Path Integral Quantization IV:

Feynman Rules in Momentum space

& Connected Green Functions

We concluded previous lecture discussing Feynman rules in coordinate space. We have seen that for every z_i internal, there is an integral $d^4 z_i$

$\Rightarrow$ building blocks are vertices & propagators $\Delta_F(x-y)$

PROPAGATORS are especially simple in momentum space

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \underbrace{\frac{i}{p^2-m^2+i\varepsilon}}_{\tilde{\Delta}_F(p)\ !}$$

$\Rightarrow$ it will be much simpler to compute generic Green functions in momentum space !

$$\int d^4x_1 \dots d^4x_n \, e^{ip_1 x_1 + \dots + ip_n \cdot x_n} \quad G_n(x_1 \dots x_n)$$

$$= (2\pi)^4 \, \delta^{(4)}(p_1 + \dots + p_n) \, \tilde{G}_n(p_1, \dots, p_n)$$

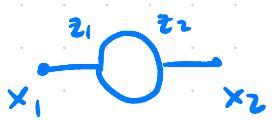
these $\tilde{G}_n$ will also be the ones that we will use to investigate directly SCATTERING PROCESSES!

We will derive FEYNMAN RULES in momentum space to compute directly $\tilde{G}_n(p_1, \ldots, p_n)$

NOTE that here p_i^0 not fixed to E_{p_i}! it's still an integration variable

the overall $\delta^{(4)}$ function makes momentum conservation manifest $\Rightarrow$ let's see how it pops out in an example:

two-point function in $g\frac{\phi^3}{3!}$ theory:



$$= -\frac{g^2}{2}\int d^4z_1 d^4z_2 \, \Delta_F(x_1-z_1)\Delta_F(x_2-z_2)\Delta_F^2(z_1-z_2)$$

take now Fourier transform

$$\rightarrow \int d^4x_1\, d^4x_2\; e^{ip_1\cdot x_1}\; e^{ip_2\cdot x_2} \left[-\frac{g^2}{2}\right] \int d^4z_1\, d^4z_2$$

$$\left[\prod_{i=1}^{4} \int \frac{d^4k_i}{(2\pi)^4} \frac{i}{k_i^2-m^2+i\varepsilon}\right] e^{-ik_1(x_1-z_1)}\; e^{-ik_2(x_2-z_2)}\; e^{-ik_3(z_1-z_2)}\; e^{-ik_4(z_1-z_2)} =$$

$$= \prod_{i=1}^{4} \int \frac{d^4k_i}{(2\pi)^4} \frac{i}{k_i^2-m^2+i\varepsilon} \left[-\frac{g^2}{2}\right] \underbrace{\int d^4x_1\; e^{ix_1\cdot(p_1-k_1)}}_{(2\pi)^4\,\delta^{(4)}(p_1-k_1)} \underbrace{\int d^4x_2\; e^{ix_2(p_2-k_2)}}_{(2\pi)^4\,\delta^{(4)}(p_2-k_2)}$$

$$\cdot \int d^4z_1\; e^{iz_1(k_1-k_3-k_4)} \int d^4z_2\; e^{iz_2(k_2+k_3+k_4)}$$

$$= \left[-\frac{g^2}{2}\right] \frac{i}{p_1^2-m^2+i\varepsilon} \frac{i}{p_2^2-m^2+i\varepsilon} \int \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4} \frac{i}{k_3^2-m^2+i\varepsilon} \frac{i}{k_4^2-m^2+i\varepsilon}$$

$$\cdot (2\pi)^4\,\delta^{(4)}(p_1-k_3-k_4)\,(2\pi)^4\,\delta^{(4)}(p_2+k_3+k_4)$$

$p_1 = k_3 + k_4$ $\Rightarrow \delta^{(4)}(p_1+p_2)$

$\longrightarrow k_4 = p_1 - k_3$ independent of k_i !

$$= \left[-\frac{g^2}{2}\right] \boxed{(2\pi)^4 \, \delta^{(4)}(p_1+p_2)} \, \frac{i}{p_1^2-m^2+i\varepsilon} \, \frac{i}{p_2^2-m^2+i\varepsilon} \,\cdot$$

$$\cdot \int \frac{d^4k_3}{(2\pi)^4} \, \frac{i}{k_3^2-m^2+i\varepsilon} \, \frac{i}{(k_3-p_1)^2-m^2+i\varepsilon}$$

overall $\delta^{(4)}(\sum_i p_i)$ as given in definition!

ALWAYS THERE: it's a consequence of the fact that $G_n(x_1,\dots,x_n)$ is TRANSLATION INVARIANT

$$G_n(x_1-y, x_2-y, \dots, x_n-y) = G_n(x_1,\dots,x_n)$$

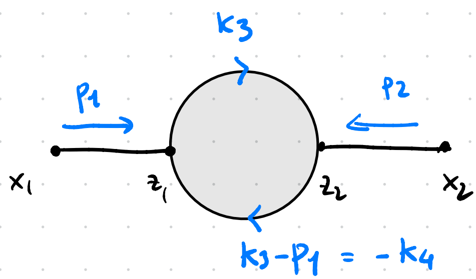
↪ true for every connected component independently!

$$\langle\Omega| T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle \Rightarrow \phi(x+y) = U(0,y)\,\phi(x)\,U^{-1}(0,y)$$

$$\langle\Omega| T\{U(0,y)\,\phi(x_1) \dots\dots \phi(x_n)\,U^{-1}(0,y)\}|\Omega\rangle$$

$+ \; U(0,y)|\Omega\rangle = |\Omega\rangle$ vacuum invariant!

Indeed, each $\delta^{(4)}$ function that we integrated out can be interpreted as momentum conservation at one of the vertices



$\delta^{(4)}$ fix : $k_1 = p_1$, $k_2 = p_2$

+ Conservation at each vertex !

↑ momentum conservation built in by choice of momenta

$$\tilde{G}_2(p_1, p_2) = \tilde{G}_2(p) = \left[-\frac{g^2}{2}\right] \frac{i}{p^2 - m^2 + i\varepsilon} \frac{i}{p^2 - m^2 + i\varepsilon}$$

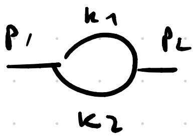
$$\underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{(k-p)^2 - m^2 + i\varepsilon}}_{\text{LOOP INTEGRAL}}$$

LOOP INTEGRAL, DOES IT CONVERGE?

I renamed $p_1 = p_2 = p$ & $k_3 = k$!

Note that we are left with ONE MOMENTUM INTEGRAL

$\Rightarrow$ started with 4 momenta = # lines

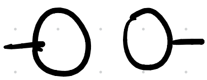


1 $\delta^{(4)}$ per vertex (int & ext)

$\Rightarrow 4 - 4 + 1$

↑ # connected components

Take



$= \underset{\text{lines}}{4} - \underset{\text{vertices}}{4} + \underset{\text{connected components}}{2} =$

= 2 integrations left!

one for each "closed loop"

$$L = P - V + n$$

↑ lines ↑ vertices ↑ connected components

L called LOOPS (k is LOOP MOMENTUM)

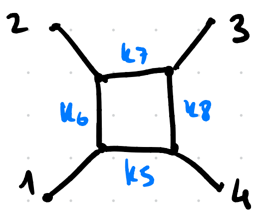
By generalizing this example, we can give momentum space FEYNMAN RULES

1. Draw all diagrams as in coordinate case

2. For every line (external or internal) associate a $\tilde{\Delta}_F(p)$ with its momentum

3. For every vertex assign a δ^4 to conserve momentum

4. Multiply by symmetry factors



$$\cancel{\prod_{i=1}^{4} \delta^{(4)}(p_i - k_i)} \quad \delta^{(4)}(k_5 - k_6 + p_1)\, \delta^{(4)}(k_6 - k_7 + p_2)$$

$$\delta^{(4)}(k_7 - k_8 + p_3)\, \delta^{(4)}(k_8 - k_5 + p_4)$$

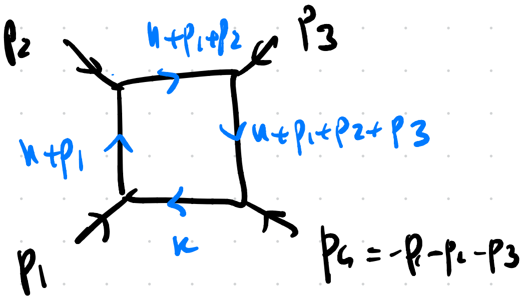
sum of all δ

$$\cancel{k_5} - \cancel{k_6} + \cancel{k_6} - \cancel{k_7} + \cancel{k_7} - \cancel{k_8} + \cancel{k_8} - \cancel{k_5} + \underline{p_1 + p_2 + p_3 + p_4 = 0}$$

$$p_4 = -p_1 - p_2 - p_3 - p_4$$

3 of k_i → in terms of 1 left (loop momentum)

in practice, already impose conservation of ext momenta out of every vertex while build diagram



$= \mathrm{Box}(p_1, p_2, p_3)$

every propagator then gives $\tilde{\Delta}_F(k) = \dfrac{i}{k^2-m^2+i\varepsilon}$

and 1 single integral left $\Rightarrow$ 1 LOOP

$$\mathrm{Box} = N_s \left[-\frac{ig}{3!}\right]^4 \frac{i}{p_1^2-m^2}\,\frac{i}{p_2^2-m^2}\,\frac{i}{p_3^2-m^2}\,\frac{i}{p_{123}^2-m^2}\,\cdot$$

$$\cdot \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2}\,\frac{i}{(k+p_1)^2-m^2}\,\frac{i}{(k+p_1+p_2)^2-m^2}\,\frac{i}{(k+p_1+p_2+p_3)^2-m^2}$$

N_s = symmetry factor $= (3!)^4$! (prove it!)

(or 1 if you include $(1/3!)^4$)

$$\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\;\;\phi(y_1)\,\phi(y_1)\,\phi(y_1) \qquad 3$$

2×2

$$\phi(y_2)\,\phi(y_2)\,\phi(y_2) \qquad 3$$

1×2

$$\phi(y_3)\,\phi(y_3)\,\phi(y_3) \qquad 3$$

1

$$\phi(y_4)\,\phi(y_4)\,\phi(y_4) \qquad 3$$

1×2

$$= 3\cdot 3\cdot 3\cdot 3\cdot 2\cdot 2\cdot 2\cdot 2 = (3!)^4$$

this cancels the $\left(\frac{1}{3!}\right)^4$ from vertices !

then: relabel $y_1 \dots y_4$ $= 4\cdot 3\cdot 2\cdot 1 = 4!$

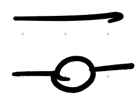
this cancels the factor $\left(\frac{1}{4!}\right)$ from expansion of exponential that defines $Z[J]$

$\Rightarrow$ Usually, one defines symmetry factor

as $\frac{1}{S} = 1$ (typical definition)

includes everything except g^3

Let us now go back to diagrams generated by $Z[J]$: for ϕ^3 theory, 4 - point we have

$\Rightarrow$ we have seen that disconnected vacuum diagrams cancel out, but there remain many other DISCONNECTED GRAPHS $\rightsquigarrow$ 

QUESTION : can we generate somehow only connected ones ?

Define $G_n^c(x_1 \dots x_n)$ connected recursively $\rightsquigarrow$

$$G_1(x) = G_1^c(x)$$

$$G_2(x_1, x_2) = G_2^c(x_1, x_2) + G_1^c(x_1)\, G_2^c(x_2)$$

$$G_3(x_1 x_2 x_3) = G_3^c(x_1 x_2 x_3) + G_2^c(x_1 x_2) G_1^c(x_3) + \text{perm}$$
$$+ G_1^c(x_1) G_1^c(x_2) G_1^c(x_3) \qquad \text{etc}$$

Interestingly, defining $Z[J] = e^{iW[J]}$

then $iW[J]$ is GENERATING FUNCTIONAL of CONNECTED GREEN FUNCTIONS!

$$G_n^c(x_1, \dots, x_n) = \frac{1}{i}\frac{\delta}{\delta J(x_1)} \cdots \frac{1}{i}\frac{\delta}{\delta J(x_n)} [iW[J]]\Big|_{J=0}$$

• Obvious for $G_1^c(x) = \frac{1}{i}\frac{\delta}{\delta J(x)} \ln Z[J]\Big|_{J=0}$

$$= \frac{1}{Z[J]}\frac{1}{i}\frac{\delta}{\delta J} Z[J]\Big|_{J=0}$$

$(Z[0]=1)$ $$= \frac{1}{Z[0]} G_1(x) = G_1(x)$$

• for $n=2$ $\quad G_2^c(x_1, x_2) = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\left[\frac{1}{Z[J]}\frac{1}{i}\frac{\delta Z}{\delta J}\right]\Big|_{J=0}$

$$G_2^c(x_1\, x_2) = -\frac{1}{Z[J]^2}\, \frac{1}{i}\frac{\delta Z}{\delta J_1}\, \frac{1}{i}\frac{\delta Z}{\delta J_2}\Big|_{J=0}$$

$$+ \frac{1}{Z[J]}\, \frac{1}{i}\frac{\delta}{\delta J_1}\, \frac{1}{i}\frac{\delta}{\delta J_2}\, Z[J]\Big|_{J=0}$$

$$= -G_1^c(x_1)\, G_1^c(x_2) + G_2(x_1, x_2) \quad \checkmark$$

- fr $n=3$ you can verify right structure

$\Rightarrow$ you can try to prove it by induction

T versus T* product

Now we come to a little subtle point that we swept under the rug: when we derived LAGRANGIAN PATH INTEGRAL, we assumed $O_x(t_x)$ does not depend on $\Pi(x)$ ⇒ in general NO DERIVATIVES

$\langle\Omega| T\{\partial_\mu \phi\, \phi\, \phi\} |\Omega\rangle$ not yet defined in terms of $\mathcal{L}$-path integral !

these cases are relevant if $\mathcal{L}_{INT}$ depends on $\partial_\mu \phi$

the problem is that $\partial_0^{(x)} = \dfrac{\partial}{\partial x_0}$ does not commute with T product !

$$T(A(x)\, B(y)) = \theta(x_0-y_0)\, A(x) B(y) + \theta(y_0-x_0)\, B(y) A(x)$$

$\partial_0^{(x)} \theta(x_0-y_0) \sim \delta(x_0-y_0)$ a "contact term" !

$$\partial_0^x T(A(x) B(y)) = \delta(x_0-y_0) A(x) B(y)$$

$$+ \theta(x_0-y_0) \partial_0^x A(x) B(y)$$

$$- \delta(x_0-y_0) B(y) A(x)$$

$$+ \theta(y_0-x_0) B(y) \partial_0^x A(x)$$

$$= T\left[\partial_0^x A(x) B(y) \right]$$

$$+ \delta(x_0-y_0) \left[A(x), B(y) \right]$$

"CONTACT TERM" $\propto$ commutator of two operators !

So we <u>DEFINE</u>

$$\langle\Omega| T^* (\partial_0^x \phi(x) \ldots) |\Omega\rangle = \partial_0^x \langle\Omega| T(\phi(x)\ldots)|\Omega\rangle$$

T^* product

pull out derivative and then act on standard T-product

From our example before then

$$\langle \Omega | T^* (\partial_0^x A(x) B(y)) | \Omega \rangle =$$

$$= \partial_0^x \langle \Omega | T (A(x) B(y)) | \Omega \rangle$$

$$= \langle \Omega | T (\partial_0^x A(x) B(y)) | \Omega \rangle + \delta(x_0 - y_0) [A(x), B(y)]$$

Applying this on products of fields & their derivatives, we get:

$$\langle \Omega | T^* \{ \partial_\mu \phi(x) \, \partial_\nu \phi(y) \} | \Omega \rangle = \text{DEFINITION}$$

$$= \partial_\mu^x \partial_\nu^y \langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle \quad \left(\begin{matrix} \text{pull } \partial \text{ out} \\ \text{of } T \end{matrix} \right)$$

$$= \partial_\mu^x \partial_\nu^y \langle \Omega | \theta(x_0 - y_0) \phi(x) \phi(y) + \theta(y_0 - x_0) \phi(y) \phi(x) | \Omega \rangle$$

$$= \partial_\nu^y \langle \Omega | \underbrace{\delta(x_0 - y_0) \delta_{\mu,0} [\phi(x), \phi(y)]}_{= 0 \text{ using comm rel. at equal times}} + T(\partial_\mu \phi(x) \phi(y)) | \Omega \rangle$$

$$= \langle \Omega | \left[\underbrace{\partial_\nu^y \, T\left(\partial_\mu \phi(x) \, \phi(y) \right)} \right] | \Omega \rangle$$

$$\Downarrow$$

$$T(\partial_\mu \phi(x) \, \partial_\nu \phi(y)) - \delta_\nu^0 \, \delta(x_0 - y_0) \, [\partial_\mu \phi, \phi]$$

$$\uparrow \; \partial_\nu^y \theta(x_0 - y_0) = -\delta(x_0 - y_0) \; !$$

$$= \langle \Omega | \, T(\partial_\mu \phi(x) \, \partial_\nu \phi(y)) \, | \Omega \rangle$$

$$- \delta_\nu^0 \, \delta(x_0 - y_0) \, \langle \Omega | \, [\partial_\mu \phi(x), \phi(y)] \, | \Omega \rangle$$

$\Uparrow$

this equal-time commutator

is $\neq 0$ only for $\partial_\mu = \partial_0$

$\Pi(x) = \partial_0 \phi(x)$ then

$$\langle \Omega | [\Pi(x), \phi(y)]_{x_0 = y_0} | \Omega \rangle = -i \, \delta^{(3)}(\vec{x} - \vec{y})$$

$$= \langle \Omega | T(\partial_\mu \phi \, \partial_\nu \phi) | \Omega \rangle + \underbrace{i \, \delta_\nu^0 \, \delta_\mu^0 \, \delta^{(4)}(x - y)}_{\text{contact term !}}$$

So ultimately we can write

$$\langle \Omega | T[\partial_\mu \phi \, \partial_\nu \phi] | \Omega \rangle =$$

$$= \langle \Omega | T^*\{\partial_\mu \phi \, \partial_\nu \phi\} | \Omega \rangle - \underbrace{i \, \delta_{\mu 0} \delta_{\nu 0} \, \delta^{(4)}(x-y)}_{\text{contact - term}}$$

this is what is computed

by PATH INTEGRAL

$$|N|^2 \int [D\phi] \exp\{i \int \mathcal{L} \, d^4x\} \, \partial_\mu \phi \, \partial_\nu \phi \quad \text{etc}$$

So whenever we write T we mean T^* !

this makes a difference ONLY WITH DERIVATIVES