QUANTUM ELECTRODYNAMICS

(RENORMALIZATION)

First question: is the theory RENORMALIZABLE?

- $e \bar{\psi} \not{A} \psi \Rightarrow [e] = 0$ dimensionless!

interaction must have $\Delta_i = 0$

- propagators

$$\longrightarrow = \frac{i}{\not{p} - m + i\varepsilon} \sim \frac{1}{p}$$

$$\sim\sim\sim = \frac{i}{p^2 + i\varepsilon}\Big(-g_{\mu\nu} + (1-\xi)\underbrace{\frac{p^\mu p^\nu}{p^2}}_{O(1)}\Big) \sim \frac{1}{p^2}$$

In lecture 20 we wrote for general propagator

$$\Delta_F^f(p) \xrightarrow{p\to\infty} \frac{1}{[p^2]^{1-S_f}} \Rightarrow \begin{array}{ll} S_f = 0 & \underline{\text{PHOTON}} \\ S_f = \frac{1}{2} & \underline{\text{FERMION}} \end{array}$$

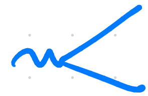
with this SUPERFICIAL DEGREE OF DIV

$$\omega(\gamma) = D - \sum_f E_f\left(\frac{D}{2} - 1 + S_f\right) - \sum_i V_i \Delta_i$$

$$\Delta_i = D - a_i - \sum_f n_{if}\left(\frac{D}{2} - 1 + S_f\right)$$

So in $D=4$ we find [$a_i = 0$ NO DERIVATIVES!]

$$\Delta_i = 4 - \sum_f n_{if}\,(1+s_f)$$

only 1 vertex 

$$\Delta_{\bar\psi\psi\gamma} = 4 - \underbrace{2\left(1+\tfrac{1}{2}\right)}_{\text{2 fermions with } s_f=\frac{1}{2}} - 1\cdot\underbrace{(1+0)}_{\text{1 photon with } s_f=0}$$

$$= 4 - 3 - 1 = 0$$

$$\Delta_{\bar\psi\psi\gamma} = 0$$

note that it is crucial that $\Pi^{F}_{\mu\nu}(p) \sim \frac{1}{p^2}$

in massive theories [massive "photon"] one would find

$$\Pi^{\mu\nu}(p) = \frac{i}{p^2+i\varepsilon}\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}\right) \overset{p\to\infty}{\sim} \underline{\underline{O(1)}}$$

Mass of boson

so this has $s_f = 1$ $\Delta_{\bar\psi\psi A_M} = 4-3-2 = -1$!

theories with massive bosons are typically NON-RENORMALIZABLE $\Rightarrow$ Standard Model has two massive bosons Z, $W^{\pm}$, but their masses are generated by SPONTANEOUS SYMMETRY BREAKING which guarantees renormalizability!

$\Rightarrow$ 't Hooft - Veltman '72

Nobel Prize in '99

$\Rightarrow$ Back to QED, we expect it should be renormalizable, NOTICE: $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$

$$\Rightarrow \quad [(\partial_\mu A^\mu)^2] = [F^{\mu\nu}F_{\mu\nu}] = 4 \qquad [\xi] = 0 \quad !$$

so let's use renormalized perturbation theory

$$\psi_0(x) = \sqrt{Z_2}\,\psi(x) \qquad A_0^\mu(x) = \sqrt{Z_3}\,A^\mu(x)$$

$$m_0 = Z_m\, m \qquad e_0 = Z_e\, e$$

what about ξ? $\Rightarrow$ For now $\xi_0 = Z_\xi\, \xi$

$$\mathcal{L} \Rightarrow -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar\psi(i\partial\!\!\!/ - m + e A\!\!\!/)\psi$$

$$-\frac{1}{4}\underbrace{(Z_3-1)}_{\delta_3} F_{\mu\nu}F^{\mu\nu} - \underbrace{\left(\frac{Z_3}{Z_\xi}-1\right)}_{\delta_\xi}\frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

$$+\underbrace{(Z_2-1)}_{\delta_2}\bar\psi i\partial\!\!\!/\psi - \underbrace{(Z_2 Z_m - 1)}_{\delta m}m\bar\psi\psi$$

δm includes "m" Typically!

$$+\underbrace{(Z_e Z_2 Z_3^{1/2}-1)}_{\delta e} e\bar\psi A\!\!\!/\psi$$

$$= \mathcal{L} + \underbrace{\mathcal{L}_{C.T.}}$$

counterterms "δ_i"

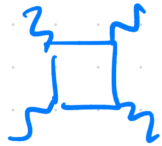
$$\left\{ Z_1 = Z_e Z_2 Z_3^{\frac{1}{2}} \right.$$

We said that $\mathcal{L}$ should contain ALL TERMS consistent with symmetries: should I add extra terms?

$$A_\mu A^\mu \qquad (\partial_\mu A_\nu)A^\mu A^\nu \qquad A^\mu A^\nu A_\mu A_\nu$$

$\Rightarrow$ consistent with Lorentz BUT they VIOLATE GAUGE INVARIANCE! $\to$ NO!

if everything works out, this would mean that

4- photon interaction  should be UV-finite

without needing an extra counter term $\propto A^\mu A^\nu A_\mu A_\nu$ etc...

In which SCHEME do we determine Z_i ?

$\Rightarrow$ $\overline{MS}$ used for high-energy $\mu \sim \sqrt{s}$

$\Rightarrow$ for "low energy" more often ON-SHELL

Let's then renormalize in on-shell scheme

Which Green functions can be DIVERGENT ?

START FROM:

$$\omega(\gamma) = 4 - \sum_f E_f (1 + s_f)$$

Used $\Delta_i = 0$

$D = 4$

E_f external lines of type f

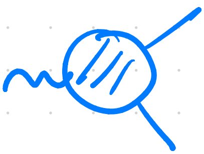
$\omega(\gamma)$ [SUP. DEGREE OF DIV]

$$4 - 2[1+0] = 2$$

DIV. (POW.)

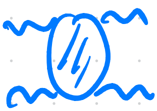
$$4 - 2\left[1+\tfrac{1}{2}\right] = 4-3 = 1$$

DIV (POW)



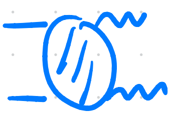
$$4 - 2\left[1+\tfrac{1}{2}\right] - 1[1+0] = 0$$

DIV (LOG.)



$$4 - 4[1+0] = 0$$

DIV (LOG)



$$4 - 2[1+0] - 2\left[1+\tfrac{1}{2}\right] = -1$$

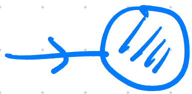
FINITE

etc

higher points FINITE !

this is MAX DIV ALLOWED $\Rightarrow$ graphs could be LESS DIVERGENT (it's the case in QED !)

QUESTION: what about 1-point functions?

 $= 0$ 

$\sim \langle \Omega | \psi | \Omega \rangle = 0$ $\sim \langle \Omega | A^\mu | \Omega \rangle = 0$

In general $|\Omega\rangle$ Poincaré invariant $\leftarrow$ violates

$$\langle \Omega | \phi(x) | \Omega \rangle = \langle \Omega | \phi(0) | \Omega \rangle$$ translation invariance

$\Rightarrow$ true $\forall$ field!

now $\langle \Omega | \psi(0) | \Omega \rangle$ must be a constant and must be zero ($U(1)$, charge conservation)

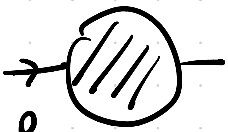
otherwise $\psi \to e^{i\alpha}\psi$ would change charge of vacuum!

if $V^\mu \neq 0$ preferred direction!

$$\langle \Omega | A^\mu(x) | \Omega \rangle = \langle \Omega | A^\mu(0) | \Omega \rangle = 0$$

A^μ not gauge invariant, its VEV would change!

So we start with TWO-POINT FUNCTION



to determine Z_2 & Z_m

ψ wave function

mass renorm.

BARE TWO-POINT FUNCTION :

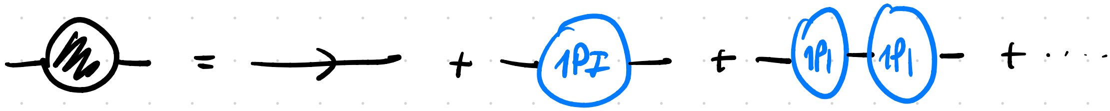
$$S_0(p) = \int d^4x\; e^{ipx} \langle \Omega | T\{\psi_0(x)\,\bar{\psi}_0(0)\} | \Omega \rangle$$

K.L. gives $\xrightarrow[p^2 \to m^2]{}$ $Z_2 \dfrac{i}{\not{p} - m + i\varepsilon}$ + "no poles"

physical on-shell mass

511 keV "physical" electron mass

as in scalar case, we can "resum" 1PI contributions $\Rightarrow$ compute it and match to KL



$-i\Sigma(p,m)$

matrix in spinor space

SELF-ENERGY

$$= \frac{i}{\not p - m} + \frac{i}{\not p - m}\left(-i\Sigma(p,m)\right)\frac{i}{\not p - m} + \dots$$

$$= \frac{i}{\not p - m - \Sigma(p,m)} \xrightarrow{p^2 \to m^2} \frac{i}{p^2 - m^2}$$

ON-SHELL REN SCHEME

pole at $p^2 = m^2$

residue $= 1$

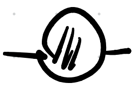
$\Rightarrow$ it's common to write

$$\Sigma(p,m) = m\,\Sigma_S(p,m)\,\mathbb{1} + \Sigma_V(p,m)\,\not p$$

$$\text{OR} \quad = \Sigma_1(p,m)\, m + \Sigma_2(p,m)\,(\not p - m)$$

Σ_s, Σ_v or Σ_1, Σ_2 are SCALAR "FORM FACTORS"

$\Rightarrow \Sigma_1, \Sigma_2$ convenient for renormalization !



$$= \frac{i}{\not{p} - m - \Sigma(p,m)}$$

$$= \frac{i}{(\not{p}-m)(1-\Sigma_2(p,m)) - \Sigma_1(p,m)\, m}$$

write $\Sigma_i \big|_{p^2=m^2} = \Sigma_i(p,m)\big|_{p^2=m^2}$

$$\Sigma_1(p,m) = \Sigma_1\big|_{p^2=m^2} + 2m \frac{\partial}{\partial p^2} \Sigma_1(p,m)\Big|_{p^2=m^2} (\not{p}-m)$$

$$\xrightarrow{p^2 \to m^2} = \frac{i}{(\not{p}-m)\left[1-\Sigma_2\big|_{p^2=m^2} - 2m^2 \frac{\partial \Sigma_1}{\partial p^2}\Big|_{p^2=m^2}\right] - m\, \Sigma_1\big|_{p^2=m^2}}$$

then if we require

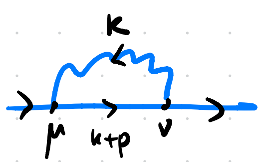
$$\Rightarrow \begin{cases} \Sigma_1 \big|_{p^2=m^2} = 0 & \text{to avoid extra pole} \\ \Sigma_2 \big|_{p^2=m^2} + 2m^2 \dfrac{\partial \Sigma_1}{\partial p^2}\Big|_{p^2=m^2} = 0 & \text{for residue} = 1 \end{cases}$$

$$\rightarrow \quad \frac{i}{\not{p} - m}\left[\frac{1}{1}\right] \quad \checkmark$$

Σ_1 & Σ_2 computed in REN PERT THEORY

$\Rightarrow$ contain counterterms & allow us to fix them

let's do this at one loop

$-i\Sigma(p,m) =$  $+ \left[\text{———}\otimes\text{———} = i\delta_2 \not{p} - i\delta_m \right]$

COUNTERTERM

we have :

$$\text{AMP} = (-e^2) \int \frac{d^D k}{(2\pi)^D} \gamma^\nu \frac{i}{\cancel{k}+\cancel{p}-m} \gamma^\mu \left[\frac{i}{k^2}\left(-g_{\mu\nu}+(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\right]$$

$$= -e^2 \int \frac{d^D k}{(2\pi)^D} \gamma^\mu \frac{1}{\cancel{k}+\cancel{p}-m} \gamma_\mu \frac{1}{k^2}$$

$$+ \underbrace{e^2(1-\xi)}_{\text{gauge!}} \int \frac{d^D k}{(2\pi)^D} \cancel{k} \frac{1}{\cancel{k}+\cancel{p}-m} \cancel{k} \frac{1}{k^4} \quad \boxed{+i\delta_2 \cancel{p} - i\delta m}$$

ADD BACK LATER

$\Rightarrow$ result is not gauge invariant

From here, notice that if we define :

$$\Sigma = \Sigma_s m + \Sigma_v \cancel{p}$$

$$= \Sigma_1 m + \Sigma_2 (\cancel{p} - m) \quad \text{then} \begin{cases} \Sigma_s = \Sigma_1 - \Sigma_2 \\ \underline{\underline{\Sigma_v = \Sigma_2}} \end{cases}$$

$\Rightarrow$ δ_2 contributes to Σ_V only !

δ_m contributes to Σ_S only !

$$\mathrm{Tr}[\Sigma] = \underbrace{\mathrm{Tr}[\mathbb{1}]}_{=4} \Sigma_S m + \underbrace{\mathrm{Tr}(\not{p})}_{=0} \Sigma_V$$

$$\mathrm{Tr}[\not{p}\Sigma] = p^2 \Sigma_V \underbrace{\mathrm{Tr}[\mathbb{1}]}_{=4} + \Sigma_S m \underbrace{\mathrm{Tr}[\not{p}]}_{=0}$$

$$\mathrm{Tr}[\Sigma] = 4m\Sigma_S$$

$$\mathrm{Tr}[\not{p}\Sigma] = 4p^2\Sigma_V$$

$$\Rightarrow \Sigma_V = \frac{1}{4p^2}\mathrm{Tr}[\not{p}\Sigma]$$

$$\Sigma_S = \frac{1}{4m}\mathrm{Tr}[\Sigma]$$

we need to compute some traces :

So we need various traces $[q = p+k]$

① $$\mathrm{Tr}\left[\gamma^\mu (\not{q}+m)\gamma_\mu\right] = m\,\mathrm{Tr}\left[\gamma^\mu\gamma_\mu\right]$$
$$= m D\,\mathrm{Tr}[\mathbb{1}] = 4mD$$

② $$\mathrm{Tr}\left[\not{p}\gamma^\mu(\not{q}+m)\gamma_\mu\right] = \mathrm{Tr}\left[\not{p}\gamma^\mu\not{q}\gamma_\mu\right]$$

use $\gamma^\mu \not{q}\gamma_\mu = (2-D)\not{q}$ in D-dimensions !

$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \Rightarrow g^\mu{}_\mu = D$!

$$\Rightarrow \mathrm{Tr}\left[\not{p}\gamma^\mu\not{q}\gamma_\mu\right] = (2-D)\,\mathrm{Tr}\left[\not{p}\not{q}\right] = 4(2-D)\,p\cdot q$$

etc if we want also ξ-dependence

and putting everything together, after the dust settles [ADDING BACK COUNTERTERMS]

$$-i\Sigma_V = -\frac{e^2}{2}\frac{1}{p^2}(2-D)\left[(p^2+m^2)\,\mathrm{Bub} - \mathrm{Tad}\right]\xi + i\delta_2$$

$$-i\Sigma_S = -e^2\left[D-1+\xi\right]\mathrm{Bub} - i\frac{\delta m}{m}$$

$\leftarrow$ normalization of Σ_S !

$$\mathrm{Tad} = \int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k+p)^2-m^2} = \int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m^2}$$

$$\mathrm{Bub} = \int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2\left((k+p)^2-m^2\right)}$$

; we know TADPOLE

$$\mathrm{Tad} = C(\varepsilon)\frac{i\,m^2}{16\pi^2}\left(\frac{1}{\varepsilon}+1+\dots\right)$$

with

$$\hookrightarrow C(\varepsilon) = (4\pi)^{\varepsilon}\,\Gamma(1+\varepsilon)\left(\frac{m^2}{\tilde{\mu}^2}\right)^{-\varepsilon} \qquad \tilde{\mu} = \sqrt{\frac{e^{\gamma_E}}{4\pi}}\,\mu$$

what about bubble ? Computing it in

$\longrightarrow$ SAME normalization

$$\text{Bub} = \frac{i}{16\pi^2} C(\varepsilon)\left[\frac{1}{\varepsilon} + 2 + \left(\frac{m^2}{p^2} - 1\right)\ln\left(1 - \frac{p^2}{m^2}\right) + \dots\right]$$

finally, introducing $\alpha = \frac{e^2}{4\pi}$ FINE STRUCTURE CONST.

and putting everything together

$$\Sigma_V = \left(\frac{\alpha}{4\pi}\right)\xi\left[-\frac{1}{\varepsilon} - \frac{m^2+p^2}{p^2} - \frac{m^4-p^4}{p^4}\ln\left(1-\frac{p^2}{m^2}\right)\right] - \delta_2$$

$$\Sigma_S = \left(\frac{\alpha}{4\pi}\right)\left[\frac{3+\xi}{\varepsilon} + 2(2+\xi) + (3+\xi)\frac{m^2-p^2}{p^2}\ln\left(1-\frac{p^2}{m^2}\right)\right] + \frac{\delta m}{m}$$

on-shell limit gives

$$\lim_{p^2\to m^2}\Sigma_V = \left(\frac{\alpha}{4\pi}\right)\xi\left[-\frac{1}{\varepsilon} - 2\right] - \delta_2$$

$$\lim_{p^2\to m^2}\Sigma_S = \left(\frac{\alpha}{4\pi}\right)\left[\frac{3+\xi}{\varepsilon} + 4 + 4\xi\right] + \frac{\delta m}{m}$$

$$\Sigma_1 = \Sigma_V + \Sigma_S \quad ; \qquad \Sigma_2 = \Sigma_V$$

$$\lim_{p^2 \to m^2} \Sigma_1 = \left(\frac{\alpha}{4\pi}\right)\left[\frac{3}{\varepsilon} + 2(2+\xi)\right] - \delta_2 + \frac{\delta m}{m}$$

ON-SHELL CONDITIONS were

$$\begin{cases} \Sigma_1 \Big|_{p^2=m^2} = 0 & \text{to avoid extra pole} \\ \Sigma_2 \Big|_{p^2=m^2} + 2m^2 \underbrace{\dfrac{\partial \Sigma_1}{\partial p^2}}_{\text{resbub}} \Big|_{p^2=m^2} = 0 & \end{cases}$$

First condition is easy

$$\delta_2 - \frac{\delta m}{m} = \left(\frac{\alpha}{4\pi}\right)\left[\frac{3}{\varepsilon} + 2(2+\xi)\right]$$

second PROBLEMATIC

$$m^2 \frac{\partial \Sigma_1}{\partial p^2}\Big|_{p^2=m^2} = \left(\frac{\alpha}{4\pi}\right)\left[-3 + 2\xi - \underbrace{\log\left(1-\frac{p^2}{m^2}\right)}_{\text{DIVERGENT}}(3+\xi)\right]$$

we encounter an IR-divergence $\Rightarrow$ if one recomputes $\frac{\partial \Sigma_1}{\partial p^2}$ directly at $p^2 = m^2$ before expanding in "ε", one would find an extra $\frac{1}{\varepsilon_{IR}}$

ON-SHELL electron wave function ren contains IR DIV !

We can bypass the problem if we renormalize instead in $\overline{MS}$ $\Rightarrow$ then we only want the counterterms to remove divergences !

POLES [UV !]

$$\left.\begin{aligned} \Sigma_v^{POLES} &= \left(\frac{\alpha}{4\pi}\right) \xi \left[-\frac{1}{\varepsilon} \right] - \delta_2 \\ \Sigma_s^{POLES} &= \left(\frac{\alpha}{4\pi}\right) \left[\frac{3+\xi}{\varepsilon} \right] + \frac{\delta m}{m} \end{aligned}\right\} \qquad \begin{aligned} \delta_2^{\overline{MS}} &= -\frac{\alpha}{4\pi} \frac{\xi}{\varepsilon} \\ \delta m^{\overline{MS}} &= -\frac{\alpha}{4\pi} m \left(\frac{3+\xi}{\varepsilon} \right) \end{aligned}$$

$\Rightarrow$ counterterms are ξ dependent !

note that

$$\delta_2 = (Z_2 - 1) \quad ; \quad \delta_m = (Z_2 Z_m - 1)\, m$$

$$Z_2 = 1 + \delta_2 = 1 - \frac{\alpha}{4\pi} \frac{\xi}{\varepsilon}$$

$$Z_m = \left(1 + \frac{\delta m}{m}\right) \frac{1}{Z_2} = 1 - \frac{\alpha}{4\pi} \frac{3}{\varepsilon}$$

Z_m gauge independent $\Leftarrow$ DEPENDENCE on ξ cancels out !

↳ actually true to all ORDERS !

the electron wavefunction renormalization constant becomes IR-divergent $\Rightarrow$ manifestation of MASSLESS PHOTONS $\longrightarrow$ issue with standard LSZ !

In practice, when computing S-matrix elements in QED we AMPUTATE and just multiply by $(Z_2^R)^{1/2}$ for each electron, in the ren. scheme we are using [R] $\Rightarrow$ this absorbs UV poles but we are left in general with IR poles

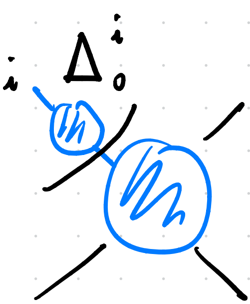
$\Rightarrow$ they only go away once we compute physical observables or cross-sections

[see AQFT course next semester !]

Remember $\Rightarrow$ For GREEN FUNCTION renormalization is

$$G_R^{(n)} = \left[\frac{1}{\sqrt{Z}}\right]^n G_0^{(n)} \quad \text{because} \quad \phi_0 = \sqrt{Z}\,\phi_R$$

but once I AMPUTATE I remove external propagators



$$G_0^{(n)} = \left(\Pi_i \Delta_0^i\right) \underbrace{\Gamma_0^{(n)}}_{\text{Amputated}}$$

$$= \left(\sqrt{Z}\right)^n \underbrace{\Pi_i \Delta_R^i \, \Gamma_R^{(n)}}_{G_R^{(n)}}$$

but how do propagators renormalize ?

$$\tilde{\Delta}_0(p) = Z\,\tilde{\Delta}_R(p)$$! [it's a 2-point function]

so for consistency

$$G_0^{(n)} = \prod_i (\Delta_0^i)\,\Gamma_0^{(n)} = (\sqrt{Z})^n \prod_i \Delta_R^i\,\Gamma_R^{(n)}$$

$$= (\sqrt{Z})^n \left(\frac{1}{Z}\right)^n \prod_i \Delta_0^i\,\Gamma_R^{(n)}$$

Comparing 2nd & 4th expression

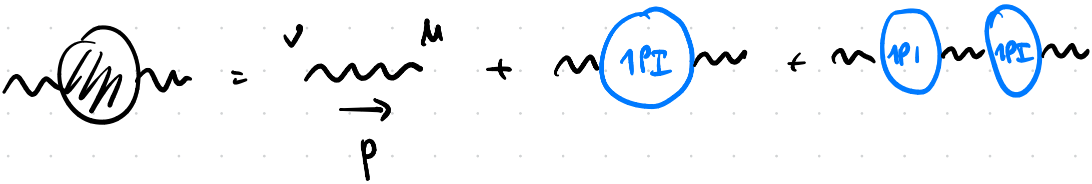
$$\Gamma_0^{(n)} = \left(\frac{1}{\sqrt{Z}}\right)^n \Gamma_R^{(n)} \Rightarrow \boxed{\Gamma_R^{(n)} = (\sqrt{Z})^n\,\Gamma_0^{(n)}}$$

OPPOSITE AS FOR $G_R^{(n)}$!

to renorm AMP. G we MULTIPLY by $\sqrt{Z}$!

Let's continue, what about Z_3, Z_ξ ?

$\Rightarrow$ they can be extracted from 



$+ \dots$

$$= \frac{i}{p^2+i\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right] + \frac{i}{p^2+i\varepsilon}\left(-g_{\mu\rho} + (1-\xi)\frac{p_\mu p_\rho}{p^2}\right) \times$$

$$\times \underbrace{\left[i\Pi_{\rho\sigma}(p)\right]}_{1PI} \frac{i}{p^2+i\varepsilon}\left(-g_{\sigma\nu} + (1-\xi)\frac{p_\sigma p_\nu}{p^2}\right) + \dots$$

computed in ren. perturbation theory

Lorentz Covariance requires

$$\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2) + p_\mu p_\nu \Pi_2(p^2)$$

one can prove that in a general gauge theory [even based on $SU(N)$ non-abelian group]

$\Pi_2(p^2) = 0$ to ALL ORDERS $\Rightarrow$ Proved it with a WARD IDENTITY $\Rightarrow$ 1PI $\Pi_{\rho\sigma}(p)$ must be TRANSVERSE !

if that's the case, we can resum series and write

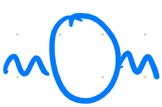
 $$= \frac{i}{p^2+i\varepsilon}\left[\left(-g_{\mu\nu}+\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{1-\Pi(p^2)} - \xi\,\frac{p_\mu p_\nu}{p^2}\right]$$

this implies

$$\delta_\xi = 0 \Rightarrow Z_3 = Z_\xi$$

no correction to gauge term

no ren needed !

$i\Pi(p^2) =$  +  + higher loops

counterterm $= -i\delta_3\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)$

From Lagrangian !

In this case, on-shell scheme defined by

$\Pi(p^2 \to 0) = 0$ such that when $p^2 \to 0$

$$\to \frac{i}{p^2 + i\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right] + \text{FINITE TERMS}$$

pole at $p^2 = 0$ from photon mass $m_\gamma = 0$!

Of course, I can also compute δ_3 (and Z_3) in $\overline{MS}$ as before

Fermion loop

$$= (ie)^2(-1)\int \frac{d^D k}{(2\pi)^D}\, \mathrm{Tr}\left[\gamma^\mu \frac{i(\not{k} + m)}{k^2 - m^2}\, \gamma^\nu \, \frac{i(\not{k} - \not{p} + m)}{(k-p)^2 - m^2}\right]$$

$$-i\,\delta_3\left[g^{\mu\nu}p^2 - p^\mu p^\nu\right] = i\,\Pi_{\mu\nu}(p)$$

integral becomes $i\Pi^{\mu\nu} =$

$$= -e^2 \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2-m^2)((k-p)^2-m^2)} \mathrm{Tr}\left(\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}-\not{p}+m)\right)$$

$$\mathrm{Tr}(\;) = 4g^{\mu\nu}\left(m^2-k^2+k\cdot p\right) + 4\left[2k^\mu k^\nu - k^\mu p^\nu - k^\nu p^\mu\right]$$

now let's use again a PROJECTION TRICK :

IF

$$\Pi^{\mu\nu} = (p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(p) + p^\mu p^\nu \Pi_2(p)$$

then

$$g_{\mu\nu}\Pi^{\mu\nu} = (D-1)\,p^2\,\Pi(p) + p^2\,\Pi_2(p)$$

$$p_\mu p_\nu \Pi^{\mu\nu} = (p^2)^2\,\Pi_2(p) \Rightarrow \Pi_2(p) = \frac{p_\mu p_\nu}{(p^2)^2}\Pi^{\mu\nu}$$

$$g_{\mu\nu}\Pi^{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Pi^{\mu\nu} = (D-1)\,p^2\,\Pi(p)$$

so

$$\Pi_2(p) = \frac{p_\mu p_\nu}{(p^2)^2} \Pi^{\mu\nu}(p)$$ ← use it to prove $\Pi_2(p) = 0$ @ 1 loop

$$\Pi(p) = \frac{1}{(D-1)p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \Pi^{\mu\nu}(p)$$

$$i\Pi_2(p) = -\frac{4e^2}{(p^2)^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2-m^2)((k-p)^2-m^2)} \left[p^2 (m^2 - k^2 - k\cdot p) + 2(k\cdot p)^2 \right]$$

$$k\cdot p = \frac{\overbrace{(k^2-m^2)}^{D_1} - (\overbrace{(k-p)^2 - m^2}^{D_2}) + p^2}{2}$$

substitute in square bracket

$$\frac{1}{D_1 D_2}\left[p^2\left(-D_1 - \frac{D_1}{2} + \frac{D_2}{2} - \cancel{\frac{p^2}{2}}\right) + \frac{1}{2}\left(D_1^2 + D_2^2 + \cancel{p^4} - 2D_1 D_2 + 2p^2 D_1 - 2p^2 D_2\right)\right]$$

$$= -\frac{4e^2}{(p^2)^2}\int\frac{d^Dk}{(2\pi)^D}\Bigg[p^2\underbrace{\left(-\frac{3}{2D_2}+\frac{1}{2D_1}\right)}_{-\frac{1}{D_1}}+\frac{D_1}{2D_2}+\frac{D_2}{2D_1}\cancel{-1}$$

$$\cancel{+p^2\left(\frac{1}{D_2}-\frac{1}{D_1}\right)}\Bigg]$$

by shifting $k \to k+p$ in first!

$$= -\frac{4e^2}{(p^2)^2}\int\frac{d^Dk}{(2\pi)^D}\left[-\frac{p^2}{D_1}+\frac{D_1}{2D_2}+\frac{D_2}{2D_1}\right]$$

$$\int\frac{d^Dk}{(2\pi)^D}\frac{D_1}{D_2} = \int\frac{d^Dk}{(2\pi)^D}\frac{k^2-m^2}{(k-p)^2-m^2} \qquad k \to -k+p$$

$$= \int\frac{d^Dk}{(2\pi)^D}\frac{(k-p)^2-m^2}{k^2-m^2} = \int\frac{d^Dk}{(2\pi)^D}\frac{D_2}{D_1}$$

$$= -\frac{4e^2}{(p^2)^2}\int\frac{d^Dk}{(2\pi)^D}\left[-\frac{p^2}{D_1}+\frac{D_2}{D_1}\right]$$

now $\Rightarrow$

$$\int \frac{d^D k}{(2\pi)^D} \frac{D_2}{D_1} = \int \frac{d^D k}{(2\pi)^D} \frac{k^2 - m^2 + p^2 - 2k\cdot p}{k^2 - m^2}$$

$$= \int \frac{d^D k}{(2\pi)^D} \left(1 + \frac{p^2}{D_1} - \frac{2k\cdot p}{D_1} \right) = \int \frac{d^D k}{(2\pi)^D} \frac{p^2}{D_1}$$

SCALELESS $\rightarrow$ $\underline{=0}$ $\qquad$ $\underline{=0}$ symmetry $k \to -k$!

which proves that $\boxed{\Pi_2(p) = 0}$ at 1 loop!

Some operations on $\Pi(p^2)$ give

$$i\Pi(p) = + \frac{2e^2}{(D-1)p^2} \int \frac{d^D k}{(2\pi)^D} \left[\underbrace{\frac{2(D-2)}{D_1}}_{\text{Tadpole}} - \underbrace{\frac{(D-2)p^2 + 4m^2}{D_1 D_2}}_{\text{Bubble}} \right]$$

results Lecture 15

$$= -i \left(\frac{\alpha}{4\pi} \right) \left[\frac{4}{3\varepsilon} - \frac{4}{9} - \frac{4}{3} \left(1 + \frac{2m^2}{p^2} \right) \underbrace{A(p^2)}_{O(\varepsilon^0) \text{ of Bubble}} \right]$$

Adding back the counterterm we get :

$$i\Pi_R(p) = i\left(\frac{\alpha}{4\pi}\right)\frac{4}{3}\left[-\frac{1}{\varepsilon}+\frac{1}{3}+\left(1+\frac{2m^2}{p^2}\right)A(p^2)\right] - i\delta_3$$

$$\boxed{\delta_3^{\overline{MS}} = -\left(\frac{\alpha}{4\pi}\right)\frac{4}{3}\frac{1}{\varepsilon}}$$

renormalization in $\overline{MS}$

$$Z_3^{\overline{MS}} = 1-\left(\frac{\alpha}{4\pi}\right)\frac{4}{3}\frac{1}{\varepsilon}$$

while in limit $p^2\to 0$ [ON-SHELL SCHEME]

$$A(p^2) = \int_0^1 dx\, \ln\left(1-\frac{p^2}{m^2}x(1-x)\right) \longrightarrow -\frac{p^2}{6m^2}+O\left(\left(\frac{p^2}{m^2}\right)^2\right)$$

$$i\Pi(p^2\to 0) = i\left(\frac{\alpha}{4\pi}\right)\frac{4}{3}\left[-\frac{1}{\varepsilon}\right] - i\delta_3$$

EXACT to $O(\varepsilon^0)$!

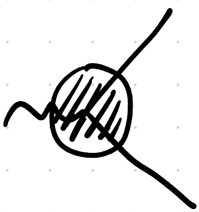
$$\Rightarrow \boxed{\delta_3^{OS} = -\left(\frac{\alpha}{4\pi}\right)\frac{4}{3}\frac{1}{\varepsilon}}$$

as well !

we miss one last renormalization constant Z_e

$\Rightarrow$ charge renormalization [beta function ⚠]

We determine it from next divergent Green funct:



= Amputated & renormalized

$ie\gamma^\mu$ one-loop $ie[\underbrace{Z_e Z_2 \sqrt{Z_3} - 1}_{\delta_e}]\gamma^\mu$

(Wave function - Field renormalization)

$$= \left[ie\, \bar{u}(p_1)\left(\gamma^\mu + \Gamma^{(1l),\mu} + \delta_e \gamma^\mu\right) u(p_2)\right] \varepsilon^*_\mu(q)$$

renormalised pert theory counterterm !

remember: this is equivalent to computing BARE Green function, amputating, and renormalizing multiplicatively: $\sqrt{Z_3}\, Z_2\, i e_0\, \bar{u}(p_1)\left[\gamma^\mu + \Gamma_0^{(n\ell),\mu}\right] u(p_2)$

$\Rightarrow$ reexpressing e_0, m_0 in terms of e_R, m_R, etc

Can we say something more about Z_2, Z_3, Z_e ?

WARD IDENTITY !

Remember, we defined

$$\psi_0(x) = \sqrt{Z_2}\,\psi(x) \qquad A_0^\mu(x) = \sqrt{Z_3}\, A^\mu(x)$$

$$m_0 = Z_m\, m \qquad e_0 = Z_e\, e$$

$$Z_1 = Z_e Z_2 Z_3^{1/2}$$

then what does the WARD ID imply ?

WARD IDENTITY for 3-point function reads

$$k_\mu \tilde{G}^{(3),\mu}_{AMP}(k,p,q) = \tilde{S}_F^{-1}(p+k) - \tilde{S}_F^{-1}(p)$$

Consider what happens when $k^\mu \to 0$ [SOFT PHOTON]

$$\tilde{G}^{(3)}(k,p,q) \to \tilde{G}^{(3)}(0,p,p)$$

$$\tilde{S}_F^{-1}(p+k) \to \tilde{S}_F^{-1}(p) + k_\mu \left.\frac{\partial \tilde{S}_F^{-1}(p+k)}{\partial k_\mu}\right|_{k=0} + \cdots$$

So WARD ID becomes

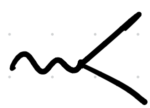
$$\lim_{k^\mu \to 0} \left[k^\mu \tilde{G}^{(3),\mu}_{AMP}(0,p,p) = k^\mu \frac{\partial \tilde{S}_F^{-1}(p+k)}{\partial k^\mu} \right]$$

$$\Rightarrow \tilde{G}^{(3),\mu}_{AMP}(0,p,p) = \left.\frac{\partial \tilde{S}_F^{-1}(p+k)}{\partial k^\mu}\right|_{k=0}$$

So we learn that

1. thanks to WARD ID 3-point function in soft limit is fully determined by DERIVATIVE of 2-point function !

I would like to interpret this as real AMPUTATED green function => need to add electric charge

 $= ie\gamma^\mu$ not just $i\gamma^\mu$!

$$\tilde{G}^{(3),\mu}_{AMP}(0,p,p) = e\,\frac{\partial \tilde{S}_F^{-1}(p+k)}{\partial k^\mu}\Bigg|_{k=0}$$

here there was a real photon !

To COMPENSATE !

some number of e Left and Right !

now let's renormalize =>

$$Z_2 Z_3^{1/2} Z_e \tilde{G}^{(3),\mu}_{AMP}(0,p,p) = \text{Finite}$$

but $\tilde{G}^{(3),\mu}_{AMP} = e \dfrac{\partial \tilde{S}_F^{-1}(p+k)}{\partial k^\mu} = Z_2 \dfrac{\partial \tilde{S}_F^{-1}}{\partial k^\mu}$

only prefactor ← e

$\Uparrow$ enough!

$$\Rightarrow \quad Z_2 \tilde{G}^{(3),\mu}_{AMP} = \underline{\text{must be finite !}}$$

this means $Z_3^{1/2} Z_e = \text{FINITE} \overset{!}{=} 1$

so $\underline{Z_e = \dfrac{1}{\sqrt{Z_3}}}$; $\underline{Z_1 = Z_e Z_2 Z_3^{1/2} \equiv Z_2}$

so charge renormalization is NOT INDEPENDENT !
at 1 LOOP, in $\overline{MS}$

$$Z_e^{\overline{MS}} = \left[1 - \left(\frac{\alpha}{4\pi}\right) \frac{4}{3} \frac{1}{\varepsilon}\right]^{-1/2} = 1 + \left(\frac{\alpha}{4\pi}\right) \frac{2}{3} \frac{1}{\varepsilon}$$

From here we can get the QED β function

$$\mu^2 \frac{d\,\alpha(\mu)}{d\mu^2} = \beta(\alpha)$$

$$\Rightarrow \frac{d\alpha}{\beta(\alpha)} = \frac{d\mu^2}{\mu^2} \qquad \alpha_0 = Z_e^2\, \tilde{\mu}^{2\varepsilon}\, \alpha_R$$

$$\mu^2 \frac{d\alpha_0}{d\mu^2} = 0$$

following same steps as for ϕ^4

but $Z_1^{\overline{MS}} \to [Z_e^{\overline{MS}}]^2 = Z_\alpha^{\overline{MS}}$

since we use $\alpha \sim e^2$!

$$[Z_e^{\overline{MS}}]^2 = \frac{1}{Z_3^{\overline{MS}}} = 1 + \left(\frac{\alpha}{4\pi}\right)\frac{4}{3}\frac{1}{\varepsilon} = Z_\alpha^{\overline{MS}}$$

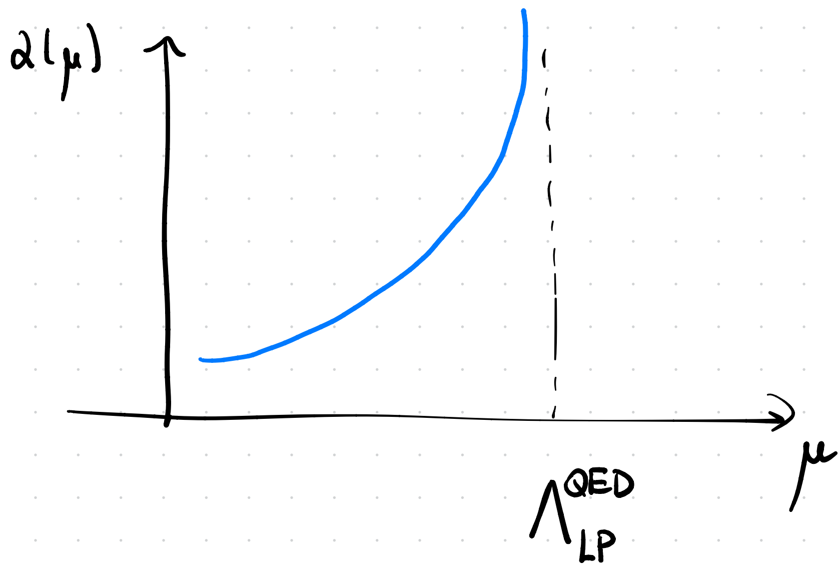
$$\beta(\alpha) = \left(\frac{\alpha}{4\pi}\right)\frac{4}{3}\,\alpha^2(\mu)$$

quadratic in α !

would be cubic in e !

all identical to ϕ^4 $\quad \beta_0^{QED} = \left(\frac{\alpha}{4\pi}\right)\frac{4}{3}$!

$$\alpha(\mu_2) = \frac{\alpha(\mu_1)}{1 - \alpha(\mu_1)\,\beta_0^{QED} \ln\left(\frac{\mu_2^2}{\mu_1^2}\right)}$$



$\alpha(0) = \frac{1}{137}$ fine structure constant

QED has a LANDAU POLE $\Rightarrow$ ill defined in UV !