the S-matrix 2:

LSZ reduction

We are finally ready to deal with one of the most important topics in QFT: how do we go from GREEN FUNCTIONS to observables that allow us to describe SCATTERING EVENTS

The crucial step is the LSZ reduction formula which under assumptions on asymptotic behavior of fields, tells us how to relate Green Functions to the S-MATRIX

We limit ourselves still to a SINGLE SCALAR FIELD and consider

$$\langle p_1 \dots p_n ; t_f \mid k_1 \dots k_m ; t_i \rangle$$

where we always assume $t_f \to +\infty$, $t_i \to -\infty$ with ASYMPTOTIC conditions in Lecture 16!

now remember that for a free field ϕ_0 we wrote

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \left[e^{-ip\cdot x} a(p) + e^{ip\cdot x} a^\dagger(p) \right]$$

which we can invert to find

$$a(k) = i \int d^3x \, e^{ik\cdot x} \overleftrightarrow{\partial_0} \phi_0$$ in fact

$$= i \int d^3x \, e^{ikx} \left[\int \frac{d^3p}{(2\pi)^3 2E_p} \left(-iE_p e^{-ip\cdot x} a(p) + iE_p e^{ip\cdot x} a^\dagger(p) \right) - \int \frac{d^3p}{(2\pi)^3 2E_p} \left(iE_k e^{-ip\cdot x} a(p) + iE_k e^{ip\cdot x} a^\dagger(p) \right) \right]$$

$$= i \int \frac{d^3p}{(2\pi)^3 2E_p} \left[-iE_p a(p) \delta(p-k) + iE_p \cancel{\delta(p+k) a^\dagger(p)} - iE_k a(p) \delta(p-k) - iE_k \cancel{\delta(p+k) a^\dagger(p)} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\, a(p)\, \delta(p-k) = a(k) \; !$$

similarly $\quad a^\dagger(k) = -i \int d^3x \, e^{-ik\cdot x} \overleftrightarrow{\partial_0} \phi_0$

now we ASSUME that when $t \to \pm\infty$ the incoming and outgoing particles are infinitely far away $\Rightarrow$ interaction can be neglected [NOT ALWAYS TRUE, SEE QCD !]

Using Källen-Lehmann repr. for $\langle \Omega | T\{\phi\phi\} | \Omega \rangle$

we can argue that in WEAK SENSE (as exp. values!)

$$\left.\begin{aligned} \phi(x) &\xrightarrow[t\to-\infty]{} Z^{1/2}\, \phi_{in}(x) \\ \phi(x) &\xrightarrow[t\to+\infty]{} Z^{1/2}\, \phi_{out}(x) \end{aligned}\right\}$$

where ϕ_{in} & ϕ_{out} are free fields with SAME constant Z

Now the formulas for $a(k)$ & $a^\dagger(k)$

$$a(k) = i\int d^3x\, e^{ik\cdot x}\overleftrightarrow{\partial_0}\phi_0$$

$$a^\dagger(k) = -i\int d^3x\, e^{-ik\cdot x}\overleftrightarrow{\partial_0}\phi_0$$

Are clearly time independent $\Rightarrow$ but the INTEGRAND does depend on time

$\Rightarrow$ I can compute it at any t, I should get the same result $\Rightarrow$ in particular, I can compute it at $t = \pm\infty$ and write

$$a^\dagger_{in}(k) = -i\int d^3x\, e^{-ik\cdot x}\overleftrightarrow{\partial_0}\phi_{in}$$

true because $\phi_{in} \sim$ FREE

$$= -i Z^{-1/2}\lim_{t\to-\infty}\int d^3x\, e^{-ik\cdot x}\overleftrightarrow{\partial_0}\phi$$

interacting field!

where $a^\dagger_{in}(k)$ act on space of INITIAL STATES

at $t = -\infty = t_i$

In the same way we have creation operators on

FINAL STATES

$$a^\dagger_{out}(k) = -i Z^{-1/2} \lim_{t \to +\infty} \int d^3x \, e^{-ik\cdot x} \overleftrightarrow{\partial_0} \phi$$

interacting field

$\Rightarrow \int d^3x \, e^{-ik\cdot x} \overleftrightarrow{\partial_0} \phi_0$ was TIME INDEPENDENT

$\longrightarrow$ $\phi_0 \to \phi$ then integral DEPENDS ON TIME !

In general, the relation between $a^\dagger_{in}$ & $a^\dagger_{out}$

is non-trivial due to the DIFFERENT LIMIT !

now let's go back to S-matrix

$$\langle p_1 \dots p_n ; t_f | k_1, \dots, k_m ; t_i \rangle =$$

$$= \langle p_1 \dots p_n ; t_f | a^{\dagger}_{in}(k_1) | k_2, \dots, k_m ; t_i \rangle$$

$$= -i Z^{-1/2} \int_{t \to -\infty} d^3x \, e^{-ik_1 \cdot x} \langle p_1, \dots, p_n ; t_f | \overleftrightarrow{\partial_0} \phi | k_2, \dots, k_m ; t_i \rangle$$

i.e. we removed "particle" k_1 and introduced in its place $\overleftrightarrow{\partial_0} \phi$ (interacting field !)

now notice that expression above is not Lorentz covariant ; one can show that :

$$= +i Z^{-1/2} \int d^4x \, e^{i k_1 \cdot x} (\Box_x + m^2) \cdot$$

$$\cdot \langle p_1, \dots, p_n ; t_f | \phi(x) | k_2, \dots, k_m ; t_i \rangle$$

Let's see how this works:

First we want to go from $\int d^3x \rightarrow \int d^4x$

we use

$$\left(\lim_{t\to+\infty} - \lim_{t\to-\infty}\right)\int d^3x\, f(x,t) = \underbrace{\int_{-\infty}^{+\infty} dt \int d^3x}_{\int d^4x}\, \frac{\partial f}{\partial t}$$

Apply this for

$$f(x,t) = -i Z^{-1/2} e^{-ik\cdot x} \overleftrightarrow{\partial_0} \phi$$

then

$$= \left(\lim_{t\to+\infty} - \lim_{t\to-\infty}\right)\left[-i Z^{-1/2}\int d^3x\, e^{-ik\cdot x}\overleftrightarrow{\partial_0}\phi\right]$$

$$= a^\dagger_{in}(k) - a^\dagger_{out}(k) = i Z^{1/2}\int d^4x\, \partial_0\left[e^{-ik\cdot x}\overleftrightarrow{\partial_0}\phi\right]$$

now

$$\int d^4x \; \partial_0 \left[e^{-ik\cdot x} \overleftrightarrow{\partial_0} \phi \right] = \int d^4x \left[e^{-ik\cdot x} \partial_0^2 \phi - \phi \partial_0^2 e^{-ik\cdot x} \right]$$

$\partial_0 e^{-ik\cdot x} \partial_0 \phi$ cancels due to $-$ sign!

now use
$$\partial_0^2 e^{-ik\cdot x} = -k_0^2 \, e^{-ik\cdot x}$$
$$= -(\vec{k}^2 + m^2) \, e^{-i\vec{k}\cdot x}$$
$$= (\vec{\nabla}^2 - m^2) \, e^{-i\vec{k}\cdot\vec{x}}$$

then use
$$\int d^4x \; \phi \, \vec{\nabla}^2 e^{-i\vec{k}\cdot x} = \int d^4x \, (\vec{\nabla}^2 \phi) e^{-ik\cdot x}$$

int by parts!

I can integrate by parts in $d^3\vec{x}$ because initial and final states assumed to be LOCALIZED in SPACE (wave packets!) $\Rightarrow$ note they are NOT LOCALIZED IN TIME !

so finally we have

$$a^{+}_{in}(k) - a^{+}_{out}(k) = i Z^{1/2} \int d^4x \, \partial_0 \left[e^{-ik\cdot x} \overleftrightarrow{\partial_0} \phi \right]$$

$$= i Z^{-1/2} \int d^4x \, e^{-ik\cdot x} (\Box + m^2) \phi(x)$$

such that

$$\langle p_1 \cdots p_n ; t_f | a^{+}_{in}(k_1) - a^{+}_{out}(k_1) | k_2, \dots, k_m ; t_i \rangle$$

$$= i Z^{1/2} \int d^4x \, e^{-ik_1\cdot x_1} (\Box_1 + m^2) \langle p_1 \cdots p_n ; t_f | \phi(x_1) | k_2 \cdots k_m ; t_i \rangle$$

$\Uparrow$ changed "x" $\to$ "x_1"

Almost what we want, except for $a^{+}_{out}(k_1)$

ASSUME $p_1, \dots, p_n \neq k_i \Rightarrow$

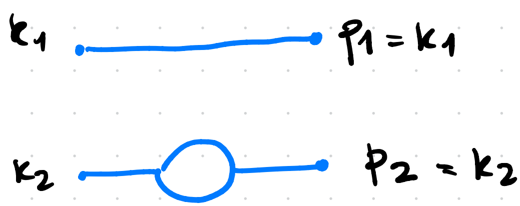
$$\langle p_1 \cdots p_n ; t_f | a^{+}_{out}(k_1) = 0$$

a^{+}_{out} only act on LEFT !

we can drop it !

the Fact that no $k_i = p_i$ means that we are computing only T in $S = 1 + iT$

$\Rightarrow$ EXCLUDING DISCONNECTED FEYNMAN DIAGRAMS



we are not considering them!

now what if we want to repeat the procedure for a FINAL STATE with momentum p_1?

We would start with

$$\langle p_1 \dots p_n ; t_f | \phi(x_1) | k_2, \dots, k_m ; t_i \rangle =$$

$$= \langle p_2, \dots, p_n ; t_f | a_{out}(p_1) \phi(x_1) | k_2, \dots, k_m ; t_i \rangle$$

this creates final state on <u>LEFT</u>

but $$a_{out}(p_1) = i Z^{-1/2} \lim_{y_0^1 \to +\infty} \int d^3y_1\, e^{ik\cdot y_1} \overleftrightarrow{\partial_0}\, \phi$$

$$a_{in}(p_1) = i Z^{-1/2} \lim_{y_0^1 \to -\infty} \int d^3y\, e^{ik\cdot y_1} \overleftrightarrow{\partial_0}\, \phi$$

$\Rightarrow$ built out of $\phi(y_1)$ at $y_0^1 = \pm\infty$ ∞

$$T\{a_{in}(p_1)\,\phi(x_1)\} = \phi(x_1)\, a_{in}(p_1)$$

$$T\{a_{out}(p_1)\,\phi(x_1)\} = a_{out}(p_1)\,\phi(x_1)$$

time ordering

with this we can write

$$\langle p_2, \dots, p_n; t_f |\, a_{out}(p_1)\, \phi(x_1)\, | k_2, \dots, k_m; t_i \rangle =$$

$$= \langle p_2, \dots, p_n; t_f |\, T\{a_{out}(p_1) - a_{in}(p_1)\}\, \phi(x_1)\, | k_2, \dots, k_m; t_i \rangle$$

$\uparrow$

use $a_{in}(p_1)\,|k_2, \dots, k_m; t_i\rangle = 0$

Again all $p_i \neq k_j$!

but we know that $a^+_{in}(k_1) - a^+_{out}(k_1)$ can be

substituted by $\sim (\Box_x + m^2)\, \phi(x)$

so we can do the same for $a_{out}(p_1) - a_{in}(p_1)$

modulo an HERMITIAN CONJUGATION

$$a_{out}(p_1) - a_{in}(p_1) \rightarrow i Z^{-1/2} \int d^4y\, e^{i p_1 y_1} (\Box_{y_1} + m^2)\, \phi(y_1)$$

but I Should remember that I am omming a Time ordering to properly insert $a_{\substack{in \\ out}}(p_1)$!

$$\langle p_1 \cdots p_n ; t_f | k_1, \dots, k_m ; t_i \rangle = (i Z^{-1/2})^2$$

$$\int d^4x_1\, e^{-i k_1 \cdot x_1} (\Box_{x_1} + m^2) \int d^4 y_1\, e^{+i p_1 \cdot y_1} (\Box_{y_1} + m^2)$$

$$\cdot \langle p_2 \cdots p_n ; t_f | T\{\phi(x_1) \phi(y_1)\} | k_2 \cdots k_m ; t_i \rangle$$

Finally we can iterate this to ALL PARTICLES

$$\langle p_1 \dots p_n ; t_f | k_1, \dots , k_m ; t_i \rangle = (i Z^{-1/2})^{n+m}$$

$$\cdot \int \prod_{i=1}^{m} d^4 x_i \prod_{j=1}^{n} d^4 y_j \, e^{i \left(\sum_{j=1}^{n} p_j \cdot y_j - \sum_{i=1}^{m} k_i \cdot x_i \right)}$$

$$\cdot (\Box_{x_1} + m^2) \dots (\Box_{y_n} + m^2) \underbrace{\langle \Omega | T \{ \phi(x_1) \dots \phi(y_n) \} | \Omega \rangle}_{\text{Green Function !}}$$

We have extracted $\Box_{x_i}$ & $\Box_{y_i}$ out of T

$\Rightarrow$ Assume T^* !

$\Rightarrow$ we will argue in a moment that here it makes no difference !

$= \langle p_1 \dots p_n | (iT) | k_1, \dots k_m \rangle$ because $p_i \neq k_j$!

now we can use the fact that

$$(\Box_{x_j} + m^2)\, G(x_1, \dots x_m, y_1, \dots, y_n) =$$

$$= -\int \prod_{i=1}^{m} \frac{d^4 \bar{k}_i}{(2\pi)^4} \prod_{j=1}^{n} \frac{d^4 \bar{p}_j}{(2\pi)^4} \, (\bar{k}_j^2 - m^2)\, e^{i\left(\sum_{j=1}^{n} \bar{p}_j \cdot y_j - \sum_{i=1}^{m} \bar{k}_i \cdot x_i\right)} \underbrace{\tilde{G}(\bar{k}_1, \dots, \bar{k}_m; \bar{p}_1, \dots, \bar{p}_n)}_{\text{Fourier Transform of Green Function}}$$

such that we can write (LSZ Formula)

$$\left(\prod_{i=1}^{m} \frac{i\sqrt{Z}}{k_i^2 - m^2}\right) \prod_{j=1}^{n} \left(\frac{i\sqrt{Z}}{p_j^2 - m^2}\right) \langle p_1 \dots p_n | iT | k_1 \dots k_m \rangle$$

$$= \underbrace{\prod_{i=1}^{m} \int d^4 x_i \, e^{-ik_i \cdot x_i} \prod_{j=1}^{n} \int d^4 y_j \, e^{iy_j \cdot p_j} \langle \Omega | T\{\phi(x_1) \dots \phi(y_n)\} | \Omega \rangle}_{\text{Fourier Transform of Green Function !}}$$

What does it mean dividing by $\frac{1}{k_i^2-m^2}$ $\frac{1}{p_j^2-m^2}$?

Bring them to other side:

Was in def of $\tilde{G}$!

$$\langle p_1 \ldots p_n | iT | k_1 \ldots k_m \rangle = (2\pi)\, \delta^4\left(\sum k_i - \sum p_j\right)$$

$$= \left(\frac{1}{i\sqrt{Z}}\right)^{n+m} \prod_{i=1}^{m} (k_i^2-m^2) \prod_{j=1}^{n} (p_j^2-m^2) \checkmark\; \tilde{G}(k_1 \ldots k_m, p_1 \ldots p_n)$$

Compute $\tilde{G}$ OFF-SHELL

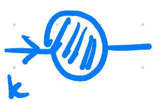
when going on-shell, $\tilde{G}$ must develop poles $\Rightarrow$ we pick their residues!

We IGNORE everything in $\tilde{G}$ which has no poles as $k_i^2 \to m^2$, $p_j^2 \to m^2$!

This operation is called AMPUTATION

in fact, remember that we proved UNDER OUR ASSUMPTIONS

2-point function is

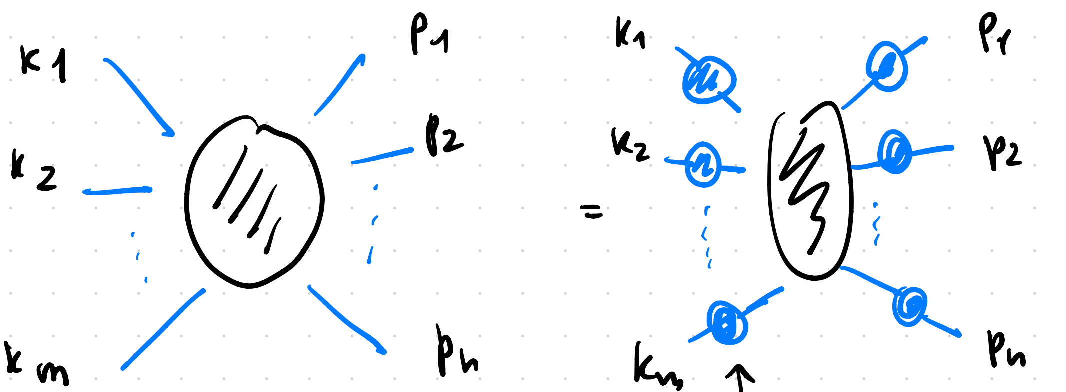


$$= \tilde{G}(k) = \frac{iZ}{k^2-m^2} + \text{non-singular terms}$$

this is what we used to

fix $\phi \sim i\sqrt{Z}\,\phi_{in/out}$!

Now imagine we are dealing with $m \to n$



Connected Green Function

exact two-point function

in FORMULAS

$$G^{conn}(k_1 \dots p_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} G^{(2)}(k_i^2)\, G^{(2)}(p_j^2)$$

$$\cdot\, G^{AMP}(k_1, \dots, p_n)$$

$\Rightarrow$ each external $\underline{\underline{G}}^{(2)}$ develops a pole at $k_i^2 = m^2$ or at $p_j^2 = m^2$!

so we use

$$\frac{1}{i\sqrt{Z}} (k_i^2 - m^2)\, \tilde{G}(k_i) \xrightarrow[\substack{\text{on-shell} \\ k_i^0 = E_{k_i}}]{} \sqrt{Z}$$

$$\downarrow$$

$$\frac{iZ}{k_i^2 - m^2} + \text{no pole}$$

so finally we can perform AMPUTATION and write

$$\langle p_1 \dots p_n | iT | k_1 \dots k_m \rangle = (2\pi)^4 \delta^{(4)}\left(\sum k_i - \sum p_j\right)$$

$$\cdot \left(\sqrt{Z}\right)^{n+m} \tilde{G}_1^{AMP}(k_1, \dots, k_m ; p_1, \dots, p_n) \Big|_{\substack{p_i^2 = m^2 \\ k_i^2 = m^2}}$$

so the T-matrix element of a scattering matrix is related to on-shell, amputated Green Functions in momentum space!

We typically write

$$\langle p_1 \dots p_n | iT | k_1 \dots k_m \rangle = (2\pi)^4 \delta^{(4)}\left(\sum k_i - \sum p_j\right)$$

$$\cdot \left(iT_{k_1 \dots k_m, p_1 \dots p_n} = i\mathcal{M}_{fi} \right)$$

so $(2\pi)^4 \delta^{(4)}$ cancels and $\boxed{iT \to \left(\sqrt{Z}\right)^{n+m} \tilde{G}}$

We can then give FEYNMAN RULES to compute directly S-matrix elements

1. Draw all connected graphs

2. Amputate external legs (neglect 2-point corrections)

3. Impose momentum conservation at each vertex

4. Include vertex factors $\Rightarrow \mathcal{L}_{INT} = -\frac{\lambda \phi^n}{n!}$

   gives $(-i\lambda)$ for each vertex!

5. For each internal line $\tilde{\Delta}_F(p) = \frac{i}{p^2 - m^2 + i\varepsilon}$

6. Integrate over all unconstrained loop momenta with $\int \frac{d^4k}{(2\pi)^4}$

7. Include symmetry factor from
   - exponential $\frac{1}{N!}$
   - $\frac{1}{n!}$ from $\frac{1}{n!}\phi^n$
   - EQUIVALENT CONTR.

NOTICE that this is true for 1 scalar field

otherwise

- a different Z_i $\forall$ field
- for ψ, A^μ one needs an extra factor for every extend line

$\Rightarrow$ the POLARIZATION $\begin{cases} u, \bar{v} & \text{spin } \frac{1}{2} \\ \varepsilon_\mu, \varepsilon^*_\mu & \text{spin } 1 \\ \text{etc} \Rightarrow \underline{\text{see later}} & \end{cases}$