begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} = \gamma^2$ [Dirac γ matrices]

$\psi^c = -i\gamma^2\psi^*$ charge conjugation on Dirac spinor

$$[\psi^c]^c = [-i\gamma^2]\left(-i\gamma^2\psi^*\right)^*$$

$$= -i\begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}\left(-i\gamma^2\right)\psi$$

$$= -\begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}\psi$$

$$= -\begin{bmatrix} -\mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}\psi = \psi$$

charge conjugation will allow us to study relation between particles and antiparticle once we quantize our field theory !

8.2 MAJORANA SPINORS (or "real" Dirac spinors)

Dirac spinors, as Weyl spinors, are COMPLEX (4 complex components)

Now if we have a complex scalar field $\phi(x)$ we can impose on it a reality condition

$$\phi^*(x) = \phi(x)$$

which is Lorentz invariant

$\Rightarrow$ apply it in one frame, it remains true

For Dirac Spinors, we CAN'T $\psi^* = \psi$ is not Lorentz invariant because Λ_D is COMPLEX!

We can define a reality condition using CHARGE CONJUGATION $\psi^c = \psi$

$\Rightarrow$ ψ that satisfies this is called a MAJORANA FIELD it must have half degrees of freedom as DIRAC

$\Rightarrow$ as many as Weyl !

$$\psi_M = \begin{pmatrix} \psi_L \\ i\sigma^2 \psi_L^* \end{pmatrix} \Rightarrow \psi_M^c = -i \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} \begin{pmatrix} \psi_L^* \\ i\sigma^2 \psi_L \end{pmatrix}$$

$$= \begin{pmatrix} \psi_L \\ i\sigma^2 \psi_L^* \end{pmatrix} = \underline{\underline{\psi_M}} \, !$$

If you continue your studies with AQFT & theoretical Particle Physics you'll see that Majorana (real) Spinors could be a possible way to explain observed neutrino masses

We will largely ignore Majorana spinors in what follows

4. VECTOR FIELDS

We can now just generalize these considerations to a vector field, just changing the repr. of the Lorentz group to $(\frac{1}{2}, \frac{1}{2})$

$V^\mu(x)$ is vector field if

$$V^\mu(x) \mapsto = V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x)$$

4-vector representation
Finite Dimensional

$(\frac{1}{2}, \frac{1}{2}) \xrightarrow{\text{PARITY}} (\frac{1}{2}, \frac{1}{2})$ so we can use it for P-invariant theory.

$\Rightarrow$ For example Electromagnetism $A^\mu(x)$

we have seen V^μ has spin 0 $\sim V^0$
spin 1 $\sim \vec{V}$

we will see the gauge invariance removes two of them!

As before

$$V'^{\mu}(x') - V^{\mu}(x) = V'^{\mu}(x-\delta x) - V^{\mu}(x)$$

$$= V'^{\mu}(x) - V^{\mu}(x) - \delta x^{\nu}\partial_{\nu} V^{\mu}(x)$$

$$= \delta\Lambda^{\mu}{}_{\nu} V^{\nu}(x)$$

so we find

$$V'^{\mu}(x) - V^{\mu}(x) = \delta\Lambda^{\mu}{}_{\nu} V^{\nu}(x) - \frac{i}{2}\omega_{\rho\sigma} L^{\rho\sigma} V^{\mu}(x)$$

we have seen that

$$\delta\Lambda^{\mu}{}_{\nu} V^{\nu}(x) = -\frac{i}{2}\omega_{\rho\sigma}[S^{\rho\sigma}]^{\mu}{}_{\nu} V^{\nu}$$

$$[S^{\rho\sigma}]^{\mu}{}_{\nu} = i\left(g^{\rho\mu}\delta^{\sigma}_{\nu} - g^{\mu\sigma}\delta^{\rho}_{\nu}\right)$$

is SPIN OPERATOR on vector representation!

so transformation is again

$$V^{\mu'}(x) - V^{\mu}(x) = -\frac{i}{2}\,\omega_{\rho\sigma}\,[J^{\rho\sigma}]^{\mu}{}_{\nu}\,V^{\nu}(x)$$

$$[J^{\rho\sigma}]^{\mu}{}_{\nu} = \underbrace{L^{\rho\sigma}\,\delta^{\mu}_{\nu}}_{\text{orbital angular momentum (infinite dimensional representation)}} + \underbrace{[S^{\rho\sigma}]^{\mu}{}_{\nu}}_{\text{spin, finite dimensional representation}}$$

POINCARE' GROUP

We have focussed on Lorentz, but theory should be invariant under full Poincaré = Lorentz + Translations

$$x^{\mu} \to \Lambda^{\mu}{}_{\nu}\,x^{\nu} \qquad ; \qquad x^{\mu} \to x^{\mu} + a^{\mu}$$

$$x'^{\mu} \sim x^{\mu} + \omega^{\mu}{}_{\nu}\,x^{\nu} \qquad\quad x'^{\mu} \sim x^{\mu} + \underbrace{\varepsilon^{\mu}}_{\text{infinitesimal translation}}$$

AS INFINITE DIM. REPRESENTATION ON FIELDS

We <u>impose</u> that every field transforms trivially under translations (independently of SPIN)

$$\phi'(x') - \phi(x) = 0 = \phi'(x+\varepsilon) - \phi(x)$$

$$= \phi'(x) - \phi(x) + \varepsilon^\mu \partial_\mu \phi = \phi'(x) - \phi(x) - i\varepsilon^\mu p_\mu \phi$$

$$\phi'(x) - \phi(x) = i\varepsilon^\mu p_\mu \phi(x)$$

$$\phi'(x'-\varepsilon) - \phi'(x') = i\varepsilon^\mu p_\mu \phi(x) \Rightarrow T(a)\phi(x):$$

$$T(a) = e^{-iP^\mu a_\mu} \underset{a^\mu = -\varepsilon^\mu}{\sim} 1 + iP^\mu \varepsilon_\mu + O(\varepsilon^2)$$

same sign conventions

Lorentz + Translations = Poincaré $ISO(1,3)$

(inhomogeneous Lorentz group)

$$U(\Lambda, a) = U(1+\omega, \varepsilon^\mu) = 1 - \frac{i}{2}\omega^{\mu\nu} J_{\mu\nu} + iP^\mu \varepsilon_\mu$$

$\Rightarrow$ clearly for $U(\Lambda, a)$ to be UNITARY $U^\dagger = U^{-1}$

we need $J^{\mu\nu}$ & P^μ to be HERMITIAN

$$(J^{\mu\nu})^\dagger = J^{\mu\nu} \quad ; \quad (P^\mu)^\dagger = P^\mu$$

Before going into REPRESENTATION THEORY notice

$[P^\mu, P^\nu] = 0$ translations commute !

Similarly, using Lorentz generators in same repr.

$$L^{\mu\nu} = i\,(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

$$\Rightarrow [P^\mu, L^{\rho\sigma}] = i\,(g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho)$$

which can then be lifted to abstract definition of the Poincaré group, whose abstract gen. we wrote $J^{\rho\sigma}$

$$[P^\mu, J^{\rho\sigma}] = i\,(g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho)$$

in any representation !

note that requiring that $\phi'(x+\varepsilon) = \phi(x)$

for every field in the only reasonable possibility

$\Rightarrow$ one can show that finite-dimensional

non-trivial representation on ϕ consistent

with Lorentz act all TRIVIALLY

In terms of $L^i = \frac{1}{2}\varepsilon^{ijk} J^{jk}$, $K^i = J^{i0}$

$P^0 = \mathcal{H}$ (Hamiltonian) , P^i

the commutation relations read

$[L^i, L^j] = i\varepsilon^{ijk} L^k$ $\qquad$ $[P^i, P^j] = 0$

$[L^i, K^j] = i\varepsilon^{ijk} K^k$ $\qquad$ $[L^i, P^j] = i\varepsilon^{ijk} P^k$

$[K^i, K^j] = -i\varepsilon^{ijk} L^k$ $\qquad$ $[K^i, P^j] = i\mathcal{H}\delta^{ij}$

$$\underbrace{[L^i, \mathcal{H}] = 0 \, ; \; [P^i, \mathcal{H}] = 0}_{\vec{L},\, \vec{P} \text{ conserved}} \; ; \; \underbrace{[K^i, \mathcal{H}] = iP^i}_{K^i \text{ not ! not used to label states}}$$

QUANTUM POINCARE' TRANSFORMATIONS

Until now we discussed fields and their transf. properties under Poincaré. In some sense, we have only cared about SPECIAL RELATIVITY and gave the ingredients to build a field theory that is Lorentz invariant.

The next question is WHAT ABOUT QUANTUM MECHANICS? How do we embed the transformation properties under Poincaré on HILBERT SPACE OF PHYSICAL STATES. And, in turn, how does covariance under Poincaré constraints the Hilbert space itself?

- In QM, physical states are RAYS in HILBERT SPACE $\mathcal{H}$

two "vectors" $|X\rangle, |Y\rangle \in \mathcal{H}$; $a|X\rangle + b|Y\rangle \in \mathcal{H}$

$a, b \in \mathbb{C}$

$$\langle X, Y \rangle = \langle Y, X \rangle^*$$

$\langle X, X \rangle \geq 0$ norm of a vector a ray is a set of normalized vectors

$$|X_R\rangle = \{ \alpha|X\rangle ; \ |\alpha| = 1 \} \qquad \langle X_R, X_R \rangle = 1$$

normalized to 1

- Observables are represented by Hermitian operators A

$$A|Y\rangle = |X\rangle ; \qquad A(a|X\rangle + b|Y\rangle) = a A|X\rangle + b A|Y\rangle$$

$$A^{+} = A \Rightarrow \langle X, AY \rangle = \langle A^{+}X, Y \rangle$$

$$= \langle Y, A^{+}X \rangle^{*}$$

- Eigenstates of A with eigenvalue α have that value of the corresponding observable

$$A|X_R\rangle = \alpha|X_R\rangle \quad \text{for the RAY !}$$

- Probability of finding system represented by $|X_R\rangle$ in state $|Y_R\rangle$ is $P(|X_R\rangle \to |Y_R\rangle) = |\langle Y_R | X_R \rangle|^2$

SYMMETRIES ON HILBERT SPACE

We want to know how to "implement" Poincaré transformations on Hilbert space.

In general, any symmetry acts on Hilbert space by modifying rays $|X_R\rangle \to |X_R'\rangle$

$|Y_R\rangle \to |Y_R'\rangle$ etc

such that probabilities are conserved

$$P(|X_R\rangle \to |Y_R\rangle) = P(|X_R'\rangle \to |Y_R'\rangle)$$

A theorem by WIGNER says that any such transformation on physical states can be represented ~~as~~ as an OPERATOR U

$$\forall\, |X\rangle \in |X_R\rangle \Rightarrow U|X\rangle \in |X_R'\rangle$$

such that :

- either U is LINEAR AND UNITARY $(U^\dagger = U^{-1})$
- or U is ANTILINEAR AND ANTIUNITARY

Now if we deal with a continuous Symmetry (A LIE GROUP) that is connected with the identity (I.E. $\exists$ a value of parameters such that $U(\alpha) = \mathbb{1}$) then U must be LINEAR AND UNITARY

$\Rightarrow$ because $U = \mathbb{1}$ is clearly unitary!

then close to the identity $U = \mathbb{1} + i\varepsilon' T$

where $U^\dagger = U^{-1}$ $\Rightarrow$ $T^\dagger = T$ Hermitian!

[the only transformation which is not UNITARY is TIME REVERSAL, which is not continuous and not part of $SO^+(1,3)$ + translations which we are considering now! $\Rightarrow$ see EX.]

Now we have studied Poincaré Group as a Lie Group and we have clarified its FINITE DIM REPS $\Rightarrow$ what about action on Hilbert ?

Let's call a generic element $T(\Lambda, a) \in ISO^{+}(1,3)$

Λ Lorentz proper

a^{μ} 4-vector translation

Wigner's theorem says $\exists$ Unitary linear $U(\Lambda, a)$

$|X_R\rangle \longrightarrow U(\Lambda, a)|X_R\rangle \Rightarrow$ $\boxed{\theta \to U(\Lambda)^{-1}\theta\, U(\Lambda)}$ FOR OPERATORS !

We will now classify $|X_R\rangle$ states depending on how they transform under $ISO^{+}(1,3)$

- We know that $[P^{\mu}, P^{\nu}] = 0$, so we can choose eigenvalues of P^{μ} to start labelling the states $|X_R\rangle = |p, \sigma\rangle$

we use label σ for any other degree of freedom.

$$\Rightarrow \quad P^\mu |p,\sigma\rangle = p^\mu |p,\sigma\rangle$$

so if we now perform a translation we get

$$(T(a) = U(\mathbb{1}, a))\, |p,\sigma\rangle$$

$$= e^{-i p^\mu a_\mu} |p,\sigma\rangle$$

- Consider now a pure Lorentz $U(\Lambda, \underset{\uparrow}{0}) = U(\Lambda)$ on such a state

no translation

$$U(\Lambda) |p,\sigma\rangle = \,?$$

We notice that

$$P^\mu U(\Lambda) |p,\sigma\rangle = U(\Lambda) [\underbrace{U^{-1}(\Lambda) P^\mu U(\Lambda)}] |p,\sigma\rangle$$

Lorentz transf on operator P^μ $\Rightarrow$ its a 4-vector

$\rightarrow$

$$U^{-1}(\Lambda)\, P^\mu\, U(\Lambda) = \Lambda^\mu{}_\nu\, P^\nu$$ such that

$$P^\mu\, U(\Lambda)|p,\sigma\rangle = \underbrace{\Lambda^\mu{}_\nu\, p^\nu}\, U(\Lambda)|p,\sigma\rangle$$

it is some state with momentum Λp

$$\Rightarrow U(\Lambda)|p,\sigma\rangle = \underbrace{\sum_{\sigma'} C_{\sigma,\sigma'}(\Lambda,p)\,|\Lambda p,\sigma'\rangle}$$

Linear combinations of states with momentum Λp

We then **DEFINE** ONE PARTICLE STATES as linear combinations of $|p,\sigma\rangle$ such that $C_{\sigma,\sigma'}(\Lambda,p)$ are IRREDUCIBLE REPRESENTATIONS OF POINCARE' GROUP

$\Rightarrow$ if we have a system made of more particles, $C_{\sigma,\sigma'}$ will be in BLOCK DIAGONAL FORM $\Rightarrow$ they DO NOT MIX under Poincare'!

to clarify these representations "$C_{\sigma\sigma'}$" notice that:

$p^2 = p^\mu p_\mu$ invariant under $\Lambda^\mu{}_\nu$

$p^0 = E$ sign of E also invariant if $p^2 \geq 0$
(E cannot become negative!)

$\forall\, p^\mu$ we can choose a "reference" momentum k^μ
such that $p = L_p k$ with $L_p \in SO^+(1,3)$:

- if $p^2 > 0 \Rightarrow k^\mu = (m, \vec{0})$ such that $\forall\, p$
  $\exists\, L_p \in SO^+(1,3)$ such that
  $$[L_p]^\mu{}_\nu k^\nu = p^\mu$$
- same for $p^2 = 0 \Rightarrow k^\mu = (E, 0, 0, E)$ or similar
  $$[L_p]^\mu{}_\nu k^\nu = p^\mu$$

then we can say $|p, \sigma\rangle = N_p\, U(L_p)\, |k, \sigma\rangle$

normalization

now we perform some generic Lorentz $U(\Lambda)$

$$U(\Lambda)\,|p,\sigma\rangle = N_p\, U(\Lambda)\, U(L_p)\,|k,\sigma\rangle$$

$$= N_p\, U(\Lambda\cdot L_p)\,|k,\sigma\rangle$$

insert $U(L_{\Lambda p})\,U(L^{-1}_{\Lambda p}) = \mathbb{1}$

$$= N_p\, U(L_{\Lambda p})\, U(\underbrace{L^{-1}_{\Lambda p}\,\Lambda\, L_p})\,|k,\sigma\rangle$$

special type of Lorentz transf.

$k \to p \to \Lambda p \to k$

$\subset$ "Little group" of k^μ

$$W^\mu{}_\nu\, k^\nu = k^\mu$$

so if W is a transf. in Little group of k^μ

$$U(W)\,|k,\sigma\rangle = \sum_{\sigma'} \underbrace{D_{\sigma,\sigma'}(W)}\,|k,\sigma'\rangle$$

one can prove $D_{\sigma,\sigma'}$ provide a representation of Little group of k^μ

so $L^{-1}_{\Lambda p} \Lambda L_p = W_{\Lambda,p}$ and we can write

$$U(\Lambda)|p,\sigma\rangle = N_p \sum_{\sigma'} \underbrace{D_{\sigma\sigma'}(W_{\Lambda,p})}_{\forall\, \sigma,\sigma' \text{ numerical coefficients}} U(L_{\Lambda p})|k,\sigma'\rangle$$

Using finally $|p,\sigma\rangle = N_p U(L_p)|k,\sigma\rangle$

$\Rightarrow$ for $U(L_{\Lambda p})|k,\sigma'\rangle = \dfrac{1}{N_{\Lambda p}} |\Lambda p,\sigma'\rangle$

we can write

$$U(\Lambda)|p,\sigma\rangle = \left[\frac{N_p}{N_{\Lambda p}}\right] \sum_{\sigma'} D_{\sigma\sigma'}(W_{\Lambda,p}) |\Lambda p,\sigma'\rangle$$

which is what we wanted

$$U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma\sigma'}(\Lambda,p) |\Lambda p,\sigma'\rangle$$

$$\Rightarrow \quad C_{\sigma\sigma'} \sim D_{\sigma\sigma'} \quad !$$

$\Rightarrow$ representations of Poincaré are clarified through $D_{\sigma\sigma'}(W_{\Lambda,p})$ representations of LITTLE GROUP !

Called method of induced representations

[we ignore normalization $N_p/N_{\Lambda p}$, see Weinberg section 2.5]

So everything we are left to do is study little group for different classes of momenta

1. $k^2 > 0$ (massive particles)

2. $k^2 = 0$ (massless particles)

3. $k^2 < 0$ (tachions, unphysical) $\Rightarrow$ IGNORE

$\Rightarrow$ of course $k^\mu = 0$ is just "trivial vacuum" and there nothing to say -

1. $k^2 = m^2 > 0$ MASSIVE PARTICLE STATES

Little group easy to see from $k^\mu = (m, 0, 0, 0)$

clearly $SO(3)$ leaves k^μ invariant

Irreps are the irreps of $SO(3) \sim SU(2)$, including spinor representations!

Irreps labelled by m & $l = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$

$\forall\, l \Rightarrow l_z = \{-l, -l+1, \dots, l\}$

"SPIN OF PARTICLE"

there are $2l+1$ STATES

$\Rightarrow$ a massive SPIN 0 has 1 state

a massive SPIN 1 has 3 states $(-1, 0, 1)$

etc

We classify these states using CASIMIRS of GROUP

$p^\mu p_\mu = p^2$ is clearly invariant $\Rightarrow$ it gives the mass!

there is a second Casimir (and ONLY a 2nd one!)
which indeed represents SPIN

PAULI-LUBANSKI VECTOR

$$W^\mu = -\frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$$

$W^\mu W_\mu$ Lorentz invariant $\Rightarrow$ $[W^\mu W_\mu, J^{\nu\rho}] = 0$

$$[W^\mu, P^\nu] = 0$$

go to rest mass frame for massive particles $P^\mu = (m, \vec{0})$

$$W^\mu = -\frac{m}{2}\,\varepsilon^{\mu\nu\rho 0} J_{\nu\rho} \Rightarrow \begin{cases} W^0 = 0 \\ W^i = \frac{m}{2}\,\varepsilon^{0ijk} J^{jk} \\ \quad = m L^i \end{cases}$$

it's the angular momentum

$$-W_\mu W^\mu = m^2 \ell(\ell+1) \Leftarrow$$

SPIN !

massless limit is DELICATE !

2. $k^2 = 0$ MASSLESS PARTICLE STATES

Here we choose our momentum as $k^\mu = (E, 0, 0, E)$

Conventional on z-axis!

What is the Little Group in this case?

$\Rightarrow$ it looks like $SO(2)$, but the story is more delicate!

Going to so-called light-cone coordinates $\begin{cases} k_+ = \dfrac{k_0 + k_z}{\sqrt{2}} \\ k_- = \dfrac{k_0 - k_z}{\sqrt{2}} \end{cases}$

we see that

$$k^\mu = (k_+, k_-, k_x, k_y) = (k_+, \underbrace{0, 0, 0}_{\text{there is more than } SO(2)})$$

One can prove that the full group is $ISO(2)$

$\Rightarrow$ Euclidean group of ROTATIONS + TRANSLATIONS

L_z A, B

Since translations commute $[A,B]=0$,
we could then try to label states with a,b

$$A|p;a,b\rangle = a|p;a,b\rangle$$
$$B|p;a,b\rangle = b|p;a,b\rangle$$

but now consider $e^{-i\theta L_z}|p;a,b\rangle = |p;a,b,\theta\rangle$

for this states, with arbitrary θ, one can prove that

$$A|p;a,b,\theta\rangle = (a\cos\theta - b\sin\theta)|p;a,b,\theta\rangle$$
$$B|p;a,b;\theta\rangle = (a\sin\theta + b\cos\theta)|p;a,b,\theta\rangle$$

states labelled by continuous degree of freedom, that we <u>have never SEEN</u>

$\Rightarrow$ Continuous Spin States, usually neglected putting $a=b=0$; they are subject of new research recently!

if we fix $a = b = 0$ then only $SO(2)$ is left!

representations $\sim U(1)$ labeled by the eigenvalue of L_z $\Rightarrow$ spin of particles, in direction of propagation called HELICITY

$$|p, h\rangle \Rightarrow e^{i\vartheta L_z}|p, h\rangle = e^{ih\vartheta}|p, h\rangle$$

h is in general CONTINUOUS! To prove that it must be quantized we need to resort to TOPOLOGY of Lorentz group $SL(2, \mathbb{C})$

$\Rightarrow$ $SL(2, \mathbb{C})$ is DOUBLE COVERING just like $SU(2)$

it means $e^{ih4\pi} = 1 \Rightarrow h = \frac{n}{2}$ HALF INTEG.

NOTE: each helicity is a DIFFERENT PARTICLE!

Why PHOTONS have 2 states: ± 1

$\Rightarrow$ because of us wanting PARITY-INVARIANCE we need to put $(h, -h)$ together!