A summary of Feynman rules for SPINORS & GAUGE BOSONS

For the past few weeks, we have been mainly concerned with the QUANTUM FIELD THEORY of SCALAR theories. We know, nevertheless, that realistic theories (like electrodynamics) involve SPIN $\frac{1}{2}$ & SPIN 1 fields. From now on, we will focus on how the results obtained for the scalar field generalize for spin $\frac{1}{2}$ & 1.

We start with an OPERATIONAL SUMMARY: in fact results can be "guessed"! [mostly ☺]

After, we will spend some time to DEMONSTRATE these formulas and use them to do some important calculations in QUANTUM ELECTRODYNAMICS.

We need:

$\boxed{1}$ Feynman Rules (in momentum space)

$\boxed{2}$ LSZ ⇒ to go from Green Fct → Amplitudes

PROPAGATORS of FERMIONS & GAUGE BOSONS

From lecture 10, we know that in order to properly quantize FREE SPIN $\frac{1}{2}$ Fields we need to impose ANTICOMMUTATION RELATIONS

$$\mathcal{L}_D = \bar{\psi}(i\not{\partial} - m)\psi \qquad \not{A} = A_\mu \gamma^\mu$$

Free Dirac $\mathcal{L}$

DIRAC MATRICES

From $\mathcal{L}_D$ we can derive FERMION PROPAGATOR

$$S_F(x-y) = \langle \Omega | T\{\psi(x)\bar{\psi}(y)\} | \Omega \rangle$$

4x4 MATRIX in SPINOR SPACE

Fourier Transform $\left[\psi = \int e^{-ip\cdot x} u_s(p)\, a_s(p) \text{ etc} \right]$

$$\Rightarrow (i\not{\partial} - m) \Rightarrow \not{p} - m \quad !$$

then inverse of $(\not{p}-m)$ gives the propagator

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \, \underline{\underline{\tilde{S}_F(p)}} \, e^{-ip(x-y)}$$

$$\tilde{S}_F(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\delta} = \text{"} \longrightarrow \text{"}$$ oriented line!

$$\underline{\tilde{S}_F(p)\,(\not{p}-m)} = \frac{i\,[\not{p}\not{p}-m^2]}{p^2-m^2+i\delta} = \underline{\underline{i}}$$

using $\not{p}\not{p} = p^\mu p^\nu \gamma_\mu \gamma_\nu = p^\mu p^\nu \frac{1}{2}\{\gamma_\mu,\gamma_\nu\} = p^2$!

$$\Rightarrow (i\not{\partial}-m)\,S_F(x-y) = i\,\delta^{(4)}(x-y)$$

so we can also write $\tilde{S}_F(p) = \frac{i}{\not{p}-m+i\delta}$

↳ we mean INVERSE OPERATOR!

Green function with Feynman prescription

What about a spin 1 ?

$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$

how do I extract propagator from here ?

$\Rightarrow$ Gauge invariance makes the inverse ill defined

$$-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \underbrace{=}_{\text{antisymm}} -\frac{1}{2} \partial_\mu A_\nu F^{\mu\nu} \underbrace{=}_{\text{IBP}} \frac{1}{2} A_\nu \partial_\mu F^{\mu\nu}$$

$$= \frac{1}{2} A_\nu \partial_\mu \left[\partial^\mu A^\nu - \partial^\nu A^\mu \right] = \frac{1}{2} A_\nu \left[\Box g^{\mu\nu} - \partial^\mu \partial^\nu \right] A_\mu$$

Propagator $\Rightarrow$ inverse of $\Pi^{\mu\nu} = \Box g^{\mu\nu} - \partial^\mu \partial^\nu$

In Fourier space $\left[-p^2 g^{\mu\nu} + p^\mu p^\nu \right] = \tilde{\Pi}^{\mu\nu}$

but now you see that this operator is not invertible, it has a zero eigenvalue

$$(-p^2 g^{\mu\nu} + p^\mu p^\nu)(p_\mu, p_\nu) = 0 \quad !$$

What does this mean for the Action ?

$$S[A] = \frac{1}{2}\int d^4x \, A_\mu(x)\left[\Box g^{\mu\nu} - \partial^\mu\partial^\nu\right]A_\nu(x)$$

$$= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \, \tilde{A}_\mu(p)\left[-p^2 g^{\mu\nu} + p^\mu p^\nu\right]\tilde{A}_\nu(p)$$

$$A_\mu(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \tilde{A}_\mu(p)$$

$$\Rightarrow \forall \; \tilde{A}_\mu(p) \sim p_\mu \quad \text{(LONGITUDINAL)}$$

$$\int [DA] \, e^{iS[A]} \xrightarrow[A_\mu \sim p_\mu]{} \int [DA] \cdot 1 = \infty$$

the action does not provide any damping effect!

the problem can be solved through FADEEV POPOV METHOD by FIXING GAUGE

$\Rightarrow$ removing gauge redundancy

To make a long story short [more later or in AQFT]

this procedure consists in adding a piece to

$$\mathcal{L} = -\frac{1}{2} F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

not unique piece that we can add $\Rightarrow$ result is that

we get a FAMILY of PROPAGATORS

$$\Pi_F^{\mu\nu}(p) = -\frac{i}{p^2+i\varepsilon}\left[g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right]$$

= ∿∿∿ wiggly line $\Rightarrow$ PHOTON

ξ parameter can be fixed to any value, its a remnant of Gauge invariance => final results for physical observables CANNOT depend on ξ

- $\xi = 0$ LANDAU GAUGE

$$\Pi_F^{\mu\nu} = \frac{-i}{p^2 + i\varepsilon}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$$

- $\xi = 1$ FEYNMAN GAUGE

$$\Pi_F^{\mu\nu} = -\frac{i g^{\mu\nu}}{p^2 + i\varepsilon}$$ (very compact!)

what about interactions?

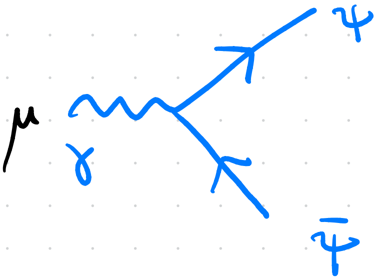
QED $\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

$D_\mu \psi = (\partial_\mu + iq\underbrace{A_\mu)\psi}$ interaction vertex

$$\mathcal{L}_{INT}^{QED} = -q A_\mu \bar{\psi} \gamma^\mu \psi \quad (\text{no "}i\text{" in } \mathcal{L})$$

$\Rightarrow$ from here QED vertex $\left[\begin{array}{l} \text{Fourier Transf.} \\ + \; e^{iS[\psi, \bar{\psi}, A]} \end{array} \right]$

brings in an "i"



$= -iq\gamma^\mu$

$$= +ie\gamma^\mu \quad \text{if } q = -e$$

for ELECTRON!

What else changes w.r.t SCALAR FEYNMAN RULES?

- since $\psi, \bar{\psi}$ are anticommuting, we will show that $\forall$ FERMION LOOP we need an additional (-1) and a TRACE

- EXTRA FACTORS when we AMPUTATE external legs in LSZ formula

In fact, for scalar field ϕ when acting on vacuum we only generate $e^{\pm ipx}$

$\Rightarrow$ + creation operator
  − annihilation operator

With spin $\neq 0$, we have also POLARIZATION VECTORS! then one can prove that AMPUTATION generates

- $\varepsilon^*_\mu(k)$ $\forall$ final-state photon of mom k^μ
- $\varepsilon_\mu(k)$ $\forall$ initial-state photon of mom. k^μ
- $u_s(p)$ $\forall$ initial-state fermion (p, s)
- $\bar{u}_s(p)$ $\forall$ final-state fermion (p, s)
- $v_s(p)$ $\forall$ final-state antifermion (p, s)
- $\bar{v}_s(p)$ $\forall$ initial-state antifermion (p, s)

$\Longrightarrow$ EACH EXTERNAL PARTICLE must be associated to its WAVE FUNCTION !

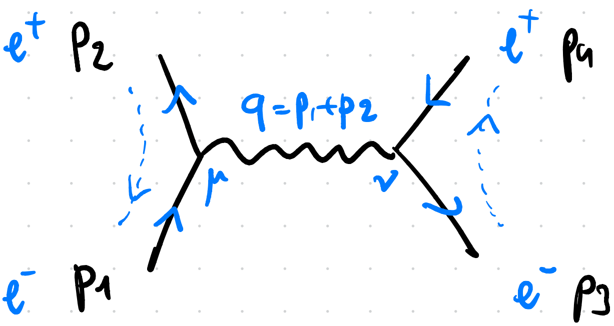
the wave function of scalar $1 \cdot e^{\pm i p\cdot x}$!

$\uparrow$ no factor !

EXAMPLE 1.

Scattering Amplitude for

$$e^-(p_1) + e^+(p_2) \to e^-(p_3) + e^+(p_4)$$



Follow INVERSE direction on fermion line :

s-channel [there is ANOTHER DIAGRAM!]

$$i\mathcal{M}_s = \bar{v}_{s_2}(p_2)\,[+ie\gamma^\mu]\,u_{s_1}(p_1)\left[\frac{-i}{q^2+i\varepsilon}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}(1-\xi)\right)\right]$$

$$\times\ \bar{u}_{s_3}(p_3)\,[+ie\gamma^\nu]\,v_{s_4}(p_4)$$

$$i\mathcal{M}_s = \frac{-ie^2}{\underset{\overset{\shortparallel}{s}}{q^2}} \bar{v}(p_2)\gamma^\mu u(p_1)\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right)$$

$$\bar{u}(p_3)\gamma^\nu v(p_4)$$

note $q_\mu = p_{1\mu} + p_{2\mu}$

$$\bar{v}(p_2)\gamma^\mu u(p_1) \cdot q_\mu = \bar{v}(p_2)\left[\not{p}_1 + \not{p}_2\right] u(p_1)$$

$$\not{p}_1 u(p_1) = m u(p_1)$$

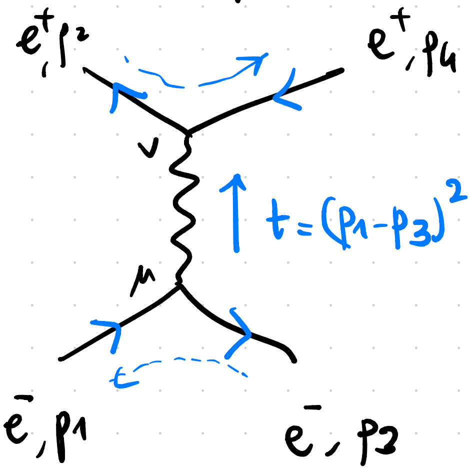
$$\bar{v}(p_2)\not{p}_2 = -m\bar{v}(p_2)$$ (from $\not{p}\, v_s(p) = -m v_s(p)$ and take $\dagger$)

$$\bar{v}(p_2)[m - m]u(p_1) = 0 \quad !$$

$(1-\xi)$ term disappears! GAUGE INVARIANCE !

$$i\mathcal{M}_s = -\frac{ie^2}{s}\bar{v}(p_2)\gamma^\mu u(p_1)\,\bar{u}(p_3)\gamma_\mu v(p_4)$$

other diagram would be t-channel



Feynman Gauge since we know ξ must cancel here!

$$i\mathcal{M}_t = \bar{u}(p_3)(+ie\gamma_\mu)u(p_1)\left[\frac{-ig^{\mu\nu}}{t}\right]$$

$$\bar{v}(p_2)(+ie\gamma_\nu)v(p_4)$$

$$= -\frac{ie^2}{t}\,\bar{u}(p_3)\gamma_\mu u(p_1)\,\bar{v}(p_2)\gamma^\mu v(p_4)$$

- How do we SQUARE DIAGRAMS ?

$\sum_{\text{spins}} |i(\mathcal{M}_s+\mathcal{M}_t)|^2 =$ take care of interferences!

$$|\mathcal{M}|^2 = \underbrace{\mathcal{M}_s^\dagger \mathcal{M}_s} + \mathcal{M}_t^\dagger \mathcal{M}_t + 2\,\mathrm{Re}\left(\mathcal{M}_s^\dagger \mathcal{M}_t\right)$$

$$\mathcal{M}_s^\dagger \mathcal{M}_s = \frac{e^4}{s^2} \overbrace{\bar{u}(p_1)\gamma^\nu v(p_2)\ \bar{v}(p_4)\gamma_\nu u(p_3)}^{\sim \mathcal{M}_s^\dagger}$$

$$\times \underbrace{\bar{v}(p_2)\gamma_\mu u(p_1)\ \bar{u}(p_3)\gamma^\mu v(p_4)}_{\sim \mathcal{M}_s}$$

using $\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right]^\dagger = u^\dagger(p_1)(\gamma^\mu)^\dagger \gamma^0 v(p_2)$

$$= \underbrace{u^\dagger(p_1)\gamma^0}_{\bar{u}(p_1)} \gamma^\mu \underbrace{\gamma^0\gamma^0}_{\mathbb{1}} v(p_2) = \underline{\underline{\bar{u}(p_1)\gamma^\mu v(p_2)}}$$

Adjoint reverses order and swaps bar - no bar !

Same for other terms ...

What when we sum over spins ?

$$\sum_{spins} \mathcal{M}_s^\dagger \mathcal{M}_s = \frac{e^4}{s^2} \sum_{spins} \boxed{\begin{matrix} \bar{u}_{s_1}(p_1)\gamma^\nu v_{s_2}(p_2) \\ \bar{v}_{s_2}(p_2)\gamma_\mu u_{s_1}(p_1) \end{matrix}} \boxed{\begin{matrix} \bar{v}_{s_4}(p_4)\gamma_\nu u_{s_3}(p_3) \\ \bar{u}_{s_3}(p_3)\gamma^\mu v_{s_4}(p_4) \end{matrix}}$$

$$\sum_{s_1, s_2} [\bar{u}_{s_1}(p_1)]_a (\gamma^\nu)_{ab} [v_{s_2}(p_2)]_b [\bar{v}_{s_2}(p_2)]_c (\gamma_\mu)_{cd} [u_{s_1}(p_1)]_d = \text{components}$$

$$= \sum_{spins} [u_{s_1}(p_1)]_{\underline{\underline{d}}} [\bar{u}_{s_1}(p_1)]_a (\gamma^\nu)_{ab} [v_{s_2}(p_2)]_b [\bar{v}_{s_2}(p_2)]_c (\gamma_\mu)_{c\underline{\underline{d}}}$$

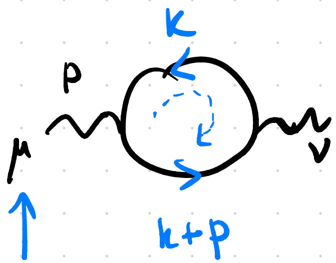
same index => TRACE in SPINOR SPACE

$$= (\not{p}_1 - m)_{da} (\gamma^\nu)_{ab} (\not{p}_2 + m)_{bc} (\gamma_\mu)_{cd}$$

$$= \mathrm{Tr}\left[(\not{p}_1 - m)\gamma^\nu(\not{p}_2 + m)\gamma_\mu\right] \; !$$

etc

we will do some of these calculations explicitly in exercises and later in these lectures now consider loop corrections, for example



= 1-loop correction to photon propagator!

AMPUTATED as for SELF ENERGY, not S-matrix!

$$\Rightarrow i\Pi^{\mu\nu}(p) = \underbrace{(-1)}_{\text{FERMION LOOP}} \int \frac{d^D k}{(2\pi)^D} \mathrm{Tr}\left[(-ie\gamma_\mu)\frac{i(\not k+m)}{k^2-m^2}(-ie\gamma_\nu) \cdot \frac{i(\not k+\not p+m)}{(k+p)^2-m^2}\right]$$

the Trace comes in naturally by combining spinor indices as before with $|\mathcal{M}|^2$!

notice Tr & (-1) require a FULL fermion loop

= no Tr and no (-1)!

[we will compute this later]

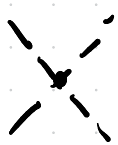
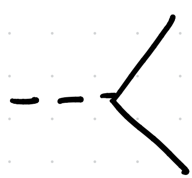
what about a different theory, where scalars interact with fermions? EXAMPLE:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{M^2}{2}\phi^2$$

$$- \frac{\lambda}{4!}\phi^4 - g\bar{\psi}\psi\phi$$

this theory has 2 fields and 2 vertices

 $= \dfrac{i(p\!\!\!/ + m)}{p^2 - m^2}$;  $= \dfrac{i}{p^2 - M^2}$

 $= -i\lambda$;  $= -ig$

external spinors as before $u, \bar{u}, v, \bar{v}$

external scalars give just 1