The Lorentz Group & Lorentz Algebra: Finite Dimensional Representations

QFT 1

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We are now ready to study the Lorentz and Poincaré groups & ALGEBRAS from a more general point of view $\Rightarrow$ the goal will be to clarify all FINITE & INFINITE DIMENSIONAL REPRESENTATIONS

the first we will use to construct FIELD THEORIES that have the right covariant properties under Lorentz (& Poincaré)

the second will allow us to define "one particle states" on the Hilbert space of the QUANTIZED VERSION of the theory

In previous lecture we have seen that $SO^+(1,3)$ is generated by 3 rotations, 3 boosts $\Rightarrow$ 6 parameters

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

$\omega_{\mu\nu}$ infinitesimal parameters

$\Rightarrow$ 4×4 antisymmetric matrix (6 independent entries!)

if we label the GENERATORS $J^{\mu\nu} = -J^{\nu\mu}$

then the EXPONENTIAL MAP implies that we can write an arbitrary Lorentz transformation as

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$$

$\Rightarrow$ $\frac{1}{2}$ is for double-counting due to $\omega_{\mu\nu} = -\omega_{\nu\mu}$

Now we would like to determine the commutation relations among the $J^{\mu\nu}$ in order to determine the Lie Algebra of the Group & study its representations!

$\Rightarrow$ the usual way to determine the Lie Algebra is through an EXPLICIT REPRESENTATION [typically, boost & rotations acting on 4-d vectors]

Consider a vector V^μ; we know Lorentz acts as $V^\mu \to \Lambda^\mu{}_\nu V^\nu$

now how do we go from ABSTRACT $\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$ to this specific representation?

Infinitesimally $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$ implies

$$\Rightarrow \quad \delta V^\rho = \Lambda^\rho{}_\sigma V^\sigma - V^\rho = \omega^\rho{}_\sigma V^\sigma$$

then what are the $[J^{\mu\nu}]^\rho{}_\sigma$ such that

$$\left(\left[e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \right]^\rho{}_\sigma - \delta^\rho_\sigma \right) V^\sigma = \omega^\rho{}_\sigma V^\sigma \; ?$$

THEY WILL REPRESENT SPIN $\Rightarrow$ call them $\underbrace{[S^{\mu\nu}]^\rho{}_\sigma = [J^{\mu\nu}]^\rho{}_\sigma}$

I WANT TO KEEP $J^{\mu\nu}$ FOR ABSTRACT GENERATORS!

expanding for small $\omega^{\mu\nu}$ we get

$$-\frac{i}{2}\omega_{\mu\nu}[S^{\mu\nu}]^\rho{}_\sigma V^\sigma = \omega^\rho{}_\sigma V^\sigma$$

$$\Rightarrow \quad \frac{1}{2}\omega_{\mu\nu}[S^{\mu\nu}]^\rho{}_\sigma = i\,\omega^\rho{}_\sigma$$

$$\Rightarrow \frac{1}{2}\omega_{\mu\nu}\,[S^{\mu\nu}]^{\rho}{}_{\sigma} = i\left[g^{\mu\rho}\,\delta^{\nu}_{\sigma}\right]\omega_{\mu\nu}$$

using antisymmetry $\omega_{\mu\nu}$

$$= \frac{i}{2}\left[g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\,\delta^{\mu}_{\sigma}\right]\omega_{\mu\nu}$$

so we find the VECTOR REPRESENTATION of the Lorentz generators

$$[J^{\mu\nu}]^{\rho}{}_{\sigma} = [S^{\mu\nu}]^{\rho}{}_{\sigma} = i\left(g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\,\delta^{\mu}_{\sigma}\right)$$

From here, we can compute explicitly the commutation relations, which can then be lifted to be true for ABSTRACT LIE GROUP

$S^{\mu\nu} \to J^{\mu\nu}$ Abstract generator

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right)$$

In this form, it's not really informative

⇒ we know actually that on 4-vectors these generators should somehow correspond to ROTATIONS + BOOSTS

To see this, let's rearrange the generators as:

$$L^i = \frac{1}{2} \varepsilon^{ijk} J^{jk} \quad ; \quad K^i = J^{i0}$$

where ε^{ijk} is Levi-Civita in 3-dimensions

$\varepsilon^{123} = 1$, fully ANTISYMMETRIC

L^i & K^i are now both 3-d vectors

for K^i it's obvious, for L^i do Lorentz:

$$L^i \to \frac{1}{2} \varepsilon^{ijk} R^{je} R^{km} J^{em}$$

where I used $\Lambda^{ie} = R^{ie}$

$R \in SO(3)$

now remember

$$R^{in} R^{je} R^{km} \varepsilon^{nem} = \varepsilon^{ijk}$$

under proper rotation (no PARITY!)

Use $R^T = R^{-1}$ to write

$$(R^{-1})^{rj}(R^{-1})^{sk} R^{in} R^{je} R^{km} \varepsilon^{nem} = (R^{-1})^{rj}(R^{-1})^{sk} \varepsilon^{ijk}$$

$$\Downarrow \qquad R^{-1} = R^T$$

$$R^{in} \varepsilon^{nrs} = R^{jr} R^{ks} \varepsilon^{ijk}$$

rename $r = \ell$, $s = m$, $n = j$ (only in RHS)

$$R^{ij} \varepsilon^{j\ell m} = \varepsilon^{ijk} R^{j\ell} R^{km}$$

so $L^i \to \frac{1}{2} \varepsilon^{ijk} R^{j\ell} R^{km} J^{\ell m}$

$$= \frac{1}{2} R^{ij} \varepsilon^{j\ell m} J^{\ell m} = R^{ij} L^j$$

transforms as a vector under rotations (as long as $\det R = 1$)

if you specialize commutation relations you get easily:

$$[L^i, L^j] = i\,\varepsilon^{ijk} L^k \quad \Leftarrow \text{SU(2) Lie Algebra}$$

$$[L^i, K^j] = i\,\varepsilon^{ijk} K^k \quad \Leftarrow K^k \text{ is a vector}$$

$$[K^i, K^j] = -i\,\varepsilon^{ijk} L^k$$

these commutation relations are more "readable" than the previous ones; if we choose these generators, then a Lorentz transf. in the VECTOR REPRESENTATION will be

$$\Lambda = e^{\left[-i\vec{\theta}\cdot\vec{L} + i\,\vec{\eta}\cdot\vec{K}\right]}$$

where the signs are conventional. Explicitly you can go from the previous $\omega_{\mu\nu} \to \theta^i, \eta^i$ as

$$\theta^i = \frac{1}{2}\varepsilon^{ijk}\omega^{jk} \quad ; \quad \eta^i = \omega^{i0}$$

which correspond to ACTIVE ROTATION:

- $\theta > 0$: we rotate a point counterclockwise of θ
- $\eta > 0$: we boost a particle giving it velocity $+\vec{v}$

REPRESENTATIONS OF LORENTZ ALGEBRA

Algebra of the group only knows about its properties close to the IDENTITY ⇒ LOCALLY

Global structure of group cannot be derived from it!

Why Algebra & not group? ⇒ Reps. of Algebra give reps. of UNIVERSAL COVERING ⇒ WHAT MATTERS IN QM!

EXAMPLE $SO(3) \sim SU(2)$ same algebra

but GLOBALLY different groups!

In $SO(3)$ a 2π rotation must take us back to where we started from

In $SU(2)$ there are SPIN $\frac{1}{2}$ ⇒ $e^{i\frac{1}{2}\varphi} = -1$

For $\varphi = 2\pi$

the Spin $\frac{1}{2}$ representations are NOT representations of the GROUP $SO(3)$ because they do not respect its global properties ($\varphi = 2\pi$ not infinitesimal)

When we study angular momentum in QM we consider SU(2), not SO(3) $\Rightarrow$ Full SPINOR ALGEBRA including spin $\frac{1}{2}$ representations

- Physics point of view $\Rightarrow$ if wave function transforms as $\psi \xrightarrow{R} (-1)\psi$, physical predictions are not impacted because only $|\psi|^2$ matter!

- Maths point of view $\Rightarrow$ a state in QM is a RAY in the HILBERT SPACE $|\psi\rangle \sim e^{i\alpha}|\psi\rangle$

$\forall$ symmetry group $\mathcal{G}$, we can allow for general PROJECTIVE REPRESENTATIONS

$$\forall\, g_1, g_2 \in \mathcal{G} \;; \quad D_R(g_1)\, D_R(g_2) = e^{i\alpha(g_1,g_2)}\, D_R(g_1 g_2)$$

IS ALLOWED !

Now, it turns out that the general projective representations of SO(3) are in 1-1 correspondence with the <u>PROPER</u> (TRUE) representations for ITS UNIVERSAL COVERING GROUP SU(2)

$\Rightarrow$ this is true <u>for any connected Lie group!</u>

in the special case of SU(2) $\Rightarrow$ DOUBLE COVER of SO(3)

this means that either we study projective reps in SO(3), or (simpler) we study STANDARD REPS in SU(2) !

SU(2) Algebra: $[L^i, L^j] = i\,\varepsilon^{ijk} L^k$

Casimir $L^2 = \vec{L}^2$; $L^2 |\psi\rangle = \ell(\ell+1)|\psi\rangle$

ℓ labels the <u>IRRED. repr.</u> $\forall \ell$ $L_z|\psi\rangle = m|\psi\rangle$

$-\ell \leq m \leq \ell$; $2\ell+1$ STATES ; $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

the FUNDAMENTAL REPR. is the SPIN $\frac{1}{2}$

$l = \frac{1}{2}$ $\quad m = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$ DIMENSION 2

typically we choose $L^i = \frac{\sigma^i}{2}$ PAULI MATRICES

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} \Rightarrow \left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\varepsilon^{ijk}\frac{\sigma^k}{2}$$

All other representations can be obtained by combining spin $\frac{1}{2}$ representations through usual ANGULAR MOMENTUM SUM RULES

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

SINGLET $l = 0$

$m = 0$ $\quad (\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2}$

VECTOR $l = 1$

$m = +1$ $\quad \uparrow\uparrow$, $\quad \downarrow\downarrow$ $\quad m = -1$

$\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}$ $\quad m = 0$

Now let's go back to $SO^{+}(1,3)$ and its algebra.

We know that

$$[L^i, L^j] = i\,\varepsilon^{ijk} L^k$$
$$[L^i, K^j] = i\,\varepsilon^{ijk} K^k$$
$$[K^i, K^j] = -i\,\varepsilon^{ijk} L^k$$

Let us then define two complex linear combinations of L, K

$$\begin{cases} A^i = \dfrac{L^i - i K^i}{2} \\ B^i = \dfrac{L^i + i K^i}{2} \end{cases}$$

it's then easy to see that

$$\begin{cases} [A^i, A^j] = i\,\varepsilon^{ijk} A^k \\ [B^i, B^j] = i\,\varepsilon^{ijk} B^k \\ [A^i, B^j] = 0 \end{cases}$$

we are left with what looks like 2 copies of $SU(2)$!

often it is then said at the ALGEBRA level

$SO^+(1,3) \sim SU(2) \oplus SU(2)$

for Algebra a $\oplus$ not $\otimes$!

this statement is not precise even at the Algebra level. The point is that the generators of $SU(2)$ can be chosen as HERMITIAN even for finite dim representations ($SU(2)$ is COMPACT!)

IN FACT (subtle point) imagine you have some unitary representation of Lorentz (not finite dim!) then $L^{i\dagger} = L^i$; $K^{i\dagger} = K^i$

On the other hand A^i, B^i are such that

$$(A^i)^\dagger = \frac{L^{i\dagger} + i K^{i\dagger}}{2} = B^i$$

$$(B^i)^\dagger = \frac{L^{i\dagger} - i K^{i\dagger}}{2} = A^i$$

A^i & B^i are NOT HERMITIAN so its not really $SU(2)$!

$(A^i)^\dagger = B^i$, $(B^i)^\dagger = A^i$ called REALITY CONDITION

With this reality condition, the algebra is actually the one of $SL(2,\mathbb{C}) = \{ g \in M_{2\times 2}(\mathbb{C}) ; \det g = 1 \}$

the only difference between $SL(2,\mathbb{C})$ and $SU(2)$ is the fact that $SU(2)$ is COMPACT, $SL(2,\mathbb{C})$ is not!

$\Rightarrow$ if we complexify $so^+(1,3)$, i.e. we give up the reality condition, then

$$so^+(1,3) \sim SL(2,\mathbb{C}) \oplus SL(2,\mathbb{C})$$

and each $SL(2,\mathbb{C})$ algebraically can be studied exactly like $SU(2)$ $\Rightarrow$ so as PHYSICISTS, we are happy to say that

$$so^+(1,3) \sim SU(2) \otimes SU(2)$$

and we classify the FINITE DIM. IRREPS of $so^+(1,3)$ through two "spin-like" labels

NOTE :

if instead the "boost generators" were antihermitian

$k^{i\dagger} = -k^i$ then $(A^i)^\dagger = A^i$; $(B^i)^\dagger = B^i$

$\Rightarrow$ this is the $SO(4)$ algebra (Euclidean rotation group!)

And we can safely say $SO(4) = SU(2) \otimes SU(2)$ both at group & algebra level!

Back to $SO^+(1,3)$ we have then discovered :

1] All Finite irreps of Lorentz can be labelled by two HALF-INTEGERS (ℓ_A, ℓ_B)

2] the dimension of irreps (ℓ_A, ℓ_B) is $(2\ell_A+1)(2\ell_B+1)$

3] the generator of rotation Subgroup is

$\vec{L} = \vec{A} + \vec{B}$ $\Rightarrow$ (ℓ_A, ℓ_B) has states with "spin"

$\ell = \{ |\ell_A - \ell_B|, \dots, \ell_A + \ell_B \}$

Simplest cases :

1] $(0,0)$ representation $\Rightarrow \vec{L}=0\,,\ \vec{K}=0$

this is the TRIVIAL SCALAR representation

2] $\left(\frac{1}{2},0\right)\ ;\ \left(0,\frac{1}{2}\right)$ they have both dim $=2$

Left-Handed

Right-Handed representations

the objects transforming under them are called

Left-handed, Right-handed } Weyl Spinors

$(\psi_L)_a$

$(\psi_R)^{\dot{a}}$

$a, \dot{a}$ indices indicate two DIFFERENT SPACES !

$\Rightarrow$ we will IGNORE dotted & undotted indices here!

take $\left(\frac{1}{2},0\right) \Rightarrow \vec{A}=\frac{\vec{\sigma}}{2}\,,\ \vec{B}=0$

$\Rightarrow \vec{L}=\vec{A}+\vec{B}=\frac{\vec{\sigma}}{2}\ ;\ \vec{K}=+i\left(\vec{A}-\vec{B}\right)=+i\frac{\vec{\sigma}}{2}$

recalling $\quad \Lambda = e^{\left[-i\vec{\theta}\cdot\vec{L} + i\vec{\eta}\cdot\vec{K}\right]}$

$\Downarrow$

$\psi_L \rightarrow \Lambda_L \psi_L$ with

$$\Lambda_L = \exp\left[-i\frac{\vec{\theta}\cdot\vec{\sigma}}{2} - \frac{\vec{\eta}\cdot\vec{\sigma}}{2}\right] = e^{(-i\vec{\theta}-\vec{\eta})\frac{\vec{\sigma}}{2}}$$

similarly for ψ_R $\quad (0, \frac{1}{2}) \quad \vec{A} = 0, \quad \vec{B} = \frac{\vec{\sigma}}{2}$

$\Rightarrow \vec{L} = \frac{\vec{\sigma}}{2} \quad ; \quad \vec{K} = -i\frac{\vec{\sigma}}{2}$

$$\Lambda_R = e^{(-i\vec{\theta}+\vec{\eta})\frac{\vec{\sigma}}{2}}$$

$\Rightarrow$ note that both Λ_L & Λ_R are complex

L, R spinors transform the same way under rotations, & the opposite way under boosts!

notice now that, since $\sigma^2 \sigma^i \sigma^2 = -(\sigma^i)^*$

then $\sigma^2 \Lambda_L^* \sigma^2 = \Lambda_R$

take then $\sigma^2 \psi_L^*$ $\Rightarrow$ clearly

$$\sigma^2 \psi_L^* \xrightarrow{\Lambda} \sigma^2 (\Lambda_L \psi_L)^* = \sigma^2 \underbrace{\Lambda_L^* \sigma^2}\, \sigma^2 \psi_L^*$$

where we used $\sigma^2 \sigma^2 = \mathbb{1}$

$$= \Lambda_R (\sigma^2 \psi_L^*)$$

so if ψ_L Left-handed $\Rightarrow$ $\sigma^2 \psi_L^*$ Right handed

the same is true with a factor "i"

Define $\psi_L^c = i \sigma^2 \psi_L^*$ CHARGE CONJUGATION

$\psi_L^c \sim \psi_R$! then complex conjugating

$$(\psi_L^c)^* = -i(-\sigma^2)\psi_L = \psi_R^*$$

$\psi_R^* = i\sigma^2 \psi_L$ multiply by $-i\sigma^2$

$$\psi_L = -i\sigma^2 \psi_R^* = \psi_R^c$$

so for consistency complex conj. gets on ψ_R with opposite sign!

$$\left.\begin{aligned} (\psi_L^c)^c &= \psi_L \\ (\psi_R^c)^c &= \psi_R \end{aligned}\right\} \Leftarrow$$

] $\left(\frac{1}{2}, \frac{1}{2}\right)$ Vector representation

it has complex dimension 4

the angular momentum $\vec{L} = \vec{A} + \vec{B} \Rightarrow$

$$\left|\frac{1}{2} - \frac{1}{2}\right| \leq \ell < \frac{1}{2} + \frac{1}{2} \Rightarrow \ell = \{0, 1\}$$

this is because a 4-vector $V^\mu = (V^0, \vec{V})$ is such that under $SO(3)$ can be split into V^0 = scalar, spin 0 ; $\vec{V}$ = vector, spin 1

But how do we see the vector V^μ from

$(\psi_L^{\dot a}, (\xi_R)_a)$, with ψ_L, ξ_R independent ?

start from $\psi_R = i\sigma^2 \psi_L^*$; $\xi_L = -i\sigma^2 \xi_R^*$

and DEFINE $\sigma^\mu = (\mathbb{1}, \sigma^i)$; $\bar\sigma^\mu = (\mathbb{1}, -\sigma^i)$

Now consider the two objects

$\xi_R^\dagger \sigma^\mu \psi_R = V^\mu$ one can prove they are contravariant

$\xi_L^\dagger \bar\sigma^\mu \psi_L = W^\mu$ 4-vectors under Lorentz

$\hookrightarrow$ in general, they are COMPLEX $\Rightarrow$ but $\Lambda^\mu{}_\nu$ REAL !

Impose a Lorentz invariant reality condition

$V^{\mu *} = V^\mu \qquad W^{\mu *} = W^\mu$

$\Rightarrow$ real 4-vector representation

(need relation among the two !)

4] $(1,0)$ & $(0,1)$ correspond each to an antisymmetric tensor $A^{\mu\nu}$ [For example $F^{\mu\nu}$ Electromagnetic Field $\Rightarrow \vec{E}, \vec{B}$, 6 vectors !]