

Loop Integrals :

Divergences & dimensional
regularization

In previous lecture we ignored a pretty serious issue $\Rightarrow$ the INTEGRALS we encountered are often **ILL DEFINED** :

$$0 \rightarrow \text{loop} \sim \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \quad \text{TADPOLE}$$

where $k \gg m \sim \int \frac{d^4 k}{k^2} \sim \int_{\#}^{\text{cut off}} \frac{dk k^3}{k^2} \sim \Lambda^2$

UV because k LARGE } POWER-LIKE UV divergence
 high frequencies $\sim$ small distances } goes as Λ^2

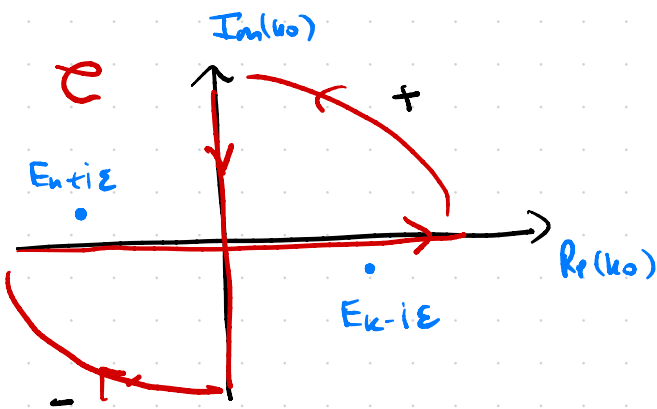
[there exist also INFRARED DIVERGENCES $\rightarrow$ later !]

more precise

$$\int \frac{d^3 k}{(2\pi)^4} \int_{-\infty}^{+\infty} dk_0 \frac{1}{k_0^2 - E_k^2 + i\epsilon}$$

$$E_k = \sqrt{\vec{k}^2 + m^2}$$

$\uparrow$
 regularizes poles on integration contour



Consider red
Contour C

$$\oint_C \frac{1}{k_0^2 - E_k^2 + i\epsilon} dk_0 = 0$$

$$= \int_{-\infty}^{+\infty} dk_0 + \cancel{\int_{ARC_+}} + \cancel{\int_{ARC_-}} + \int_{+i\infty}^{-i\infty} \Rightarrow$$

sending arcs to ∞

$$\int_{-\infty}^{+\infty} dk_0 = \int_{-i\infty}^{+i\infty} dk_0 \quad \text{integral along imaginary axis}$$

$$k_4 = -i k_0 \Rightarrow k_4 \in [-\infty, +\infty] \quad dk_0 = +i dk_4$$

$$= +i \int_{-\infty}^{+\infty} dk_4 \quad \text{called WICK ROTATION}$$

then
$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \rightarrow i \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{-k_E^2 - m^2}$$

$$= -i \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2}$$

↑ euclidean momenta now

$$k_E^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2$$

with this, we can go to polar coordinates in $D(4)$

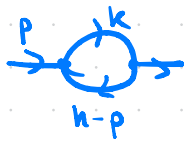
$$= -i \frac{\Omega(4)}{(2\pi)^4} \int_0^\Lambda dk_E \frac{k_E^3}{k_E^2 + m^2}$$

$$= -i \frac{2\pi^2}{(2\pi)^4} \left[\frac{1}{2} \left(\Lambda^2 + m^2 \ln \left(\frac{m^2}{m^2 + \Lambda^2} \right) \right) \right]$$

↑
QUADRATIC
DIVERGENCE

↑
~ $\ln \Lambda$ there
is also a logarithmic
divergence!

Similarly



BUBBLE

contained integral (Bubble Integral)

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(k-p)^2 - m^2 + i\epsilon}$$

when $k \gg m, p$ we can approximate

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \cdot \frac{1}{k^2} \sim \int_{\#}^{\Lambda} dk \frac{k^3}{k^4} \sim \ln \Lambda$$

Bubble contains
only a LOGARITHMIC
DIVERGENCE!

Soon we will see what to do with these

divergences $\Rightarrow$ FIRST WE NEED A WAY TO EXTRACT THEM

SAFELY

Cutoff Λ physically appealing $\Rightarrow$ theory might be only defined up to some energies $k \sim \Lambda$

$\Rightarrow$ problem: breaks Lorentz invariance

Calculations become a mess at high orders

does not regulate INFRARED

The most powerful regularization technique we possess
 $\hookrightarrow$ called DIMENSIONAL REGULARIZATION (Dim Reg)

Idea: use # dimensions of integration as regulator
make it a continuous parameter

$\Rightarrow$ Analytic continuation $d^4k \rightarrow d^Dk; D \in \mathbb{C}$

Might seem crazy but it's not.

Consider TADPOLE after WICK ROT, but now in $\mathbb{R}^D$

$$T(D, m^2) = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2 + m^2} \quad \left[\text{remove } -i \text{ for simplicity} \right]$$

we can go to spherical $\mathbb{R}^D$ coordinates

$$T(D, m^2) = \frac{\Omega(D)}{(2\pi)} \int_0^\infty dk k^{D-1} \frac{1}{k^2 + m^2}$$

$$\Omega(n) = \frac{2\pi^{D/2}}{(D-1)!} \quad ; \quad \text{note for } D < 2, \text{ integral converges!}$$

We want to compute it as function of "D"

easy $\Rightarrow$ Analytically continue this expression to $\mathbb{C}$

$$\frac{2\pi^{D/2}}{(\frac{D}{2}-1)!} \longrightarrow \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})}$$

$$\Gamma(x) = \int_0^\infty dt e^{-t} t^{x-1}$$

Gamma function
Analytic continuation
of FACTORIAL
from $N \rightarrow \mathbb{C}$

$$\Gamma(1+x) = x \Gamma(x)$$

"FACTORIAL" PROPERTY

So I DEFINE tadpole for $D \in \mathbb{C}$

$$T(D, m^2) \equiv \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} \frac{1}{(2\pi)^D} \underbrace{\int_0^\infty dk k^{D-1} \frac{1}{k^2 + m^2}}$$

scale $k = m\sqrt{t}$ then $dk = \frac{m}{2} \frac{dt}{\sqrt{t}}$

$$= \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} \frac{1}{(2\pi)^D} \frac{1}{2} [m^2]^{\frac{D-2}{2}} \int_0^\infty dt \frac{t^{\frac{D-1}{2}} \cdot t^{-1/2}}{t+1}$$

$$= \frac{\pi^{D/2}}{\Gamma(\frac{D}{2})} \frac{1}{(2\pi)^D} (m^2)^{\frac{D-2}{2}} \underbrace{\int_0^\infty dt \frac{t^{\frac{D}{2}-1}}{(1+t)}}$$



Beta Function

$$B(x, y) = \int_0^\infty dt \frac{t^{x-1}}{(1+t)^{x+y}} = \int_0^1 du u^{x-1} (1-u)^{y-1}$$

$$\frac{D}{2} + y = 1$$

One can prove that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

combination of Γ
functions

then we get

$$T(D, m^2) = \frac{\pi^{D/2}}{\cancel{\Gamma(\frac{D}{2})}} \frac{1}{(2\pi)^D} (m^2)^{\frac{D-2}{2}} \frac{\cancel{\Gamma(\frac{D}{2})} \Gamma(1-\frac{D}{2})}{\cancel{\Gamma(1)}}$$

$$T(D, m^2) = \frac{\pi^{D/2}}{(2\pi)^D} \Gamma(2-\frac{D}{2}) (m^2)^{\frac{D-2}{2}}$$

meromorphic function !

$\Gamma(x)$ has POLES $\forall x = \{0, -1, -2, \dots\}$

$\Gamma(2-\frac{D}{2})$ has pole at $D=2, D=4, D=6, \dots$

$$\Gamma(2-\frac{D}{2}) = \frac{2}{2-D} \Gamma(1+\frac{2-D}{2}) = \frac{2}{2-D} \Gamma(\frac{4-D}{2}) =$$

$\uparrow$ EXPLICIT $D=2$

BUT $D=3$

$$\frac{\pi^{3/2}}{(2\pi)^3} \Gamma(-\frac{1}{2}) (m^2)^{\frac{1}{2}}$$

$$= -\frac{1}{16\pi^2} \sqrt{m^2} \quad \text{FINITE} < 0!$$

$$\Gamma\left(\frac{2-D}{2}\right) = \frac{2}{2-D} \frac{2}{4-D} \Gamma\left(\frac{6-D}{2}\right)$$

$\uparrow$ pole at $D=2$ $\uparrow$ pole at $D=4$ $\uparrow$ regular @ $D=4$

$$T(D, m^2) = \frac{\pi^{D/2}}{(2\pi)^D} \frac{2}{(D-2)} \frac{2}{D-4} \Gamma\left(\frac{6-D}{2}\right) (m^2)^{\frac{D-2}{2}}$$

$$D = 4 - 2\varepsilon \quad \varepsilon > 0 \text{ for definiteness}$$

↓

$$= \frac{[4\pi]^\varepsilon}{16\pi^2} \Gamma(1+\varepsilon) \frac{1}{1-\varepsilon} \frac{1}{\varepsilon} (m^2) (m^2)^{-\varepsilon}$$

$$= \underbrace{[(4\pi)^\varepsilon \Gamma(1+\varepsilon)]}_{C(\varepsilon) \rightarrow 1 \text{ when } \varepsilon \rightarrow 0} (m^2)^{-\varepsilon} \underbrace{\left(-\frac{m^2}{16\pi^2}\right) \frac{1}{1-\varepsilon} \frac{1}{\varepsilon}}_{\text{expand in } \varepsilon}$$

$$T(D=4-2\epsilon, m^2) = C(\epsilon) \left[\frac{m^2}{16\pi^2} \right] \left(-\frac{1}{\epsilon} - 1 - O(\epsilon) \right)$$

$\uparrow$
UV pole

$\nwarrow$
finite part

IMPORTANT $\Rightarrow$ never expand $C(\epsilon)$

it produces a big mess

$$T(D=4-2\epsilon, m^2) = \left[\frac{m^2}{16\pi^2} \right] \left(-\frac{1}{\epsilon} - 1 + \gamma_E + \ln(m^2) - \ln(4\pi) + O(\epsilon) \right)$$

γ_E = Euler Mascheroni constant = 0,577216...

$\ln(4\pi)$; $\ln(m^2)$ = logarithm of "mass variable"

$\Downarrow$

very disturbing

having continued $d^4 k \rightarrow d^D k$ we can keep
 "right dimensions" as in $D=4$ by multiplying

$$\int d^4 k \rightarrow (\mu^2)^{\frac{4-D}{2}} \int d^D k$$

$[\mu]$ mass = 1
 "new arbitrary
 scale"

then $(\mu^2)^{\frac{4-D}{2}} = (\mu^2)^\epsilon$ and

$$\begin{aligned} T(W=4-2\epsilon, m^2) &= (4\pi)^\epsilon \Gamma(1+\epsilon) \left(\frac{m^2}{\mu^2}\right)^{-\epsilon} \left(-\frac{m^2}{16\pi^2}\right)^{\frac{1}{1-\epsilon}} \frac{1}{\epsilon} \\ &= \left[\frac{m^2}{16\pi^2}\right] \left(-\frac{1}{\epsilon} - 1 + \gamma_E + \ln\left[\frac{m^2}{\mu^2}\right] - \ln(4\pi)\right. \\ &\quad \left.+ O(\epsilon)\right) \end{aligned}$$

$\uparrow$
 DIMENSIONLESS

We will ALWAYS assume

$$\int d^4 k \rightarrow (\mu^2)^\epsilon \int d^D k !$$

still important point is : eventually we will have to REMOVE POLES " $\frac{1}{\epsilon}$ " $\Rightarrow$ compute FINITE quantities ; in those cases , if we always collect out $C(\epsilon)$, we can put $\epsilon=0$ and just substitute $C(0) = 1$

How do we generalize this to ARBITRARY FRYNMAN INTEGRALS ?

AXIOMATIC DIMENSIONAL REGULARIZATION

Define an integration $\int d^n k$ which fulfills

$$\bullet \int d^n k [a f(k) + b g(k)] = a \int d^n k f(k) + b \int d^n k g(k)$$

for $a, b \in \mathbb{C}$ Linearity

$$\cdot \int d^n k f(k) = \lambda^{-n} \int d^n \tilde{k} f(\tilde{k}) \quad \tilde{k} = \lambda k$$

rescaling $\lambda \in \mathbb{C}$

$$\cdot \int d^n k f(k+p) = \int d^n k f(k)$$

translation invariance

$$\text{In general } \int d^n k f(k) = \int d^n k |\det A|^{-1} f(A^{-1}k - A^{-1}b)$$

general linear transformation

Does not have to be separately postulated

Follows from analytic continuation of Lebesgue measure

$$\cdot \int d^n k \frac{\partial}{\partial k^\mu} [f(k)] = 0$$

integration by parts identities $\Rightarrow$ no boundary terms!

these "axioms" have various interesting consequences

MOST IMPORTANT : ALL SCALELESS INTEGRALS ARE ZERO

Imagine $f(k)$ depends ONLY on k , no extra "scale"

then :

$$\begin{aligned}\int d^n k f(k) &\xrightarrow{k \rightarrow \lambda k} \int d^n k f(\lambda k) \\ &= \lambda^{-n} \int d^n k f(k)\end{aligned}$$

now, though, $f(k)$ must be homogeneous

$$\begin{aligned}f(\lambda k) &= \lambda^a f(k) \quad a \neq -n \text{ in general} \\ a &\in \mathbb{Z} \text{ while } n \in \mathbb{R}!\end{aligned}$$

$$\text{then } \int d^n k f(\lambda k) = \lambda^a \int d^n k f(k)$$

$$\lambda^a \int d^n k f(k) = \lambda^{-n} \int d^n k f(k) \quad \forall \lambda, n \in \mathbb{R}$$

$\rightarrow$ only possible if $\int d^D k f(k) = 0$

EXAMPLE: the massless TADPOLE

$$\int \frac{d^D k}{k^2 + m^2} \rightarrow \int \frac{d^D k}{k^2} = 0 \quad \nabla$$

notice that for $D \in \mathbb{N}$ this integral would
NEVER CONVERGE

$$\sim \int_0^\infty dk \frac{k^{D-1}}{k^2} \rightarrow \int_0^\infty du k^{D-3}$$

$D > 1$ Diverges at $k = \infty$

$D \leq 1$ Diverges at $k = 0$ (INFRARED)

so... it's just zero! Consistency of analytic

continuation saves us!

what about more complicated integrals?

TADPOLE "easy" because all angular integrations are

$\Rightarrow$ in general, various tricks to get rid of angular integration and map $\int d^D k \rightarrow$ STANDARD integral over finite-dimensional simplex

MOST FAMOUS $\Rightarrow$ FEYNMAN - SCHWINGER PARAMETRIZATION

Trick:
$$\frac{1}{A \cdot B} = \int_{\Delta} dx dy \frac{1}{[Ax + By]^2}$$

$\Delta =$ projective 1-simplex

$$= \{ (x, y) \in \mathbb{R}_{\geq 0}^2; (x, y) \sim (tx, ty); t > 0 \}$$

$=$ projective line

$$\frac{1}{(Ax+By)^2} \quad \text{homogeneous degree } -2 \quad x, y \rightarrow tx, ty$$

$$dx dy \quad \text{homogeneous of degree } +2 \quad x, y \rightarrow tx, ty$$

$$\frac{\underbrace{d\mu}_{dx dy}}{(Ax+By)^2} = \text{scale invariant} \quad (x, y) \rightarrow (tx, ty)$$

An integral of this type is called a

PROJECTIVE INTEGRAL

$\Rightarrow$ we can parametrize ^[simplex] projective line as we prefer

$\Rightarrow$ DEPROJECTIVIZE INTEGRAL

• For example choosing $1 = \frac{x}{y} \quad (x, y) \rightarrow \underbrace{\left(\frac{x}{y}, 1\right)}_t$

$$\int_{\Delta} d\mu \frac{1}{(Ax+By)^2} = \int_0^{\infty} dx \frac{1}{(Ax+B)^2} = \frac{1}{A \cdot B} \quad \begin{array}{l} y=1 \\ d\mu = dx \end{array}$$

• but I can also choose $l = \frac{1}{x}$ then $x=1$
 $dy = dx$

$$\int_{\Delta} d\mu \frac{1}{(Ax+By)^2} = \int_0^1 dy \frac{1}{(A+By)^2} = \frac{1}{A \cdot B}$$

• or I can also choose $l = \frac{1}{x+y} \Rightarrow \left[\underbrace{\frac{x}{x+y}}_{x'}, \underbrace{\frac{y}{x+y}}_{y'} \right]$

$$x'+y' = \frac{x+y}{x+y} = 1$$

$$\int_0^{\infty} dx' dy' \frac{\delta(1-x'-y')}{(Ax'+By')^2} = \int_0^1 dx \frac{1}{[Ax+B(1-x)]^2}$$



Form in which it was
 originally proposed by
FEYNMAN

this can be generalized to

$$\int_{\Delta} \frac{d^n \mu}{[A_1 x_1 + \dots + A_n x_n]^n} = \frac{1}{A_1 A_2 \dots A_n}$$

$$\Delta = \{ (x_1, \dots, x_n) \in \mathbb{R}_{\geq 0}^n; (1-x_1, \dots, 1-x_n) \sim (x_1, \dots, x_n) \\ 1 \in \mathbb{R}^+ \}$$

typically written more explicitly as

$$\int_0^{\infty} dx_1 \dots dx_n \frac{\delta(1 - \sum_{j \in S} x_j)}{[A_1 x_1 + \dots + A_n x_n]^n} = \frac{1}{A_1 \dots A_n}$$

S = any subset of the x_j !

$$\frac{1}{A_1 A_2 A_3} = \int_0^{\infty} dx_1 \int_0^1 dx_2 \frac{1}{[A_1 x_1 + A_2 x_2 + A_3 (1-x_2)]^3}$$

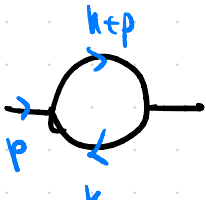
but also

$$\frac{1}{A_1 A_2 A_3} = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{[A_1 x_1 + A_2 x_2 + A_3 (1-x_1-x_2)]^3}$$

etc.. etc..

x_i called **FEYNMAN PARAMETERS**

Why useful? **EXAMPLE**



$$\propto (\mu^2)^\epsilon \int \frac{d^D k}{(2\pi)^D} \frac{1}{\underbrace{(k^2 - m^2)}_A \underbrace{((k+p)^2 - m^2)}_B} \quad (m^2 \rightarrow m^2 - i\epsilon)$$

$$= (\mu^2)^\epsilon \int_{\Delta} dx dy \int \frac{d^D k}{(2\pi)^D} \frac{1}{[(k^2 - m^2)x + ((k+p)^2 - m^2)y]^2}$$

$$= (\mu^2)^\epsilon \int_{\Delta} dx dy \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2(x+y) + 2k \cdot p y + p^2 y - m^2(x+y)]^2}$$

we can get rid of **angles** with Feynman Parameters

$$k^2(x+y) + 2k \cdot p y = \left[k + p \frac{y}{x+y} \right]^2 (x+y) - \frac{p^2 y^2}{(x+y)^2}$$

choose $x+y = 1$ parametrisation + $k \rightarrow k - p \frac{y}{x+y}$

$$(\mu^2)^\epsilon \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[(k + p(1-x))^2 - \underbrace{p^2 y^2 + p^2 y}_{= p^2 y(1-y) = p^2 x(1-x)} - m^2 \right]^2} \Big|_{y=1-x}$$

$$(\mu^2)^\epsilon \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[k^2 + p^2 x(1-x) - m^2 + i\delta \right]^2}$$

$\uparrow$ Feynman prescription

$$\text{WICK ROT} = i(\mu^2)^\epsilon \int_0^1 dx \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{\left[k_E^2 + p_E^2 x(1-x) + m^2 \right]^2} \quad \underline{\underline{p_E^2 = -p^2}}$$

$$= (\mu^2)^\epsilon \int_0^1 dx \left(p_E^2 (1-x)x + m^2 \right)^{\frac{D-4}{2}} \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[k^2 + 1 \right]^2}$$

renamed $k_E = k$ $\rightarrow$

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2+1)^2} = \frac{\Omega(D)}{(2\pi)^D} \int_0^\infty \frac{dk k^{D-1}}{(k^2+1)^2}$$

$$= \frac{\cancel{2\pi}^{D/2}}{\Gamma(\frac{D}{2})} \frac{1}{(2\pi)^D} \frac{1}{\cancel{2}} \int_0^\infty \frac{dt t^{\frac{D}{2}-1}}{(t+1)^2}$$

$$x-1 = \frac{D}{2} - 1$$

$$x+y = 2$$

$$x = \frac{D}{2} \quad y = 2 - \frac{D}{2}$$

$$= \frac{\pi^{D/2}}{\Gamma(\frac{D}{2})} \frac{1}{(2\pi)^D} \frac{\Gamma(\frac{D}{2}) \Gamma(\frac{4-D}{2})}{\Gamma(\cancel{2})} = 1$$

$$= \frac{\pi^{D/2}}{(2\pi)^D} \Gamma\left(\frac{4-D}{2}\right) = \frac{\pi^{D/2}}{(2\pi)^D} \frac{2}{4-D} \Gamma\left(\frac{6-D}{2}\right)$$

$\uparrow$ pole $D=4$

$$\bigcirc = \frac{(4\pi)^\epsilon \Gamma(1+\epsilon)}{16\pi^2} \frac{(\mu^2)^\epsilon}{\epsilon} \int_0^1 dx (p_E^2(1-x)x + m^2)^{-\epsilon}$$

$$= \frac{(4\pi)^\epsilon \Gamma(1+\epsilon)}{16\pi^2} \frac{1}{\epsilon} \left[\frac{\mu^2}{m^2} \right]^\epsilon \int_0^1 dx \left[1 + \frac{p_E^2}{m^2} x(1-x) \right]^{-\epsilon}$$

$$0 = C(\epsilon) \frac{1}{16\pi^2 \epsilon} \left[\int_0^1 dx - \epsilon \int_0^1 dx \ln \left(1 + \underbrace{\frac{p^2}{m^2} x(1-x)}_{\substack{V1 \\ 0 \text{ if } p^2 > 0 \\ \text{but } p^2 = -p^2 < 0}} \right) + O(\epsilon) \right]$$

$$C(\epsilon) = (4\pi)^{\epsilon} \Gamma(1+\epsilon) \left(\frac{\mu^2}{m^2} \right)^{\epsilon}$$

$$\text{use } \ln \left(1 - \frac{p^2}{m^2} x(1-x) + i\delta \right) \left\{ \begin{array}{l} x = \frac{1}{2} \pm \frac{\sqrt{p^2(p^2 - 4m^2)}}{p^2} = x_{\pm} \\ \text{solutions if } p^2 > 4m^2. \end{array} \right.$$

$$= \ln \left(\left| 1 - \frac{p^2}{m^2} x(1-x) \right| \right) + i\pi \Theta(p^2 - 4m^2) \Theta(x - x_-) \Theta(x_+ - x)$$

$$= \frac{C(\epsilon)}{16\pi^2} \left[\frac{1}{\epsilon} - \int_0^1 dx \ln \left(\left| 1 - \frac{p^2}{m^2} x(1-x) \right| \right) - i\pi \Theta(p^2 - 4m^2) \int_{x_-}^{x_+} dx + O(\epsilon) \right]$$

↑
imaginary part if $p^2 > 4m^2$
threshold for internal particles!

$$\int_{x_-}^{x_+} dx = \sqrt{\frac{p^2 - 4m^2}{p^2}} \quad \text{is fully}$$

$$\text{---}\bigcirc\text{---} = \frac{C(\varepsilon)}{16\pi^2} \left[\frac{1}{\varepsilon} - \int_0^1 dx \ln \left(\left| 1 - \frac{p^2}{m^2} x(1-x) \right| \right) \right. \\ \left. - i\pi \sqrt{\frac{p^2 - 4m^2}{p^2}} \theta(p^2 - 4m^2) \right]$$

imaginary part above threshold