From the S-matrix to Decay rates & Cross-sections

we have seen how to relate S-matrix elements to GREEN FUNCTIONS ; we write

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^{(4)}\Big(\sum_{i=1}^{m} k_i - \sum_{j=1}^{n} p_j\Big) i \underbrace{\mathcal{M}_{fi}}_{\text{iT transfer MATRIX (connected graphs!)}}$$

for $\underbrace{k_1 + \dots + k_m}_{P_i} \to \underbrace{p_1 + \dots + p_n}_{P_f}$

In QUANTUM MECHANICS we know how to compute decay rates or cross sections from $\mathcal{M}$ $\Rightarrow$ let us first make a distinction between

M non-relativistic case

$\mathcal{M}$ relativistic case

what changes between M & $\mathcal{M}$ is normalization of one-particle states (in/out states) in fixed volume V

$\Rightarrow$ remember, we always work in fixed V and eventually send $V \to \infty$

In NON-RELATIVISTIC QM we write

$$\psi_{\vec{p}}(\vec{x}) = \frac{1}{\sqrt{V}} e^{i \vec{p} \cdot \vec{x}}$$ normalized such that

$$\int_V d^3\vec{x} \, |\psi_{\vec{p}}(\vec{x})|^2 = 1$$ in finite volume

and $$\int_V d^3\vec{x} \, \psi^*_{\vec{p}_1}(\vec{x}) \psi_{\vec{p}_2}(\vec{x}) = \delta_{\vec{p}_1, \vec{p}_2}$$ discrete because

$$\vec{p} = \frac{2\pi \vec{n}}{L}$$

$\Downarrow$

$$\langle \vec{p}_1 | \vec{p}_2 \rangle_{N.R.} = \delta_{\vec{p}_1, \vec{p}_2}$$

non-relativistic normalization

Quantized in box of $L^3 = V$!
(with periodic boundary conditions)

now if we go to QFT, we normalized relativistic

one-particle states differently

$$\langle \vec{p}_1 | \vec{p}_2 \rangle_R = 2E_{p_1} (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$$

[what we wrote in $V \cdot T \to \infty$]

using some V & $\int d^3\vec{p} = \left(\frac{2\pi}{L}\right)^3 \sum_n$ Discretized version

$$\Rightarrow (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) = L^3 \delta_{\vec{p},\vec{q}} = V \delta_{\vec{p},\vec{q}}$$

Such that in fixed V, relativistic norm is

$$\langle \vec{p}_1 | \vec{p}_2 \rangle_R = 2E_{p_1} V \delta_{\vec{p}_1,\vec{p}_2}$$

$$\Rightarrow |\vec{p}\rangle_R = [2E_p V]^{1/2} |\vec{p}\rangle_{NR}$$

$$|p_1 \ldots p_n\rangle_R = \prod_{i=1}^{n} [2E_{p_i} V]^{1/2} |p_1 \ldots p_n\rangle_{N.R}$$

this also gives a relation between $\mathcal{M}$ & $\mathcal{U}$

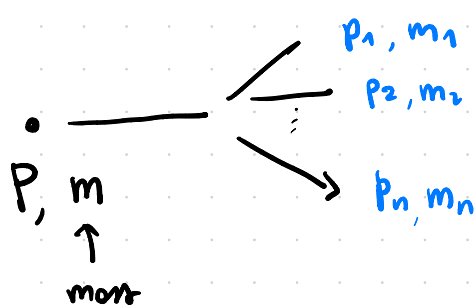
$$\mathcal{M}_{fi} = \prod_{i=1}^{n} (2E_{p_i} V)^{-1/2} \prod_{j=1}^{m} (2E_{k_j} V)^{-1/2} \mathcal{U}_{fi}$$

with this we can convert QM non-relativistic formulas into relativistic QFT ones!

Let's do this explicitly for 2 important cases

1] DECAY RATES

Consider a decay process:



Assume all final state particles assumed to be DISTINGUISHABLE (different masses!)

Quantum mechanics : square $|M|^2$ and sum over final states

since $f \neq i$ there is never interference with $\delta_{f,i}$

By naively squaring then we get

$$\left|(2\pi)^4 \delta^{(4)}\Big(P - \sum_{i=1}^{n} p_i\Big)\, i \underset{\uparrow}{M_{fi}}\right|^2 \Rightarrow \Big[\delta^{(4)}\Big(P - \sum_i p_i\Big)\Big]^2 \; ?$$

Q.M. matrix element

For us this is simply (in FINITE VOLUME & for finite T)

$$(2\pi)^4 \delta^{(4)}\Big(P - \sum_i p_i\Big) \cdot \underbrace{(2\pi)^4 \delta^{(4)}(0)}_{V \cdot T \text{ in Finite volume}} = (2\pi)^4 \delta^{(4)}\Big(P - \sum_i p_i\Big)\, VT$$

$$= (2\pi)^4 \delta^{(4)}\Big(P - \sum_i p_i\Big)\, VT\, |M_{fi}|^2$$

which we need to sum over $\vec{p}_i$!

in finite V, $\vec{p}_i$ are "discrete", but we take here

the same approach as in STATISTICAL PHYSICS

$$\sum_{p_i} \xrightarrow[\text{LARGE } V]{} \frac{V}{(2\pi)^3} \int d^3 p_i$$

this is nothing but STAT. PHYS. integration measure

$$\int \frac{d^3 x_i \, d^3 p_i}{(2\pi)^3} = V \int \frac{d^3 p_i}{(2\pi)^3}$$

because M does not depend on $\vec{x}_i$!

volume of phase-space cell

So finally the probability for $I \to p_1 + \dots + p_n$

where each $p_i \in [p_i, p_i + dp_i]$ and $-\frac{T}{2} < t < \frac{T}{2}$

$$dw = (2\pi)^4 \, \delta^{(4)}(P - \sum_i p_i) \, VT \, |M_{fi}|^2 \prod_{i=1}^{n} \frac{V d^3 \vec{p}_i}{(2\pi)^3}$$

we are interested in DECAY RATE : probability per time!

$$d\Gamma = \frac{dw}{T} = (2\pi)^4 \delta^{(4)}(P - \sum_i p_i)\, |M_{fi}|^2\, V \prod_{i=1}^{n} \frac{V d^3\vec{p}_i}{(2\pi)^3}$$

which is now independent of 'T'

nicely, going to relativistic $|\mathcal{M}^R_{fi}|^2 \propto V^{1+n} |M^{N.R.}_{fi}|^2$

removes them

$$d\Gamma = (2\pi)^4 \delta^{(4)}(P - \sum_i p_i)\, |\mathcal{M}^R_{fi}|^2 \frac{1}{2E_p} \overbrace{\prod_{i=1}^{n} \frac{d^3\vec{p}_i}{(2\pi)^3\, 2E_{p_i}}}^{\text{Lorentz inv.}}$$

energy of initial particle

energies of final state p_i

$|\mathcal{M}^R_{fi}|^2$ is exactly our relativistic matrix element

from LSZ reduction → Green functions!

Defining the n-body PHASE SPACE

$$d\Phi_n = (2\pi)^4 \, \delta^{(4)}\left(P - \sum_i p_i\right) \prod_{i=1}^{n} \frac{d^3\vec{p}_i}{(2\pi)^3 \, 2E_{p_i}}$$ rel. invariant

$$\Rightarrow \quad \boxed{d\Gamma = \frac{1}{2E_p} |\mathcal{M}_{fi}|^2 \, d\Phi_n}$$

- Note that if $m < \sum_i m_i$ then $\delta^{(4)}\left(P - \sum_i p_i\right)$ forbids the process, as expected!

- In rest frame of P: $\frac{1}{2E_p} = \frac{1}{2m}$

if we boost to another frame $E_p = \gamma m = \frac{m}{\sqrt{1-v^2}}$ "velocity v"

$d\Gamma$ is reduced to $\frac{d\Gamma}{\gamma}$ in boosted frame

$\Rightarrow$ Life-time of P is $\propto \frac{1}{d\Gamma} \propto \gamma$ LIFE-TIME increases!

2] CROSS-SECTIONS

In this case we are interested in $2 \to n$ scattering.

Realistically, we consider two beams of particles

m_1, n_1, v_1 m_2, n_2, v_2

$$\left.\begin{array}{l} n_i = \dfrac{\#\ \text{particles}}{V} \\ m_i = \text{mass}, \quad v_i = \text{velocity} \end{array}\right\} \text{not Lorentz invariant}$$

Go to REST FRAME of m_2 $\Rightarrow$ $v_2 = 0$, n_2^0

$v_1 = v_1^0$, n_1^0

Assume also UNIFORM DISTRIBUTION n_1^0 & n_2^0

then call N is number of scattering events $\longrightarrow$

$$dN = \sigma\, v_1^0\, n_1^0\, n_2^0\, dV\, dt \qquad (*)$$

$$\frac{\#\text{ events}}{dV\,dt} \propto v_1^0 \cdot n_1^0 = \text{flux}$$

$$n_2^0 = \text{density of target}$$

this formula DEFINES $\underline{\sigma = \text{CROSS SECTION}}$

$$[\sigma] = \text{Area} \quad \text{because} \quad v_1^0 \sim \frac{L}{T}$$

$$n_i^0 \sim \frac{1}{L^3} \qquad dV \sim L^3$$

$$\#\text{ particles} = [\sigma]\cdot\frac{L}{\cancel{T}}\cdot\frac{1}{L^{\cancel{3}\,2}}\,\frac{1}{\cancel{L^3}}\;\cancel{L^3}\;\cancel{T} \qquad \underline{[\sigma] = L^2}$$

how does $(*)$ generalize to ARBITRARY FRAME ?

dN, σ, $dV \cdot dt$ are all Lorentz invariant

[work now with $c = \hbar = 1$!]

Consider a general frame where n_1, n_2
$\vec{v}_1, \vec{v}_2$

then $n_1 n_2 \sqrt{(\vec{v}_1-\vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2}$

becomes $n_1^\circ n_2^\circ v_1^\circ$ in rest frame of 2

one can prove that it's the unique generalization !

note that $\sqrt{(\vec{v}_1-\vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2} = \frac{1}{E_{k_1} E_{k_2}} \sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2}$

where $k_i = (E_{k_i}, \vec{k}_i)$ $\vec{v}_i$ velocity of $\vec{k}_i$

which says that $E_1 \cdot E_2 \cdot \sqrt{(\vec{v}_1-\vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2}$ = is Lorentz invariant !

notice that if $\vec{v}_1 \parallel \vec{v}_2$ then

$$\sqrt{(\vec{v}_1-\vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2} \rightarrow |\vec{v}_1 - \vec{v}_2| = |v_1 - v_2|$$

it's common to introduce $I = \sqrt{(\vec{k}_1 \cdot \vec{k}_2)^2 - m_1^2 m_2^2}$ such that

$$dN = \frac{I}{E_1 E_2} n_1 n_2 \, dV \, dt \; \sigma$$

$$= \sigma \frac{I}{V E_1 E_2} \underbrace{(n_1 V)}_{N_1} (n_2 \, dV) \, dt$$

N_1 (uniform density)

$\int dV \cdot n_2 = N_2$

so from here we can obtain the # events per unit particle of type 1, unit particle type 2 in a total time T $\Rightarrow$ event PROBABILITY

$$P(\text{event}) = \frac{\sigma I T}{V E_1 E_2} = \text{S-matrix element summed over final states}$$

$$= (2\pi)^4 \delta^{(4)}(P_i - P_f) \, V T \int |M_{fi}|^2 \prod_{i=1}^{n} \frac{V d^3 p_i}{(2\pi)^3}$$

$P_i = p_1 + p_2$; P_f = all outgoing !

non-relativistic

$\Rightarrow$ going to relativistic normalization

removes all factors of $\{V, E_1, E_2\}$

$$d\sigma = (2\pi)^4\, \delta^{(4)}(P_i - P_f)\, \frac{1}{4I}\, |\mathcal{M}_{fi}|^2 \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2E_{p_i}}$$

we recognize again $d\phi_n$ with $P \to P_i$ = sum of all incoming particles

$$d\sigma = \frac{1}{4I}\, |\mathcal{M}_{fi}|^2\, d\phi_n$$

$$4I = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$$

called

FLUX FACTOR

Remember: we always assumed "n" final state particles are DISTINGUISHABLE $\Rightarrow$ if INDISTING. we must divide by $\frac{1}{n!}$

A very common situation is $n=2$

- $P \to p_1 + p_2$ (decay to two bodies)
- $k_1 + k_2 \to p_1 + p_2$ (production of two particles)

Compute $d\Phi_2$ FOR DECAY P with mass m

$$d\Phi_2 = (2\pi)^4 \delta^{(4)}(P - p_1 - p_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_{p_1}} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_{p_2}}$$

$$= \cancel{(2\pi)^4} \cancel{\delta^{(4)}(P - p_1 - p_2)} (2\pi)\theta(p_{10}) (2\pi)\theta(p_{20})$$

$$\delta(p_1^2 - m_1^2)\,\delta(p_2^2 - m_2^2) \frac{d^4 p_1}{(2\pi)^4} \cancel{\frac{d^4 p_2}{(2\pi)^4}}$$

$$= (2\pi)^2 \theta(p_{10})\,\theta(P_0 - p_{10}) \frac{d^4 p_1}{(2\pi)^4}$$

$$\delta(p_1^2 - m_1^2)\,\delta((P-p_1)^2 - m_2^2)$$

must be Lorentz invariant $\Rightarrow$

compute it in P rest frame $P = (m, \vec{0})$ then

$$d\phi_2 = (2\pi)^2 \frac{d^4 p_1}{(2\pi)^4} \theta(p_{10}) \theta(m - p_{10}) \delta(p_1^2 - m_1^2) \delta(m^2 + m_1^2 - m_2^2 - 2m p_{10})$$

$$= \frac{d^3 \vec{p}_1}{(2\pi)^2} d p_{10}\, \theta(p_{10}) \theta(m - p_{10}) \frac{1}{2m} \delta\left(p_{10} - \frac{m^2 + m_1^2 - m_2^2}{2m}\right)$$

$$\cdot \delta(p_{10}^2 - \vec{p}_1^{\,2} - m_1^2)$$

as long as $0 < \frac{m^2 + m_1^2 - m_2^2}{2m} < m \Rightarrow m^2 + m_1^2 - m_2^2 < 2m^2$

$m_1^2 - m_2^2 < m^2$

the δ-function has support $\Rightarrow$ this should always be the case because we want $m^2 > (m_1 + m_2)^2$ to have enough energy to produce the two particles at rest!

$$= \frac{d^3 p_1}{(2\pi)^2} \frac{1}{2m} \delta\Big(\underbrace{\left(\frac{m^2 + m_1^2 - m_2^2}{2m}\right)^2}_{\mu^2} - \vec{p}_1^{\,2} - m_1^2\Big)$$

now $\quad \frac{d^3p_1}{(2\pi)^2} = d\Omega \, \frac{d|\vec{p}_1|}{(2\pi)^2} \, |\vec{p}_1|^2$

and $\quad E_{p1} = \sqrt{|\vec{p}_1|^2 + m_1^2} \qquad dE_{p1} = \frac{1}{\cancel{2} E_{p1}} \cancel{2} \, d|\vec{p}_1| \cdot |\vec{p}_1|$

$$dE_{p1} = \frac{|\vec{p}_1|}{E_{p1}} \, d|\vec{p}_1|$$

so finally we find:

$$d\phi_2 = \frac{d\Omega}{8\pi^2 m} \, \frac{E_{p1} |\vec{p}_1|^{\cancel{2}}}{\cancel{|\vec{p}_1|}} \, dE_{p1} \, \underbrace{\delta(E_{p1}^2 - \mu^2)}_{\delta((E_{p1}-\mu)(E_{p1}+\mu))} \qquad E_{p1} > 0 \, !$$

$$= \frac{d\Omega}{8\pi^2 m} \, \cancel{E_{p1}} \sqrt{E_{p1}^2 - \mu^2} \, \frac{1}{2\cancel{\mu}} \, \delta(E_{p1} - \mu)$$

$$= \frac{d\Omega}{16\pi^2 m} \sqrt{\mu^2 - m_1^2} = \frac{d\Omega}{32\pi m^2} \sqrt{(m^2 - m_{12}^2)(m^2 - \mu_{12}^2)}$$

$m_{12} = m_1 + m_2 \; ; \; \mu_{12} = m_1 - m_2$

simple, useful limits

- $m_2 = m_1$ then $\mu_{12} = 0$; $m_{12} = 2m_1$

$$d\phi_2 = \frac{d\Omega}{32\pi m^2}\sqrt{m^2 - 4m_1^2} = \frac{d\Omega}{32\pi^2}\sqrt{1 - \frac{4m_1^2}{m^2}}$$

well defined only if $m > 2m_1$!

- $m_2 = 0$ then $\mu_{12} = m_{12} = m_1$

$$d\phi_2 = \frac{d\Omega}{32\pi^2 m^2}\left[m^2 - m_1^2\right] = \frac{d\Omega}{32\pi^2}\left[1 - \frac{m_1^2}{m^2}\right]$$

we see that (IN $D=4$!) the phase-space always goes to zero when $m^2 = m_{12}^2$ $\Rightarrow$ maximum mass that can be produced (what about $D=2$?)

What changes for CROSS-SECTION $2 \to 2$?

$k_1 + k_2 \to p_1 + p_2$ from now on use

$p_1 + p_2 \to p_3 + p_4$ (keep k^μ for LOOPS !)

$p_i^2 = m_i^2$ introduce MANDELSTAM VARIAB.

$$\left.\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_2 - p_3)^2 \end{aligned}\right\} \quad s + t + u = \sum_{i=1}^{4} m_i^2$$

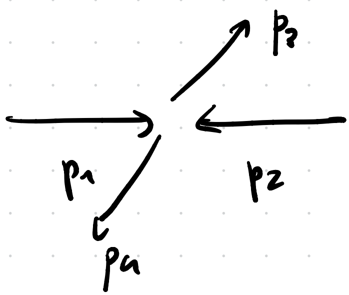
Not much changes for $d\phi_2$ except $P = p_1 + p_2$

$\Rightarrow$ I can still go to Center of Mass of $p_1 + p_2$

then $$d\phi_2 = \frac{d\Omega}{32\pi^2 s} \sqrt{(s - m_{34}^2)(s - \mu_{34}^2)}$$

since in CoM $p_1 + p_2 = (\sqrt{s}, \vec{0})$!

now notice that in CoM frame



$$p_1 = (E_1, \vec{p})$$
$$p_2 = (E_2, -\vec{p})$$

$$\left.\begin{aligned} p_3 &= (E_3, \vec{p}\,') \\ p_4 &= (E_4, -\vec{p}\,') \end{aligned}\right\} \text{by def!}$$

and standard calculation gives

$$|\vec{p}\,'| = \frac{1}{2\sqrt{s}} \sqrt{(s - m_{34}^2)(s - \mu_{34}^2)} \quad \text{such that}$$

$$d\phi_2 = \frac{d\Omega}{(2\pi)^4} \frac{|\vec{p}\,'|}{4\sqrt{s}} \qquad \left[\begin{aligned} & E_1 + E_2 = \sqrt{s} \\ & (E_1+E_2)^2 = s = E_1^2 + E_2^2 + 2E_1E_2 \\ & E_1E_2 = \tfrac{1}{2}(s - E_1^2 - E_2^2) \end{aligned}\right.$$

Finally, still in CoM $I = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$

$$= \sqrt{(E_1E_2 + |\vec{p}|^2)^2 - m_1^2 m_2^2} = \frac{1}{2}\sqrt{(s - m_{12}^2)(s - \mu_{12}^2)}$$

$$= \sqrt{s} \cdot |\vec{p}| \quad \text{AS FOR } |\vec{p}\,'| \,!$$

$\Rightarrow$ finally we can write

$$d\sigma = \frac{1}{64\pi^2 s} |\mathcal{M}_{fi}|^2 \frac{|\vec{p}\,'|}{|\vec{p}|} d\Omega$$

and if $m_1 = m_3$, $m_2 = m_4$ then $|\vec{p}\,'| = |\vec{p}|$ ELASTIC SCATTERING

$$d\sigma_{el} = \frac{1}{64\pi^2 s} |\mathcal{M}_{fi}|^2 d\Omega$$

Finally, it turns out that all these formulas remain true with SPIN or other degrees of freedom $|in\rangle = |\underset{}{p_1}\, \underset{\text{SPIN}}{s_1}\, \underset{\text{EXTRA}}{a_1}, \ldots, p_n\, s_n\, a_n\rangle$ etc

we ignore a_i for now (could be COLOR!)

But what is important, is that if we want to IGNORE S_i in experiments, we must

sum over final S_i and AVERAGE over initial ones!

$$\overline{|\mathcal{M}_{fi}|^2} = \underbrace{\frac{1}{(2S_1+1)}\,\frac{1}{(2S_2+1)}}_{\text{average factor over INITIAL SPINS}} \sum_{\substack{\text{in} \\ \text{spin}}} \; \sum_{\substack{\text{final} \\ \text{spin}}} |\mathcal{M}_{fi}|^2$$

Similarly, we can <u>average</u> & <u>sum</u> over other degrees of freedom, if we want to!