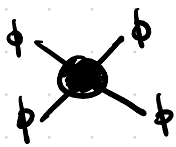
Renormalization 3:
the renormalization "group"

We have seen how renormalization works in ϕ^n Scalar theories. One important observation we can make is that RENORMALIZED RESULTS depend on new "MASS-SCALES" $\Rightarrow$ Λ cut off

or μ in DIM REG

these new scales have also some peculiar effect

Go back to  for example; in $\overline{MS}$

$$\tilde{G}^{(4)}_{AMP} = -i\lambda_{\overline{MS}}\left[1-\frac{\lambda_{\overline{MS}}}{32\pi^2}\left(-3\ln\left(\frac{m^2}{\mu^2}\right)-A(s)-A(t)-A(u)\right)\right]$$

where

$$A(s) = \int_0^1 dx\,\ln\left[1-\frac{s}{m^2}x(1-x)\right] \xrightarrow[s\gg m^2]{m^2\to 0} \ln\left[-\frac{s}{m^2}\right]$$

$$A(t) = \ln\left(-\frac{t}{m^2}\right) \qquad A(u) = \ln\left(-\frac{u}{m^2}\right)$$

So the amplitude becomes

$$i\mathcal{M} \sim -i\lambda_{\overline{MS}}\left[1+\frac{\lambda_{\overline{MS}}}{32\pi^2}\left(\ln\left(-\frac{s}{\mu^2}\right)+\ln\left(-\frac{t}{\mu^2}\right)+\ln\left(-\frac{u}{\mu^2}\right)\right)\right]$$

where $\lambda_{\overline{MS}} = \lambda(\mu)$ depends on μ !

this means that next order $\propto$ $\lambda(\mu)\cdot\ln\left(-\frac{q^2}{\mu^2}\right)$

[when $m^2 \to 0$]

with $q^2=\{s,t,u\}$

$\Rightarrow$ what about μ? depending on the value I choose, I might get a very large log!

say $\mu^2 = m^2$ $\Rightarrow$ $\ln\left[-\frac{s}{m^2}\right] \gg 1$

such that $\lambda_{\overline{MS}}\ln\left(-\frac{s}{m^2}\right) \gg 1$

We say PERTURBATION THEORY BREAKS DOWN !

Next-to-Leading Order is as big as Leading order !

on the other hand, if we choose $\mu^2 \sim s$

then all $\left| -\frac{s}{\mu^2}, -\frac{t}{\mu^2}, -\frac{u}{\mu^2} \right| \sim 1 \Rightarrow$ NO BIG LOGS

$\Rightarrow$ perturbation theory then makes sense, but

we should make sure that $\lambda_{\overline{MS}} = \lambda(\mu)$

is still SMALL if $\mu \sim O(\sqrt{s})$

So we need to compute the SCALE DEPENDENCE of

λ [and possibly of other parameters!]

to do that, we go back to BARE $\lambda_0 \Rightarrow$

λ_0 must be μ-independent

$$\boxed{\mu^2 \frac{d}{d\mu^2} \lambda_0 = 0}$$

But remember from lecture 19 we defined

$$\lambda_0 = Z_\lambda^R \tilde{\mu}^{2\varepsilon} \lambda_R$$

$\uparrow$ use $\tilde{\mu}$ to land on $R = \overline{MS}$

then $$\mu^2 \frac{d}{d\mu^2} \lambda(\mu) = \mu^2 \frac{d}{d\mu^2}\left[\mu^{-2\varepsilon} \left[Z_\lambda^{\overline{MS}}\right]^{-1} \underset{\mu\text{-indep}}{\lambda_0} \right]$$

call $\tilde{\mu} = \mu$ EVERYWHERE

$$= \mu^2 \left[-\varepsilon (\mu^2)^{-\varepsilon-1} \left(Z_\lambda^{\overline{MS}}\right)^{-1} \lambda_0 - \underbrace{\frac{1}{\left(Z_\lambda^{\overline{MS}}\right)^2} \mu^{-2\varepsilon} \lambda_0}_{\frac{1}{Z_\lambda^{\overline{MS}}} \lambda(\mu)} \frac{d Z_\lambda^{\overline{MS}}}{d\mu^2} \right]$$

$$= -\varepsilon \lambda(\mu) - \frac{1}{Z_\lambda^{\overline{MS}}} \lambda(\mu)\, \mu^2 \frac{d Z_\lambda^{\overline{MS}}}{d\mu^2}$$

$\overset{\text{DEFINE}}{\equiv} \beta(\lambda)$ BETA FUNCTION ; FINITE AS $\varepsilon \to 0$

so we get a differential equation for $\lambda(\mu)$

$$\mu^2 \frac{d\lambda(\mu)}{d\mu^2} = \beta(\lambda) \Rightarrow \frac{d\lambda}{\beta(\lambda)} = \frac{d\mu^2}{\mu^2}$$

this is what we call a RENORMALIZATION GROUP EQUATION

Formal solution is
$$\ln\left(\frac{\mu_2^2}{\mu_1^2}\right) = \int_{\lambda(\mu_1)}^{\lambda(\mu_2)} \frac{d\lambda}{\beta(\lambda)}$$

if we know β-function, we can compute how λ changes from $\mu_1 \to \mu_2$

Take our
$$Z_\lambda^{\overline{MS}} = 1 + \frac{\lambda}{32\pi^2}\frac{3}{\varepsilon} + O(\lambda^2)$$

$$\beta(\lambda) = -\varepsilon\lambda(\mu) - \frac{1}{Z_\lambda^{\overline{MS}}}\lambda(\mu)\mu^2 \frac{d Z_\lambda^{\overline{MS}}}{d\mu^2} \longrightarrow$$

$$\beta(\lambda) = -\varepsilon\lambda - \left[1 + \frac{\lambda}{32\pi^2}\frac{3}{\varepsilon} + \dots\right]^{-1} \underbrace{\lambda \frac{3}{\varepsilon}\frac{1}{32\pi^2}}\ \underbrace{\mu^2 \frac{d\lambda}{d\mu^2}}_{\beta(\lambda)\text{ again!}}$$

→ iterate, keeping $O(\lambda^2)$ because this starts at $O(\lambda)$ and our current $Z_\lambda^{\overline{MS}}$ is only valid to $O(\lambda)$!

$$= -\varepsilon\lambda - \left(1 - \frac{\lambda}{32\pi^2}\frac{3}{\varepsilon}\right)\lambda\frac{3}{\varepsilon}\frac{1}{32\pi^2}\left(-\varepsilon\lambda + \dots\right)$$

$$= -\varepsilon\lambda + \lambda^2\frac{3}{32\pi^2} + O(\lambda^3)$$

$$\lim_{\varepsilon\to 0}\beta(\lambda) = \underbrace{\frac{3}{32\pi^2}}\lambda^2(\mu) = \beta_0\lambda^2$$

this first coefficient usually called $\underline{\underline{\beta_0}}$

So now we can go back to ren-group equation

$$\frac{d\lambda}{\beta(\lambda)} = \frac{d\mu^2}{\mu^2} \Rightarrow \ln\left(\frac{\mu_2^2}{\mu_1^2}\right) = \int_{\lambda(\mu_1^2)}^{\lambda(\mu_2^2)} d\lambda' \frac{1}{\beta_0 \lambda'^2}$$

$$= \frac{1}{\beta_0}\left[\frac{1}{\lambda(\mu_1)} - \frac{1}{\lambda(\mu_2)}\right] \quad \text{or}$$

$$\frac{1}{\lambda(\mu_2)} = \frac{1}{\lambda(\mu_1)} - \beta_0 \ln\left(\frac{\mu_2^2}{\mu_1^2}\right)$$

$$\Rightarrow \boxed{\lambda(\mu_2) = \frac{\lambda(\mu_1)}{1 - \lambda(\mu_1)\beta_0 \ln\left(\frac{\mu_2^2}{\mu_1^2}\right)}}$$

running of coupling constant

given value of μ_1 $\rightarrow$ get value of μ_2 !

[in $\overline{\text{MS}}$ scheme !]

$\lambda(\mu)$ is not the only parameter that "RUNS"

$\Rightarrow$ we have seen that also $m_{\overline{MS}}(\mu)$!

Let's compute running of mass

$m_0 = Z_m^{\overline{MS}}\, m(\mu)$ $\Rightarrow$ remember we wrote

$$m_0^2 = m_R^2 + \delta m^2 = \left(Z_m^R\, m_R\right)^2$$

gives relation between the two

then similarly in $\overline{MS}$

$$\mu^2 \frac{dm_0}{d\mu^2} = 0$$

$$\Rightarrow \mu^2 \frac{d}{d\mu^2}\Big[\underbrace{\frac{1}{Z_m^{\overline{MS}}}\, m_0}_{m(\mu)}\Big] = \underbrace{\left(-\frac{1}{\left(Z_m^{\overline{MS}}\right)}\mu^2 \frac{d}{d\mu^2} Z_m^{\overline{MS}}\right)}_{\gamma_m^{\overline{MS}}(\lambda)} m(\mu)$$

Can $\gamma_m^{\overline{MS}}(\lambda)$ depend on m? $\Rightarrow$ no in $\overline{MS}$ because $k \to \infty$, no mass !

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so we have $\quad m_{\overline{MS}} = m(\mu)$

$$\frac{dm(\mu)}{m(\mu)} = \frac{d\mu^2}{\mu^2}\,\gamma_m(\lambda) = \frac{d\lambda}{\beta(\lambda)}\,\gamma_m(\lambda)$$

using $\beta(\lambda)$ we make all dependent on λ only !

$$\Rightarrow \ln\left[\frac{m(\mu_2)}{m(\mu_1)}\right] = \int_{\lambda(\mu_1)}^{\lambda(\mu_2)} d\lambda\, \frac{\gamma_m(\lambda)}{\beta(\lambda)}$$

which gives running of mass in $\overline{MS}$

$$\delta m^2 = \frac{\lambda}{32\pi^2}\, m^2\, \frac{1}{\varepsilon} + O(\lambda^2) \qquad \text{in } \phi^4 \text{ then}$$

$$Z_m^{\overline{MS}} = 1 + \frac{\lambda}{64\pi^2}\,\frac{1}{\varepsilon} \qquad \text{such that}$$

$$Z_m^{\overline{MS}}\, m_{\overline{MS}} = m_{\overline{MS}} + \frac{\lambda}{64\pi^2}\, m_{\overline{MS}}\, \frac{1}{\varepsilon} \Rightarrow$$

$$\left(Z_m^{\overline{MS}}\, m_{\overline{MS}}\right)^2 = m_{\overline{MS}}^2 + \frac{\lambda}{32\pi^2}\, m_{\overline{MS}}^2\, \frac{1}{\varepsilon} + O(\lambda^2)$$

$$\gamma_m(\lambda) = -\left[1+\frac{\lambda}{64\pi^2}\frac{1}{\varepsilon}\right]^{-1} \underbrace{\mu^2\frac{d\lambda}{d\mu^2}}_{\beta(\lambda)=\beta_0\lambda^2} \left[\frac{1}{64\pi^2}\frac{1}{\varepsilon}\right]$$

$$= -\left(1-\frac{\lambda}{64\pi^2}\frac{1}{\varepsilon}\right)\left(-\varepsilon\lambda + O(\lambda^2)\right)\left[\frac{1}{64\pi^2}\frac{1}{\varepsilon}\right]$$

$$= +\frac{\lambda}{64\pi^2} + O(\lambda^2)$$

$$\ln\left[\frac{m(\mu_2)}{m(\mu_1)}\right] = \int_{\lambda(\mu_1)}^{\lambda(\mu_2)} d\lambda\, \frac{\cancel{\lambda}}{\underset{2}{\cancel{64\pi^2}}} \cdot \frac{1}{\frac{3}{\cancel{32\pi^2}}\lambda^{\cancel{2}}}$$

$$= \int_{\lambda(\mu_1)}^{\lambda(\mu_2)} d\lambda\, \frac{1}{6\lambda} = \frac{1}{6}\ln\left[\frac{\lambda(\mu_2)}{\lambda(\mu_1)}\right]$$

$$\left[\frac{m(\mu_2)}{m(\mu_1)}\right] = \left(\frac{1}{1-\lambda(\mu_1)\beta_0 \ln\left(\frac{\mu_2^2}{\mu_1^2}\right)}\right)^{1/6} \quad (*)$$

$$m(\mu_2) = m(\mu_1)\left[1 + \frac{\beta_0}{6}\lambda \cdot \ln\left(\frac{\mu_2^2}{\mu_1^2}\right) + \dots\right]$$

each extra power of λ comes with $\ln\left(\frac{\mu_2^2}{\mu_1^2}\right)$

$\Rightarrow$ if $\log\left(\frac{\mu_2}{\mu_1}\right) \gg 1$ then this expansion makes no sense $\Rightarrow$ on the other hand,

eq $(*)$ RESUMS ALL THESE LOGS THROUG dependence on renormalized coupling $\lambda(\mu)$

$$\frac{m(\mu_2)}{m(\mu_1)} = \left[\frac{\lambda(\mu_2)}{\lambda(\mu_1)}\right]^{1/6}$$

Now what does the scale dependence $\lambda(\mu)$ tell us about the theory?

$\Rightarrow$ Depending on how λ evolves, we have different possibilities $\Rightarrow$ determined by the SIGN of β_0

take the case we just computed

$$\lambda(\mu_2) = \frac{\lambda(\mu_1)}{1-\beta_0 \lambda(\mu_1) \ln\left(\frac{\mu_2^2}{\mu_1^2}\right)} \qquad \beta_0 = \frac{3}{32\pi^2} > 0$$

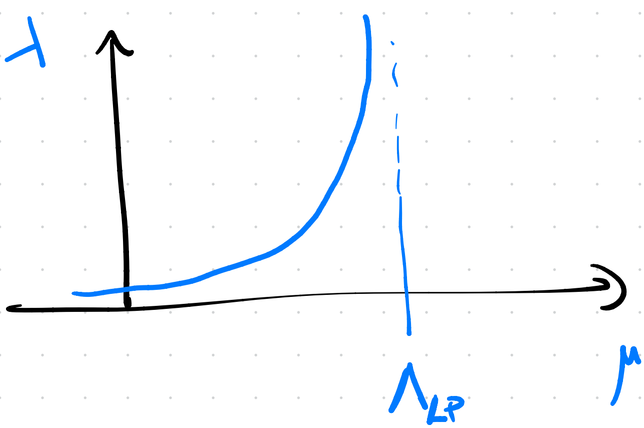
note that in this result all LEADING LOGS are accounted for so $\lambda \cdot \ln\left(\frac{\mu_2}{\mu_1}\right)$ does <u>NOT</u> have to be small!

We could fix $\mu_1 = m \quad \mu_2 = \sqrt{s}$ with $s \gg m^2$! no problem!

we see that $\lambda(\mu_2)$ INCREASES with μ_2 and diverges at $\mu_1 = \Lambda_{LP}$ such that

$$1 - \beta_0 \lambda \ln\left(\frac{\Lambda_{LP}^2}{\mu_1^2}\right) = 0 \qquad \Lambda_{LP} = \mu_1 \cdot \exp\left[\frac{1}{2\beta_0 \lambda(\mu_1)}\right]$$

Λ_{LP} is called LANDAU POLE

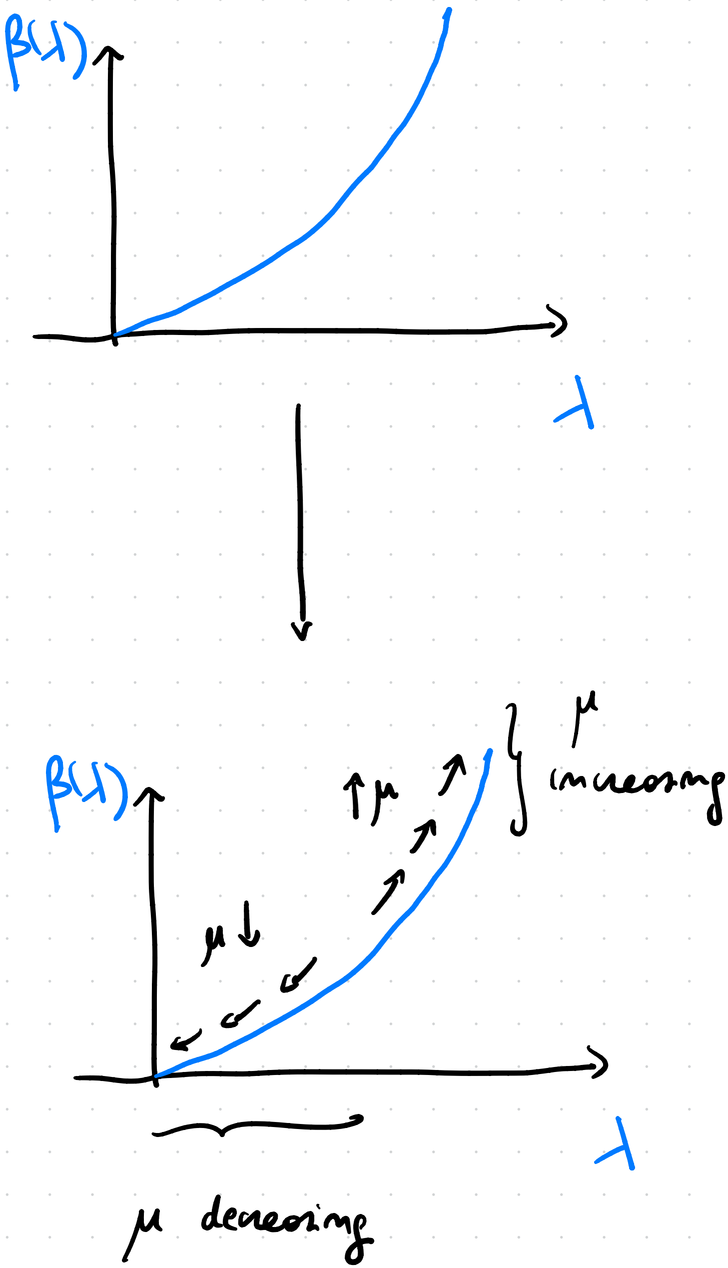


Λ_{LP} is EXPONENTIALLY LARGE if $\lambda(\mu_1)$ is "small" at low energies

for this conclusion to hold, $\lambda(\mu)$ needs to be small for $\mu \in [\mu_1, \mu_2]$ $\Rightarrow$ so what really happens close to Λ_{LP} is unclear and beyond perturbation theory

$\Rightarrow$ if true, theory ILL-DEFINED despite RENORMALIZABLE!

notice that in this case $\beta(\lambda)$ grows with λ



since λ grows with μ, this means that β grows with μ

at $\lambda_* = 0$

$\beta(\lambda_*) = 0$

RUNNING STOPS

$\lambda_* = 0$ called "trivial IR Fixed Point"

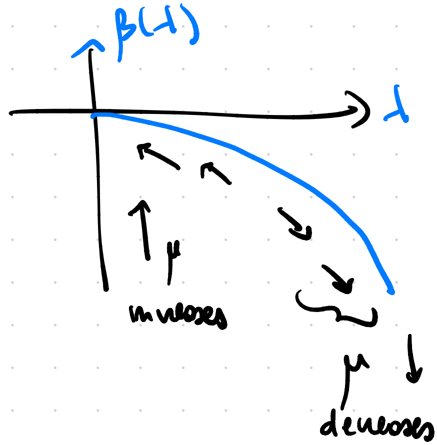
$\Downarrow$

Trivial because $\beta = 0$

$\lambda = 0$, the theory becomes FREE

• What if β_0 had opposite sign? $\beta_0 < 0$

$$\lambda(\mu_2) = \frac{\lambda(\mu_1)}{1 + [-\beta_0]\,\lambda(\mu_1)\ln\left(\frac{\mu_2^2}{\mu_1^2}\right)}$$



λ because SMALLER when μ increases

$\Rightarrow$ TRIVIAL UV FIXED POINT $\beta = 0\,;\ \lambda = 0$ at large μ!

this is phenomenon of

ASYMPTOTIC FREEDOM ! $\Rightarrow$ As in QCD strong interactions

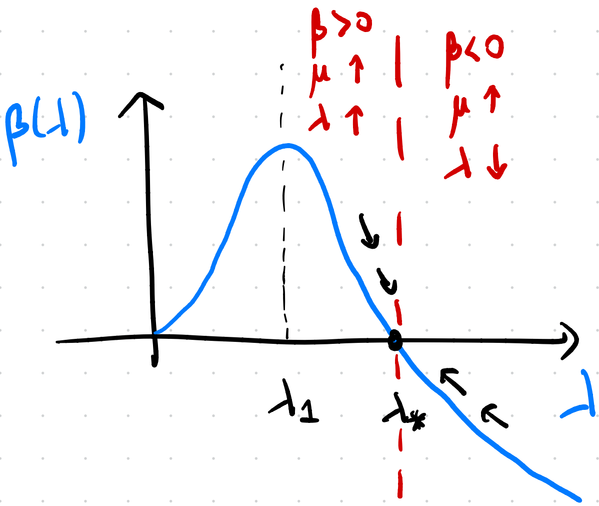
* Can there be NON-TRIVIAL FIXED POINTS?

we need $\beta(\lambda_*) = 0$ at $\lambda_* \neq 0$

There are two possibilities

1] UV stable fixed point (i.e. when $\mu \to \infty$)

idea is that if I start from some range $\lambda \in [\lambda_1, \lambda_2]$
then when $\mu \to \infty$, $\lambda \to \lambda_*$ with $\beta(\lambda_*) = 0$
irrespective of exact value of λ in that interval



when $\mu \to \infty$
$\beta(\lambda) \to 0$ as
$\lambda \to \lambda_*$, because β
changes sign at λ_*

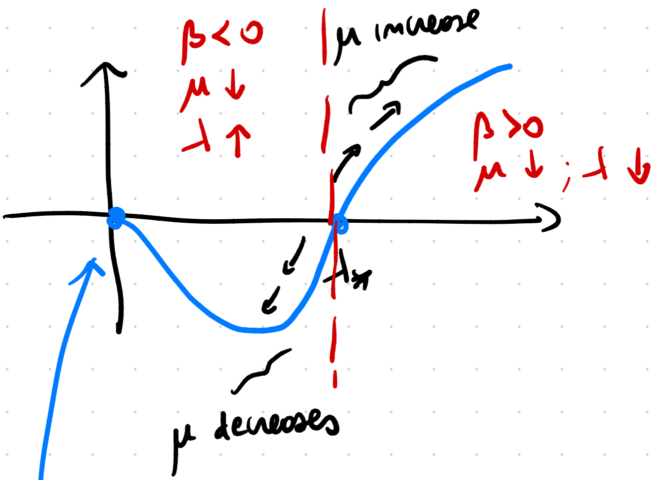
β function pulls $\lambda \to \lambda^*$ as $\mu \to \infty$

note also that in drawing above, $\lambda = 0$ is an

IR trivial fixed-point as $\mu \to 0$ $\beta(\lambda) \to 0$ if $\underline{\lambda < \lambda_*}$

Also $\Rightarrow$ behavior above "could" happen in ϕ^4 if
loop corrections large enough $\Rightarrow$ currently excluded

2] IR-stable non-trivial Fixed Point



λ attracted to FIXED POINT when $\mu \to 0$ [repelled when $\mu \to \infty$]

$\lambda = 0$ is trivial UV fixed point, as $\mu \to \infty$

3] What if $\beta(\lambda) > 0$ but $\beta \sim \lambda^k$ $0<k<1$

$$\Rightarrow \text{then} \quad \int_{\lambda_0}^{\infty} \frac{d\lambda}{\beta(\lambda)} = +\infty = \ln\left[\frac{\mu_\infty^2}{\mu_{low}^2}\right]$$

then $\lambda_* = \infty$ is a UV fixed point !

What if I have a theory with multiple λ_i?

if RENORMALIZABLE : $[\lambda_i] = \Delta_i = 0$; $\vec{\lambda} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{pmatrix}$

We can generalize discussion and will get

$$\mu^2 \frac{d}{d\mu^2} \lambda_i(\mu) = \beta_i(\vec{\lambda}) = \beta_i^{jk} \lambda_j \lambda_k$$

coupled system of differential equations

then a FIXED POINT in space of $\vec{\lambda}$ is a $\vec{\lambda}_*$

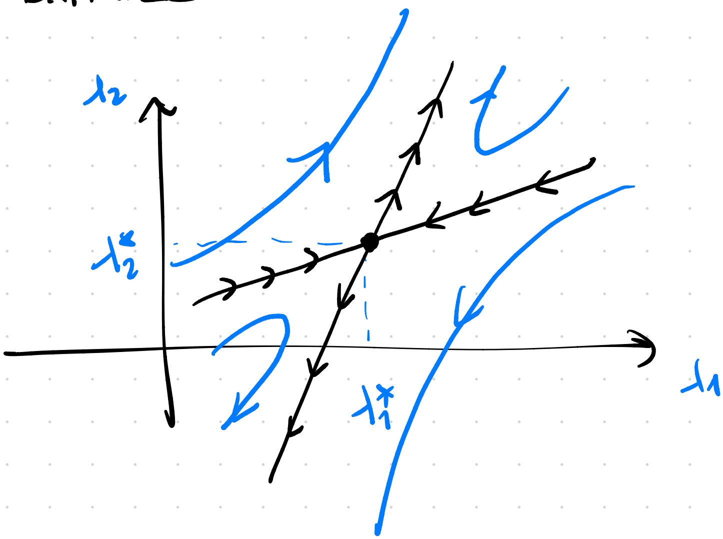
with $\beta_i(\vec{\lambda}_*) = 0 \quad \forall i$

Fixed point can be attractive or repulsive in IR or UV, depending on eigenvalues of linearized system

$$\mu^2 \frac{d}{d\mu^2} \lambda_i = \left[\frac{\partial \beta_i}{\partial \lambda_k} \Bigg|_{\vec{\lambda} = \vec{\lambda}_*} \right] (\lambda_k - \lambda_k^*) + \cdots$$

Attractive in directions of NEGATIVE eigenvalues of $\left[\frac{\partial \beta^i}{\partial \lambda_k}\right]_{\lambda^*} = A_{ik}$; Repulsive for POSITIVE eigen.

EXAMPLE



Renormalization FLOW for increasing μ [UV fixed points]

One could even have CYCLE BEHAVIOUR !

Now, what happens if we add Dimensionful couplings $\Rightarrow$ NON-RENORMALIZABLE THEORIES

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - \frac{\lambda_4}{4!}\phi^4 - \frac{\lambda_6}{6!}\phi^6 - \dots$$

$+ \text{ c.t.}$

• introduce dimensionless couplings in $D=4$

$$\bar{\lambda}_2 = \frac{m^2(\mu)}{\mu^2} \; ; \; \bar{\lambda}_4 = \lambda_4(\mu) \; ; \; \bar{\lambda}_6 = \mu^2 \lambda_6(\mu) \text{ etc}$$

then we get for coupled system of ren group eq.

$$\mu^2 \frac{d}{d\mu^2} \bar{\lambda}_i = \beta_i(\vec{\bar{\lambda}})$$

$$= \underbrace{-\frac{\Delta_i}{2} \bar{\lambda}_i}_{} + \beta_i^{jk} \bar{\lambda}_j \bar{\lambda}_k + \cdots$$

this determines μ dependence
for small $\bar{\lambda}_i$

=> this comes from rescaling we have made by $[\mu^2]^{-\frac{\Delta_i}{2}}$ to make them dimensionless!

it tells us that for small λ_i; $\bar{\lambda}_i \sim (\mu^2)^{\frac{\Delta_i}{2}}$

now ASSUME

1. all $\bar{\lambda}_i$ small enough that we can use perturbation theory

2. theory valid to $\mu \sim \Lambda$ very large

$\Lambda \gg E$ experimental energy.

then $$\frac{d\bar{\lambda}_i}{\bar{\lambda}_i} = -\frac{\Delta_i}{2}\frac{d\mu^2}{\mu^2} + \dots$$

$$\Rightarrow \quad \bar{\lambda}_i(\mu) = \bar{\lambda}_i(\Lambda)\left[\frac{\mu}{\Lambda}\right]^{-\Delta_i} + \text{corrections}$$

So for $\boxed{\text{non-renormalizable theories}}$ $\Delta_i < 0$

and $\bar{\lambda}_i(\mu) \ll 1$ for $\mu \sim E \ll \Lambda$ (accessible energies)

Low-Energy phenomena described by renormalizable theories! operator with such coupling $O_i \bar{\lambda}_i$

$\Rightarrow [O_i] > 4 \quad (\Delta_i < 0)$ called IRRELEVANT operators (interactions)

- For RENORMALIZABLE theories $\Delta_i = 0$

  $[O_i] = 0$ MARGINAL $\Rightarrow$ it runs with second order $\sim O(\lambda^2)$, runs SLOWLY

- For $[O_i] < 0$; $\Delta_i > 0$ then

  $\bar{\lambda}_i(\mu)$ becomes LARGE at small $\mu \sim E \ll \Lambda$

  O_i Called RELEVANT, like mass term $m^2\phi^2$

  $$m^2(\mu) = \mu^2 \bar{\lambda}_2(\mu) = \Lambda^2 \bar{\lambda}_2(\mu) \sim O(\Lambda^2)$$

  if $m^2 \ll \Lambda^2$ it is expected there should be either some SYMMETRY or FINE TUNING

  $\Rightarrow$ Problem with Higgs mass, too small ?

Up to this point we considered evolution of couplings and masses

$$\beta_i(\bar{\lambda}) = \mu^2 \frac{d}{d\mu^2} \lambda_i$$

$$\gamma_{m_i}(\bar{\lambda}) = \frac{\mu^2}{m_i} \frac{dm_i}{d\mu^2}$$

we have also a Z_ϕ for each field $\Rightarrow$ define

$$\gamma_{\phi_j}(\bar{\lambda}) = \frac{\mu^2}{Z_{\phi_j}} \frac{dZ_{\phi_j}}{d\mu^2}$$

γ_{m_i} called mass ANOMALOUS DIMENSIONS

γ_{ϕ_j} called field ANOMALOUS DIMENSIONS

Why these strange names? to understand this, let's consider now a generic n-point Green function $\tilde{G}^{(n)}(p_i, m_i, \lambda_j)$ in ren pert. theory

this will have some mass dimension, say D_G

So what happens if I rescale all $p_i \to a\, p_i$?

$$\tilde{G}^{(n)}(a p_i, m_i, \lambda_j) \overset{?}{\to} \underbrace{[a]^{D_G}}_{\text{Dimensional analysis ?}} \tilde{G}^{(n)}(p_i, \frac{m_i}{a}, \lambda_j)$$

$\Rightarrow$ we FORGOT that $\tilde{G}^{(n)}$ depends also on μ !

$$\to [a]^{D_G} \tilde{G}^{(n)}(p_i, \frac{m_i(\mu)}{a}, \lambda_i(\mu), \frac{\mu}{a})$$

and if $\mu \sim p_i$, then we will produce

large logarithms $\ln\left[\frac{\mu}{a}\right] \sim \ln\left[\frac{p_i}{a}\right]$ if

a is taken too large [$a \gg p_i$ typical scales]

$\Rightarrow$ can we write a renormalization group eq.

for the whole $\tilde{G}^{(n)}$ to bring $\frac{\mu}{a} \xrightarrow{\text{back}} \mu$?

As for λ_0, m_0, we use the fact that

$\tilde{G}_0^{(n)}$ [BARE GREEN FUNCTION] must be μ-indep!

Take simple case of 1 FIELD ϕ, 1 λ, 1 m, then:

$$\mu^2 \frac{d}{d\mu^2}\Big[\underbrace{\left(\sqrt{Z_\phi}\right)^n \tilde{G}^{(n)}(p_i, m(\mu), \lambda(\mu), \mu)}_{\tilde{G}_0^{(n)}}\Big] = 0$$

bare written in terms of renormalized one!

$$\Big[\mu^2\frac{\partial}{\partial\mu^2} + \overbrace{\mu^2\frac{\partial\lambda}{\partial\mu^2}}^{\beta(\lambda)}\frac{\partial}{\partial\lambda} + \overbrace{\mu^2\frac{\partial m}{\partial\mu^2}}^{\gamma_m(\lambda)m}\frac{\partial}{\partial m}$$

$$+ \frac{n}{2}\underbrace{\frac{\mu^2}{Z_\phi}\frac{\partial Z_\phi}{\partial\mu^2}}_{\gamma_\phi(\lambda)}\Big]\tilde{G}^{(n)}(p_i, m(\mu), \lambda(\mu), \mu) = 0$$

so we can write

$$\left[\mu^2\frac{\partial}{\partial\mu^2}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\gamma_m(\lambda)\, m\frac{\partial}{\partial m}+\frac{n}{2}\gamma_\phi(\lambda)\right]\tilde{G}^{(n)}=0$$

Called CALLAN - SYMANZIK - GELL-MANN - LOW EQUATION

Now imagine I choose $\bar{m}(\mu)$, $\bar{\lambda}(\mu)$ such that they solve exactly R.G.E. $\beta(\bar{\lambda})=\mu^2\frac{d}{d\mu}\bar{\lambda}$

$$\gamma_m(\bar{\lambda})=\frac{\mu^2}{\bar{m}^2}\frac{\partial\bar{m}}{\partial\mu^2}$$

with some boundary value at $\mu=\mu_1$; $\lambda(\mu_1)\equiv\lambda_1$

$m(\mu_1)\equiv m_1$

Consider now $F(\mu) := \tilde{G}^{(n)}(p_i, \bar{\lambda}(\mu), \bar{m}(\mu), \mu)$

so $\tilde{G}^{(n)}$ evaluated along trajectory $\bar{\lambda}(\mu), \bar{m}(\mu)$

then by construction, clearly:

$$\mu^2 \frac{d}{d\mu^2} F(\mu) = \mu^2 \left[\frac{\partial}{\partial\mu^2} + \frac{\partial\bar{\lambda}}{\partial\mu^2}\frac{\partial}{\partial\lambda} + \frac{\partial m}{\partial\mu^2}\frac{\partial}{\partial m} \right] \tilde{G}^{(n)}$$

and we have then reabsorbed β & γ_m:

$$\left[\mu^2 \frac{d}{d\mu^2} + \frac{n}{2}\gamma_\phi(\lambda(\mu)) \right] F(\mu) = 0$$

$\hookrightarrow$ we already SOLVED RGEs for λ & m !

whose solution is

$$F(\mu) = F(\mu_1) \exp\left\{ -\frac{n}{2} \int_{\mu_1^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \gamma_\phi(\lambda(\mu')) \right\}$$

$$\tilde{G}^{(n)}(p_i, m(\mu), \lambda(\mu), \mu) = \tilde{G}^{(n)}\Big|_{\mu_1} \exp\left\{ -\frac{n}{2} \int_{\mu_1^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \gamma_\phi \right\}$$

now go back to original equation:

$$\tilde{G}^{(n)}(\alpha p_i, m, \lambda, \mu) = [\alpha]^{D_G}\, \tilde{G}^{(n)}\left(p_i, \frac{m(\mu)}{\alpha}, \lambda(\mu), \frac{\mu}{\alpha}\right)$$

and use solution of R.G.E in left-hand side

$$\tilde{G}^{(n)}(\alpha p_i, m_1, \lambda_1, \mu_1)\, e^{-\frac{n}{2}\int_{\mu_1^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \gamma_\phi(\lambda(\mu'))}$$

$$= [\alpha]^{D_G}\, \tilde{G}^{(n)}\left(p_i, \frac{m(\mu)}{\alpha}, \lambda(\mu), \frac{\mu}{\alpha}\right)$$

and choose $\mu = \alpha \mu_1$; then rename $\mu_1 \Rightarrow \mu$
(and bring exp to R.G. side)

$$\tilde{G}^{(n)}(\alpha p_i, m(\mu), \lambda(\mu), \mu) = [\alpha]^{D_G}\, e^{\frac{n}{2}\int_{\mu^2}^{\alpha^2\mu^2} \frac{d\mu'^2}{\mu'^2} \gamma_\phi(\lambda(\mu'))}$$

$$\times\, \tilde{G}^{(n)}\left(p_i, \frac{m(\alpha\mu)}{\alpha}, \lambda(\alpha\mu), \mu\right)$$

no ratio!

using $r = \frac{\mu'}{\mu}$; $\frac{d\mu'^2}{\mu'^2} = 2\cancel{\mu'}\frac{d\mu'}{\mu'^{\cancel{2}}} = 2\frac{dr}{r}$

$$\frac{n}{2}\int_{\mu^2}^{\alpha^2\mu^2}\frac{d\mu'^2}{\mu'^2}\,\gamma_\phi(\lambda(\mu')) = n\int_1^\alpha \frac{dr}{r}\,\gamma_\phi(\lambda(r\mu))$$

we can finally write

$$\tilde{G}^{(n)}(\alpha p_i, m(\mu), \lambda(\mu); \underline{\underline{\mu}})$$

SAME SCALE (no large logs)

$$= \alpha^{D_G}\, e^{\int_1^\alpha \frac{dr}{r} n\,\gamma_\phi(\lambda(r\mu))}\,\tilde{G}^{(n)}\left(p_i, \frac{m(\alpha\mu)}{\alpha}, \lambda(\alpha\mu), \underline{\underline{\mu}}\right)$$

now we have the same scale μ LEFT & RIGHT

$\Rightarrow$ price to pay, scaling becomes ANOMALOUS

$$\tilde{G}^{(n)} \sim \alpha^{D_G}\, e^{\int_1^\alpha \frac{dr}{r} n\,\gamma_\phi(\lambda(r\mu))}$$

Anomalous dimension changes "naive dim-scaling" !

take a 2-point function as <u>EXAMPLE</u>

$\tilde{G}^{(2)}(p, m(\mu), \lambda(\mu), \mu)$ $\qquad D_G = -2$

[like scalar propagator!]

by calling $p = q$ & Scaling $q \to \alpha q$; $\alpha = \frac{p}{q}$

we can read out from previous formula

$$\tilde{G}^{(2)}(p, m(\mu), \lambda(\mu); \mu) = \left[\frac{p}{q}\right]^{-2} \exp\left[\int_1^{p/q} \frac{dr}{r} 2\gamma_\phi(\lambda(r\mu))\right]$$

$$\times \tilde{G}^{(2)}\left(p, \frac{m(\alpha\mu)}{\alpha}, \lambda(\alpha\mu); \mu\right)$$

then consider two cases

$\boxed{1}$ λ small, $\qquad \gamma_\phi = \gamma_0 \lambda + \dots$

$\qquad \beta = \beta_0 \lambda^2 + \dots$

then

$$\exp\left[\int_1^{\alpha} \frac{dr}{r}\, 2\gamma_\phi(\lambda(r\mu))\right] = \exp\left[\int_{\mu^2}^{(\alpha\mu)^2} \frac{d\mu'^2}{\mu'^2}\, \gamma_\phi(\lambda(\mu'))\right]$$

$$= \exp\left[\int_{\lambda(\mu)}^{\lambda(\alpha\mu)} d\lambda' \,\frac{\gamma_\phi(\lambda')}{\beta(\lambda')}\right] \sim \exp\left[\int_{\lambda(\mu)}^{\lambda(\alpha\mu)} d\lambda' \,\frac{\gamma_0}{\beta_0\lambda'}\right]$$

$$= \exp\left[\frac{\gamma_0}{\beta_0}\ln\left(\frac{\lambda(\alpha\mu)}{\lambda(\mu)}\right)\right] = \left(\frac{\lambda(\alpha\mu)}{\lambda(\mu)}\right)^{\gamma_0/\beta_0}$$

such that

$$\tilde{G}^{(2)}(p, m(\mu), \lambda(\mu); \mu) = \frac{q^2}{p^2}\left(\frac{\lambda(\alpha\mu)}{\lambda(\mu)}\right)^{\gamma_0/\beta_0} \tilde{G}^{(2)}\left(q, \frac{m(\alpha\mu)}{\mu}, \lambda(\alpha\mu); \mu\right)$$

$\alpha = \frac{p}{q}$ send now $p \to \infty$ $[\alpha \to \infty]$

$$\sim \frac{1}{p^2}\,[\lambda(p)]^{\frac{\gamma_0}{\beta_0}}\left(c + O\left(\lambda, \frac{m^2}{p^2}\right)\right)$$

c: p-indep

can change scaling of propagator as $p \to \infty$

now imagine that the ren. group flow has a non-trivial fixed point for $\partial \to \infty$; $\lambda \to \lambda^*$ where $\beta(\lambda^*) = 0$, so theory stops running

$$\exp\left[\int_1^{\partial} \frac{dr}{r}\, 2\gamma_\phi(\lambda(r\mu))\right] \sim \exp\left[2\gamma_\phi(\lambda^*) \int_1^{\partial} \frac{dr}{r}\right]$$

$$= \left[\frac{p^2}{q^2}\right]^{\gamma_\phi(\lambda^*)}$$

which also gives

$$\tilde{G}^{(2)}(p, m(\mu), \lambda(\mu); \mu) \sim \frac{1}{[p^2]^{1-\gamma_\phi(\lambda^*)}} \left(1 + O\left(\lambda, \frac{m^2}{p^2}\right)\right)$$

the FIELD DIMENSION change near fixed point!

this also implies that a FIXED POINT in the UV may change behavior of propagators $\Rightarrow$ it may make a non-renorm theory $\Rightarrow$ renormalizable ; this usually requires a NON-PERTURBATIVE ANALYSIS.