

Recap of Group theory :

the Lorentz Group

QFT 1

WS 1025/26

Lorenz
Tourolle

In last lecture we have seen that dealing with FIELDS as fundamental objects, is a possible way to write a theory that treats $\vec{x}$ & t in the very same way $\Rightarrow$ relativistic covariance is possible!

So our goal now is to write a Relativistic Covariant (Quantum) Field Theory. We start with "Quantum" in brackets, since we will first see how to get such theories at the Classical level, and then discuss their quantization.

the first step to get there, is to study the Group of Transformations of SR, the Poincaré group, and understand how they act on

FIELDS

In Particular, we will start with the Lorentz group, then consider the addition of translations

$\Rightarrow$ Poincaré Group. Our goal will be that of writing a Poincaré-invariant LAGRANGIAN to describe fields of different type.

We will then ask ourselves what does this tell us about PARTICLES.

To understand symmetries in physics we need GROUP THEORY & especially LIE GROUPS. Let's start with a quick recap.

A GROUP is a set G , in which a multiplication operation is defined, such that

1. $\forall a, b, c \in G \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. $\exists e \in G ; \forall a \in G \quad e \cdot a = a \cdot e = a$
3. $\forall a \in G ; \exists a^{-1} \in G ; a \cdot a^{-1} = a^{-1} \cdot a = e$

A group is called **ABELIAN** if multiplication is commutative $\Rightarrow \forall a, b \in G \quad a \cdot b = b \cdot a$

Otherwise it is **NON ABELIAN**

A **LIE GROUP** is a group G whose elements depend in a smooth way (continuous & differentiable) on a set of real parameters $\theta^a, a = 1, \dots, N$

$\Rightarrow$ A **LIE GROUP** is also a **DIFFERENTIABLE MANIFOLD**

$g = g(\theta^a)$ and $e = g(0)$ by convention

A linear **REPRESENTATION** R of a group is a way to assign to each $g \in G$ a **linear operator**

$D_R(g)$ on some vector space V such that

$$1. D_R(e) = \mathbb{1}$$

$$2. D_R(g_1) D_R(g_2) = D_R(g_1 g_2) \quad \forall g_1, g_2 \in G$$

$$3. D_R(g^{-1}) = [D_R(g)]^{-1} \quad \forall g \in G$$

if V has dimension n , g is represented by a $n \times n$ matrix $[D_R(g)]_{ij}$

$\Rightarrow$ the representation has then dimension " n "

if e_i is BASIS of V then

$$e_i = [D_R(g)]_{ji} e_j$$

in fact $\forall \psi \in V \quad \psi = \psi_i e_i$ then

$$[D_R(g)] \psi = \psi_i [D_R(g)] e_i = \psi_i (D_R(g))_{ji} e_j$$

$$= \sum_j \underbrace{[D_R(g)]_{ji} \psi_i}_{\text{components of } \psi} e_j$$

components of any vector transform as expected !

A representation R is **FAITHFUL** if $D_R(g_1) = D_R(g_2) \Rightarrow g_1 = g_2$

A representation R is said to be **REDUCIBLE** if

$\exists$ a subspace of V , $U \subset V$, such that

$\forall \varphi \in U, g \in G; D_R(g)\varphi \in U \Rightarrow U$ is an

INVARIANT SUBSPACE

If this is NOT THE CASE, the representation is called

IRREDUCIBLE $\Rightarrow$ no invariant subspaces!

A reducible representation can be written, with

a suitable choice of basis, as the direct sum of irreducible representations

$$[D_R(g)]_{ij} = \begin{bmatrix} [n \times n] & & 0 \\ & [m \times m] & \\ 0 & & \ddots \\ & & & [\dots] \end{bmatrix}$$

matrix in
block form
each block
 $1 \times 1 = p$

we say also that two representations R, R' are equivalent if $\forall g \in G \exists S$ invertible such that

$$[D_{R'}(g)] = S^{-1} [D_R(g)] S$$

$\Rightarrow$ switching between two equivalent representations is like changing basis e_i in V !

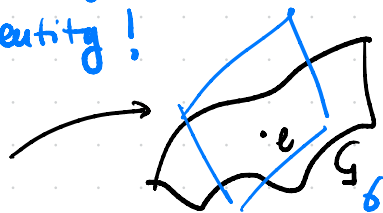
Now, since Lie groups are smooth, and we assumed that $g(\theta^a) \in G$; $g(0) = e$, then for ANY REPRESENTATION R if $\theta^a \ll 1$ we can write

$$D_R[g] = D_R[\theta^a] = 1 + i\theta^a T_R^a + O((\theta^a)^2)$$

with $T_R^a = -i \frac{\partial D_R}{\partial \theta^a} \Big|_{\theta^a=0}$ the GENERATORS of the group

T_R^a SPAN the LIE ALGEBRA $\Rightarrow$ tangent space to Lie group manifold at the identity!

TANGENT SPACE !



note that the "i" is CONVENTIONAL in Physics

One can then show that, from the identity, this equation can be "resummed" and we can write

$$D_R(g(\theta)) = e^{i\theta^a T^a} \quad \text{EXPONENTIAL MAP}$$

the "physics i" is because in Q.M., when we study symmetries, we want the groups to

act on STATES in a UNITARY way $\Rightarrow$ conserve probabilities ($D_R^\dagger = D_R^{-1}$)

With these conventions $D_R(g)$ UNITARY

$$\Rightarrow T_R^a \text{ are HERMITIAN}$$

$$(T_R^a)^\dagger = T_R^a$$

If G non ABELIAN, what can we say about $[T^a, T^b] = ?$

$\Rightarrow$ take two different elements of the group

$$D_R(g_1) = e^{i\alpha^a T_R^a}$$

$$D_R(g_2) = e^{i\beta^a T_R^a}$$

of course we have

$$D_L(g_1) D_L(g_2) = D_L(g_1 g_2) = e^{i \delta^a T_R^a}$$

$\delta^a = \delta^a(\alpha^a, \beta^a)$

What can we say about $\delta^a(\alpha, \beta)$?

$$e^{i \alpha^a T_R^a} e^{i \beta^a T_R^a} \neq e^{i (\alpha^a + \beta^a) T_R^a}$$

because in general T_R^a do not commute !

We can write though

$$\log(e^{i \delta^a T_R^a}) = \log(e^{i \alpha^a T_R^a} e^{i \beta^a T_R^a})$$

$\Rightarrow$ if $\alpha^a, \beta^a, \delta^a$ are all INFINITESIMAL, then

$$i \delta^a T_R^a = \log \left[\left(1 + i \alpha^a T_R^a - \frac{1}{2} (\alpha^a T_R^a)^2 + \dots \right) \right. \\ \left. \left(1 + i \beta^a T_R^a - \frac{1}{2} (\beta^a T_R^a)^2 + \dots \right) \right]$$

$$= \log \left[1 + i(a^a + b^a) T_R^a - a^a \beta^b T_R^a T_R^b - \frac{1}{2} a^a a^b T_R^a T_R^b - \frac{1}{2} \beta^a \beta^b T_R^a T_R^b + \dots \right]$$

now we use $\log(1+x) = x - \frac{x^2}{2} + O(x^3)$

$$= i(a^a + \beta^a) T_R^a - T_R^a T_R^b \left(a^a \beta^b + \cancel{\frac{1}{2} a^a a^b} + \cancel{\frac{1}{2} \beta^a \beta^b} \right)$$

$$+ \underbrace{\frac{(a^a + \beta^a) T_R^a]^2}{2}}_{\text{bracketed term}} + O(a^3, \beta^3)$$

$$+ \frac{1}{2} (\cancel{a^a a^b} + \cancel{\beta^a \beta^b} + a^a \beta^b + a^b \beta^a) T_R^a T_R^b$$

$$= i(a^a + \beta^b) T_R^a - \frac{1}{2} a^a \beta^b T_R^a T_R^b + \frac{1}{2} a^b \beta^a T_R^b T_R^a$$

swapped $a \leftrightarrow b$
in second term

so finally we get

$$i[\delta^a - \alpha^a - \beta^a] T_R^a = -\frac{1}{2} \alpha^a \beta^b [T_R^a, T_R^b]$$

or

$$\alpha^a \beta^b [T_R^a, T_R^b] = i[-2(\delta^c - \alpha^c - \beta^c)] T_R^c$$

renamed
dummy
index

this relation should hold for all values of α^a, β^a

$$\Rightarrow -2(\delta^c - \alpha^c - \beta^c) \equiv C^{abc} \alpha^a \beta^b \quad \text{such that}$$

LINEAR IN α^a, β^a

$$[T_R^a, T_R^b] = i C^{abc} T_R^c$$

C^{abc} are called STRUCTURE
CONSTANTS

they characterize the Lie Group up to isomorphisms

clearly, C^{abc} must be antisymmetric in $a \leftrightarrow b$

but in general NOT IN ALL INDICES!

Of course, for an ABELIAN GROUP the structure constants are trivially zero since all elements of the group commute, which implies that also the generators commute

$$[T^a, T^b] = 0 \Rightarrow C^{abc} = 0 \quad \text{for an ABELIAN GROUP}$$

In a given group, there can be SPECIAL ELEMENTS, or operators, constructed from the T^a , that COMMUTE WITH ALL GENERATORS

$\Rightarrow$ these are called CASIMIR OPERATORS $\bar{C}$

$$[\bar{C}, T^a] = 0 \quad \forall T_R^a$$

Second Schur's Lemma guarantees that such operators must be PROPORTIONAL TO THE IDENTITY

$\bar{c} = c \cdot e$; this is true for ABSTRACT group elements!

Once we specify to a REPRESENTATION

$\bar{c}_R = c_R \cdot \mathbb{1}_R$ c_R characterizes the representation.

EXAMPLE :

if you recall $SO(3) \sim SU(2)$ (Angular momentum)

then we have $[L_i, L_j] = i \epsilon_{ijk} L_k$

$L^2 = L_x^2 + L_y^2 + L_z^2$ is the CASIMIR

$$[L^2, L_i] = 0 \Rightarrow L^2 \cdot |l, m\rangle = \underbrace{\hbar l(l+1)} |l, m\rangle$$

with $\hbar = 1$ Casimir is $l(l+1)$

irreducible repr labelled by l ; $c_l = l(l+1)$

We have said that a Lie group is a manifold.
 If this manifold is COMPACT, then the Lie group
 is COMPACT

EXAMPLE: $SU(2)$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{C})$

with $U^\dagger = U^{-1}$

$\det U = 1$

A matrix in $SU(2)$ $U = \begin{bmatrix} z_1 + iz_2 & z_3 + iz_4 \\ -z_3 + iz_4 & z_1 - iz_2 \end{bmatrix}$

From $U(2) \Rightarrow e^{i\theta} U$
 one more parameter!

$U^\dagger = \begin{bmatrix} z_1 - iz_2 & z_3 + iz_4 \\ -z_3 - iz_4 & z_1 + iz_2 \end{bmatrix}$

$\det U = \sum_{i=1}^4 z_i^2 = 1$

$\Rightarrow$ equation of sphere S^3
 COMPACT MANIFOLD!

$UU^\dagger = \begin{bmatrix} z_1^2 + z_2^2 + z_3^2 + z_4^2 & 0 \\ 0 & z_1^2 + z_2^2 + z_3^2 + z_4^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note that, if the group is COMPACT, we can choose the generator T^a to form an ORTHONORMAL BASIS using the TRACE as "scalar product"

$$\text{Tr}[T^a T^b] = C \delta^{ab}$$

where $C \rightarrow C_R$ in specific representation $\Rightarrow$ Fixing C ,
Fixes also C_R in every representation $\forall$

$$\left[\begin{array}{l} \text{For example} \\ \text{in } \text{SU}(N) \end{array} \Rightarrow \begin{array}{l} \text{Tr}[T_F^a T_F^b] = \frac{1}{2} \delta^{ab} \text{ (Fundamental)} \\ \text{Tr}[T_A^a T_A^b] = N \delta^{ab} \text{ (Adjoint)} \end{array} \right]$$

With this choice, one can show that the structure constants C^{abc} are TOTALLY ANTISYMMETRIC

In this case, we typically call them f^{abc}

Once the normalization is fixed, the structure constants are INDEPENDENT OF THE REPRESENTATION
 $\Rightarrow$ they completely characterize the Lie algebra!

the Lorentz group is NOT COMPACT, as we will see explicitly in a moment.

Note that not only compact groups can have totally antisymmetric structure constants !

the condition on the scalar product can be generalized to semi-simple groups as the Lorentz group But Not to Poincare !

IMPORTANT : there is a theorem about compact groups that says that if the group is NON COMPACT, then there are NO FINITE DIMENSIONAL UNITARY REPRESENTATIONS, except for the trivial one where $\text{Tr}^a = 0$

→ In Quantum Mechanics we want Unitary representations & Hermitian generators to represent physical observables !

$\Rightarrow$ generator's Eigenvalues are OBSERVABLES $\Rightarrow$ Ang. Mom!

if we deal with a non-compact group like Lorentz (or Poincaré), we will be forced

to work with INFINITE-DIMENSIONAL representations which act on the HILBERT SPACE of ONE-PARTICLE STATES

IMPORTANT

We need UNITARY REPS ONLY WHEN ACTING ON STATE VECTORS in HILBERT SPACE

$\Rightarrow$ the physical states (one-particle states!)

the Fields that we will use are "bookkeeping"

devices, so in working with them we can

(and we WILL!) use NON-UNITARY Finite-Dim.

Representations of Lorentz & Poincaré!

But let us now finally look at Lorentz Group:

THE LORENTZ GROUP

Consider a 4-vector x^μ ; the Lorentz group $O(1,3)$ is defined such that $\forall \Lambda \in O(1,3)$

$$x^\mu \rightarrow \Lambda^\mu_\nu x^\nu; \quad x^\mu x_\mu \rightarrow x'^\mu x'_\mu = x^\mu x_\mu$$

$$\text{since } x^\mu x_\mu = x^\mu x^\nu g_{\mu\nu} \rightarrow \Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho x^\sigma g_{\mu\nu}$$

$$\Rightarrow \Lambda \in O(1,3) \text{ implies}$$

$$\Lambda^\mu_\rho \Lambda^\nu_\sigma g_{\mu\nu} = g_{\rho\sigma} \quad (*)$$

$$g = \Lambda^T g \Lambda \quad \text{in matrix notation}$$

$$\Rightarrow \det(g) = \det(\Lambda^T g \Lambda) = [\det \Lambda]^2 \det g$$

$$\det \Lambda = \pm 1$$

Transformations that reverse 1 x_i
reverse ALL $\vec{x}$ (PARITY)
reverse t (TIME REVERSAL)

are examples of Λ with $\det \Lambda = -1$

the subset $\Lambda \in O(1,3)$ with $\det \Lambda = +1$ is called
PROPER LORENTZ GROUP $SO(1,3)$

$\forall \tilde{\Lambda} \in O(1,3)$ can be written as $\Lambda \cdot Q$; $\Lambda \in SO(1,3)$
 $Q =$ reverse odd # coordinates

Moreover, the 00 component of (*) gives

$$\Lambda^\mu_0 \Lambda^\nu_0 g_{\mu\nu} = 1$$

$$\Rightarrow 1 = (\Lambda^0_0)^2 - \sum_{i=1}^3 [\Lambda^i_0]^2$$

which implies $(\Lambda^0_0)^2 \geq 1$

so either $\Lambda^0_0 \geq 1$ or $\Lambda^0_0 \leq -1$

transformations with $\Lambda^0_0 \geq 1$ are called

ORTHOCHRONOUS ; those with $\Lambda^0_0 \leq -1$ NON-ORTHOCHRONOUS

they reverse flow of time !

these two components are DISCONNECTED, there is no continuous transf. that can switch Λ^0_0 or the determinant $\Rightarrow$ we obtain NON-ORTHO by combining ORTHO + time reversal !

$\Rightarrow$ We can then focus on $SO(1,3)$ with $\Lambda^0_0 \geq 1$

PROPER ORTHOCHRONOUS LORENTZ GROUP

" $SO^+(1,3)$ "

INFINITESIMAL LORENTZ TRANSFORMATIONS

Consider now an infinitesimal transformation in $SO^+(1,3)$
we write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \quad \text{and putting this into}$$

$$g_{\rho\sigma} = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma$$

$$\Rightarrow \cancel{g_{\rho\sigma}} = g_{\mu\nu} [\delta^\mu{}_\rho + \omega^\mu{}_\rho] [\delta^\nu{}_\sigma + \omega^\nu{}_\sigma]$$
$$= \cancel{g_{\rho\sigma}} + \omega_{\sigma\rho} + \omega_{\rho\sigma}$$

$$\omega_{\rho\sigma} = -\omega_{\sigma\rho} \quad \text{it must be antisymmetric!}$$

Antisymmetric 4×4 matrix $\omega_{\mu\nu}$ has 6 components

1] Fix $t = x^0$, $SO^+(1,3) \mapsto SO(3)$
rotation group has 3 parameters

2] Fix each $x_i, x_j \quad \forall i, j = 1, 2, 3$

$\Rightarrow$ transformations that leave $t^2 - x_k^2$ invariant

there are 3 of them in $(t, x), (t, y), (t, z)$

Sub planes $\Rightarrow$ called BOOSTS

For example in $(t, x) \Rightarrow t^2 - x^2$ invariant

$$\begin{cases} t \rightarrow \gamma(t + vx) \\ x \rightarrow \gamma(x + vt) \end{cases}$$

indeed for $v \rightarrow 0$

$$x \sim x + vt + O(v^2)$$

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

$$-1 < v < 1$$

velocity in
units of c

$\Rightarrow$ 3 ROTATIONS + 3 BOOSTS

We can parametrize boosts or hyperbolic rotations

$$\begin{cases} t = (\cosh \eta)t + (\sinh \eta)x \\ x = (\sinh \eta)t + (\cosh \eta)x \end{cases}$$

$\Rightarrow t^2 - x^2$ is
conserved because

$$\cosh^2 \eta - \sinh^2 \eta = 1$$

then a boost in t, x plane is

$$\Lambda(\eta) = \begin{bmatrix} \cosh \eta & \sinh \eta & 0 & 0 \\ \sinh \eta & \cosh \eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$-\infty < \eta < +\infty$$

hence set

$$\Lambda(\eta) \subset M_{4 \times 4}(\mathbb{R})$$

IS UNBOUNDED

$\Rightarrow SO^+(1,3)$ CANNOT BE COMPACT!

So $SO^+(1,3)$ is a Lie group, continuously parametrized through 3 angles and 3 boosts, but it is not compact $\Rightarrow$ there are NO FINITE-DIM UNITARY REPRESENTATIONS