Path Integral Quantization 3:

Green Functions for free & interacting Scalar theories

In last lecture we have computed the Generating Functional of a free scalar theory

$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$ and found

$$Z_0[J] = \exp\left\{-\frac{1}{2}\int d^4x\, d^4y\; J(x)\,\Delta_F(x-y)\,J(y)\right\}$$

with

$$\boxed{\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i}{p^2-m^2+i\varepsilon}}$$

We have also seen that $Z_0[J]$ allows us to compute all Green Functions through

$$G_0(x_1,\dots,x_n) = \left[\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right]\cdots\left[\frac{1}{i}\frac{\delta}{\delta J(x_n)}\right] Z_0[J]\Bigg|_{J=0}$$

Let us then try to apply this formula and see what we get. We start with

- One-Point Green Function

$$G_0(z) = \frac{1}{i} \frac{\delta}{\delta J(z)} Z_0[J] \Big|_{J=0}$$

$$= \frac{1}{i} \Big[-\frac{1}{2} \int d^4y \, \Delta_F(z-y) J(y)$$

$$- \frac{1}{2} \int d^4x \, \underbrace{J(x)}_{\text{one } J(x) \text{ left}} \Delta_F(x-z) \Big] Z_0[J] \Big|_{J=0}$$

$G_0(z) = 0$ which remains true for all odd-point Green functions

$G_0(x_1, \dots, x_{2n+1}) = 0$ there is always one J left!

this is expected, if you think about our one-dimensional example

$$\int_{-\infty}^{+\infty} dq\, q^{2n+1} e^{-\frac{m^2}{2}q^2} = 0$$ since by $q \to -q$ integrand is ODD

So let's focus on EVEN GREEN FUNCTIONS

- 2-point ($|\Omega\rangle$ vacuum of free theory here!)

$$\langle\Omega| T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle = \left(\frac{1}{i}\right)^2 \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \cdot$$

$$\cdot \exp\left\{ -\frac{1}{2} \int d^4y_1\, d^4y_2\, J(y_1)\, \Delta_F(y_1-y_2)\, J(y_2) \right\}\Big|_{J=0}$$

$$= -\frac{\delta}{\delta J(x_1)} \left[-\frac{1}{2} \Delta_{x_2,y_2} J_{y_2} - \frac{1}{2} J_{y_1} \Delta_{y_1,x_2} \right]$$

$$\cdot \exp\left\{ -\frac{1}{2} J_{y_1} \Delta_{y_1,y_2} J_{y_2} \right\}\Big|_{J=0}$$

where I introduced the compact notation

$$\Delta_{x,y} = \Delta_F(x-y) \, , \quad J_x = J(x) \quad \text{and}$$

$$\Delta_{x,y} J_y = \int d^4x \; \Delta_F(x-y) J(y)$$

"repeated indices are integrated"

Now let's differentiate again

$$\langle \Omega | T\{\phi(x_1)\phi(x_2)\} | \Omega \rangle =$$

$$\Big[\frac{1}{2}\Delta_F(x_1-x_2) + \frac{1}{2}\Delta_F(x_1-x_2) + \underbrace{O(J)}_{\uparrow} \Big] Z_0[J] \Big|_{J=0}$$

$$= \Delta_F(x_1-x_2)$$

these terms come from $\frac{\delta}{\delta J(x_1)}$ on $e^{-\frac{1}{2} J\cdot\Delta\cdot J}$ and drop when we put $J=0$!

so we find the first interesting result

$$\langle \Omega | T\{ \phi(x_1)\, \phi(x_2) \} | \Omega \rangle = \Delta_F(x_1 - x_2)$$

$\Rightarrow$ the two-point function in the free theory coincides with the FEYNMAN PROPAGATOR !

Let's continue

- 4-point Green Function $\quad \phi(x_i) = \phi_i$

$$\langle \Omega | T\{ \phi_1 \phi_2 \phi_3 \phi_4 \} | \Omega \rangle = \left[\frac{1}{i}\right]^4 \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} \frac{\delta}{\delta J_3} \frac{\delta}{\delta J_4} e^{-\frac{1}{2} J_{y_1} \Delta_{y_1 y_2} J_{y_2}} \Big|_{J=0}$$

$$= \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} \frac{\delta}{\delta J_3} \Big[\underbrace{-\frac{1}{2} \Delta_{x_4, y_2} J_{y_2} - \frac{1}{2} J_{y_1} \Delta_{y_1, x_4}}_{\text{using } \Delta_F(x-y) = \Delta_F(y-x) \text{ \& renaming } y_2 = y_1 = y} \Big] e^{-\frac{1}{2} J_{y_1} \Delta J_{y_2}} \Big|_{J=0}$$

$$= \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} \frac{\delta}{\delta J_3} \left[- J_y \Delta_{y, x_4} \right] e^{-\frac{1}{2} J_{y_1} \Delta_{y_1 y_2} J_{y_2}} \Big|_{J=0}$$

$$= \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} \left[- \Delta_{x_3, x_4} + J_y \Delta_{y, x_4} \; J_z \Delta_{z, x_3} \right] e^{-\frac{1}{2} J \cdot \Delta \cdot J} \Bigg|_{J=0}$$

$$= \frac{\delta}{\delta J_1} \Big[\Delta_{x_3, x_4} J_y \Delta_{y, x_2} + \Delta_{x_2, x_4} J_z \Delta_{z, x_3}$$

$$+ J_y \Delta_{y, x_4} \Delta_{x_2, x_3} + J_y \Delta_{y x_4} J_z \Delta_{z x_3} J_t \Delta_{t x_2} \Big] e^{-\frac{1}{2} J \cdot \Delta \cdot J} \Bigg|_{J=0}$$

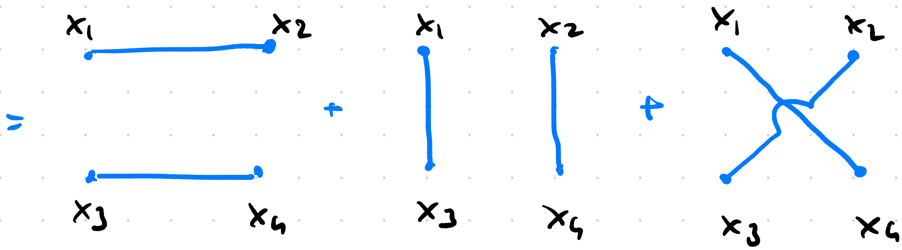
$$= \left[\Delta_{x_3 x_4} \Delta_{x_1 x_2} + \Delta_{x_2 x_4} \Delta_{x_1 x_3} + \Delta_{x_1 x_4} \Delta_{x_2 x_3} + O(J) \right] e^{-(\;)} \Bigg|_{J=0}$$

$$= \Delta_{34} \Delta_{12} + \Delta_{24} \Delta_{13} + \Delta_{14} \Delta_{23}$$

$$= \sum_{\substack{\text{pairwise} \\ \text{contractions}}} \phi_1 \phi_2 \phi_3 \phi_4 = \phi_1 \phi_2 \phi_3 \phi_4 + \phi_1 \phi_2 \phi_3 \phi_4$$

$$+ \phi_1 \phi_2 \phi_3 \phi_4$$

every pairwise contraction gives rise to a Feynman Propagator $\Rightarrow$ GRAPHICAL DEPICTION



these are our first (trivial) examples of Feynman Diagrams in coordinate space!

$\Rightarrow$ In canonical Quantization this goes under name of WICK THEOREM ! We get it for free!

WHY IS FEYNMAN PROPAGATOR CALLED A PROPAGATOR?

(A digression)

$$\Delta_F(x-y) = \langle\Omega| T\{\phi(x)\phi(y)\}|\Omega\rangle$$

$$= \langle\Omega|\, \theta(x_0-y_0)\,\phi(x)\,\phi(y) + \theta(y_0-x_0)\,\phi(y)\,\phi(x)\, |\Omega\rangle$$

now remember that

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \left[\underbrace{a(p)\, e^{-ip\cdot x}}_{\phi^{(+)}(x)} + \underbrace{a^\dagger(p)\, e^{ip\cdot x}}_{\phi^{(-)}(x)} \right]$$

$$= \phi^{(+)}(x) + \phi^{(-)}(x)$$

using $a(p)|\Omega\rangle = 0$ $\qquad$ $\langle\Omega| a^\dagger(p) = 0$

we see that

$$\Delta_F(x-y) = \theta(x_0-y_0)\langle\Omega|\phi^{(+)}(x)\,\phi^{(-)}(y)|\Omega\rangle \quad \text{I}$$

$$+\, \theta(y_0-x_0)\langle\Omega|\phi^{(+)}(y)\,\phi^{(-)}(x)\,|\Omega\rangle \quad \text{II}$$

I Amplitude to create particle at y, $t=y_0$, and destroy it at x, $t=x_0>y_0$

II the opposite: create @ x, $t=x_0$
destroy @ y, $t=y_0>x_0$

evaluating explicitly the two terms we get indeed

$$\Delta_F(x-y) = \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6\, 2E_p\, 2E_q} \Big[\theta(x_0-y_0)\, e^{-i(px-qy)} \langle\Omega|\, a(p)\, a^\dagger(q)\,|\Omega\rangle$$

$$+\, \theta(y_0-x_0)\, e^{-i(qy-px)} \langle\Omega|\, a(q)\, a^\dagger(p)\,|\Omega\rangle \Big]$$

then use $[a(p), a^\dagger(q)] = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p}-\vec{q})$

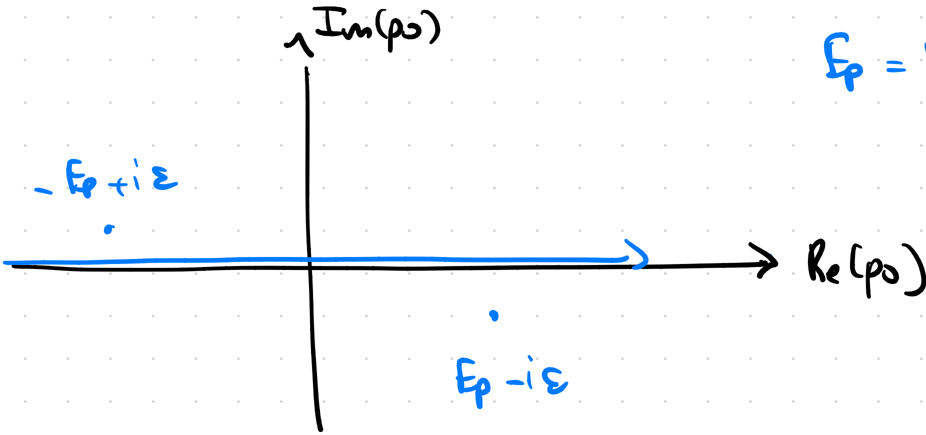
and $\langle\Omega|\Omega\rangle = 1$

$$\Delta_F(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \Big[\theta(x_0-y_0)\, e^{-ip\cdot(x-y)} + \theta(y_0-x_0)\, e^{ip(x-y)} \Big]$$

$$\stackrel{?}{=} \int \frac{d^4\bar{p}}{(2\pi)^4}\, e^{-ip\cdot(x-y)} \frac{i}{p^2-m^2+i\varepsilon}$$

yes! we can see this by integrating in $dp_0 \Rightarrow$

$$\int \frac{d^3\vec{p}}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi}\, e^{-ip_0(x_0-y_0)+i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{i}{p_0^2 - E_p^2 + i\varepsilon}$$



$$E_p = \sqrt{\vec{p}^{\,2} + m^2}$$

separate the two poles of

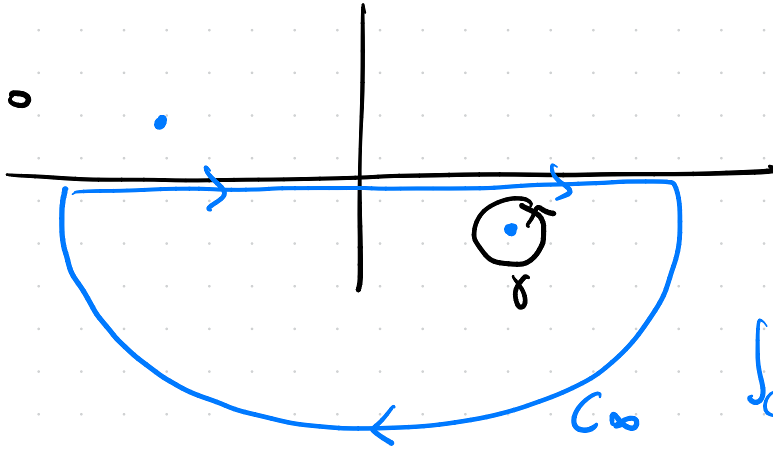
$$\frac{1}{p_0^2 - E_p^2 + i\varepsilon} = \frac{1}{2E_p}\left[\frac{1}{p_0 - E_p + i\varepsilon} - \frac{1}{p_0 + E_p - i\varepsilon}\right]$$

So we have two integrals

$$\mathrm{I}]\quad \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi}\, e^{-ip(x-y)} \frac{1}{p_0 - E_p + i\varepsilon}$$

$$\mathrm{II}]\quad \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, e^{-ip(x-y)} \frac{1}{p_0 + E_p - i\varepsilon}$$

depending on $(x_0 - y_0) > 0$ or < 0 we can close BELOW or ABOVE if $x_0 - y_0 > 0$



$$\int_{C_\infty} = 0$$

$$e^{-i(p_0 - i\Gamma)} \sim e^{-\Gamma} \to 0$$

Clockwise

$$\int_{-\infty}^{+\infty} \frac{dp_0}{2\pi}\, e^{-ip(x-y)} \frac{1}{p^2 - m^2 + i\varepsilon} = -2\pi i \oint_\gamma \frac{dp_0}{2\pi} \frac{e^{-ip(x-y)}}{2E_p} \frac{1}{p_0 - E_p + i\varepsilon}$$

$$= -i \left. \frac{e^{-ip(x-y)}}{2E_p} \right|_{p_0 = E_p}$$

if $x_0 - y_0 < 0$ instead I must close ABOVE (counterclockwise) $+2\pi i$

$$\int_{-\infty}^{+\infty} \frac{dp_0}{2\pi}\, e^{-ip(x-y)} \frac{1}{p^2 - m^2 + i\varepsilon} = - \left. \frac{i\, e^{-ip(x-y)}}{2E_p} \right|_{p_0 = -E_p}$$

so finally

$$\Delta_F(x-y) = \cancel{i} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \cancel{(-i)} \left[\theta(x_0-y_0)\, e^{-iE_p(x_0-y_0)} + \underbrace{\theta(y_0-x_0)\, e^{iE_p(x_0-y_0)}}_{\substack{\text{send } \vec{p} \to -\vec{p} \\ e^{-i\vec{p}(\vec{x}-\vec{y})} \to e^{+i\vec{p}(\vec{x}-\vec{y})}}} \right]$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p} \left[\theta(x_0-y_0)\, e^{-ip(x-y)} + \theta(y_0-x_0)\, e^{-ip(y-x)} \right]_{p_0=E_p}$$

As we wanted to demonstrate !

Now to make more sense of this idea of "PROPAGATION", let's see what happens in two cases :

1] $x_0 - y_0 = 0$; $|\vec{x} - \vec{y}| > 0$ SPACE-LIKE TRANSITION

Do we expect $\Delta_F = 0$?

the integral becomes

$$\Delta_F(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p} \left[e^{+i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right]$$

Call $\vec{x} - \vec{y} = \vec{r}$ $\quad \vec{p}\cdot\vec{r} = |\vec{p}|\, r \cos\theta$

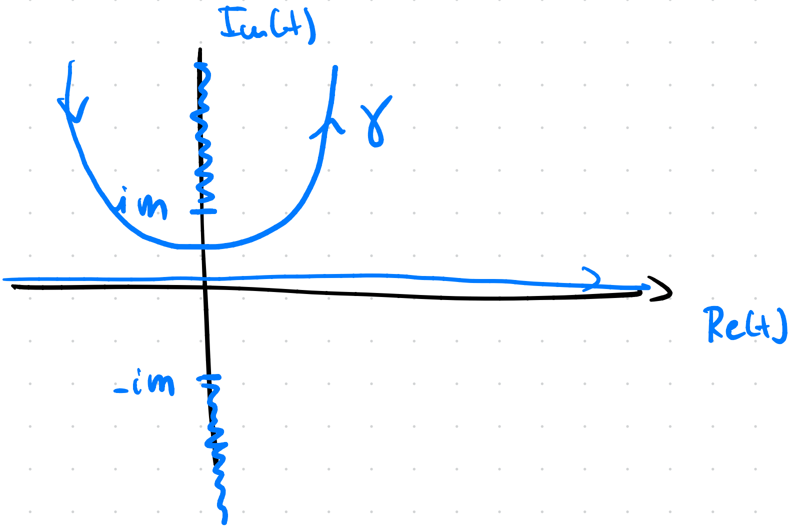
$$\Delta_F = \int_0^\infty \frac{d|\vec{p}|\cdot|\vec{p}|^2}{(2\pi)^2} \frac{1}{2E_p} \int_0^1 d\cos\theta \left[e^{i|\vec{p}| r \cos\theta} + e^{-i|\vec{p}| r \cos\theta} \right]$$

$$= \int_0^\infty \frac{d|\vec{p}|\, |\vec{p}|^{\cancel{2}}}{(2\pi)^2\, 2E_p} \left[\frac{e^{i|\vec{p}| r} - e^{-i|\vec{p}| r}}{i\cancel{|\vec{p}|}\, r} \right] = \quad |\vec{p}| = t$$

$$= -\frac{i}{2(2\pi)^2 r} \int_0^\infty dt \frac{t\left(e^{itr} - e^{-itr}\right)}{\sqrt{t^2 + m^2}}$$

send $t \to -t$ in second piece

$$= -\frac{i}{2(2\pi)^2 r} \int_{-\infty}^{+\infty} dt \, \frac{t}{\sqrt{t^2+m^2}} \, e^{itr}$$



if $r>0$

we can close contour around upper branch cut

$$\int_{-\infty}^{+\infty} dt \, \frac{t}{\sqrt{t^2+m^2}} \, e^{itr} = 2 \int_{im}^{+i\infty} dt \, \frac{t}{\sqrt{t^2+m^2}} \, e^{itr}$$

$t = +iu \Rightarrow u = -it$ $\quad u \in [m, \infty)$ $\quad dt = -i\,du$

$$= -2i \int_{m}^{\infty} du \, \frac{u}{\sqrt{u^2-m^2}} \, e^{-ur}$$

(there is a i coming from root too!)

$$\Delta_F(x-y) = \frac{1}{(2\pi)^2 r} \int_m^\infty du \, \frac{u}{\sqrt{u^2-m^2}} \, e^{-ur}$$

as $r \to \infty$ one can show that

$$\Delta_F(x-y) \sim \frac{e^{-mr}}{r^{3/2}} \to 0$$ EXPONENTIALLY

but <u>not zero</u>!

A particle can propagate outside the light-cone with probability that is EXPONENTIALLY SUPPRESSED

$\Rightarrow$ Quantum theory ! $\quad m \cdot r \sim 1$

$\Delta p \sim m \Rightarrow \Delta x \sim r$

Uncertainty principle allows this !

2] time like $\Delta t = x_0 - y_0 > 0$; $|\vec{x} - \vec{y}| = 0$

$$\Delta_F(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3 \, 2E_p} \left[\theta(x_0-y_0) \, e^{-ip\cdot(x-y)} + \cancel{\theta(y_0-x_0) \, e^{ip(x-y)}} \right]$$

$$\Delta_F(x-y) = \int \frac{d^3\bar{p}}{(2\pi)^3 2E_p} \underbrace{\Theta(\Delta t)}_{1} e^{-iE_p\Delta t}$$

$\uparrow$ oscillate $\Delta t \to \infty$

if $\Delta t \to \infty$, main contribution from STATIONARY PHASE

$$\nabla_{\vec{p}} E_p = 0 \quad \to \quad \frac{\vec{p}}{E_p} = 0 \quad \Rightarrow \quad \vec{p} = 0 \quad \text{then}$$

$$E_p \simeq m + \frac{p^2}{2m} + \dots \Rightarrow e^{-iE_p\Delta t} \sim e^{-im\Delta t} e^{-i\frac{p^2}{2m}\Delta t}$$

$$\Delta_F(x-y) \sim \frac{e^{-im\Delta t}}{(2\pi)^3 2m} \left[\int d^3\vec{p}\ e^{-i\left(\frac{p^2}{2m}\right)\Delta t} = \left[\frac{2\pi m}{i\Delta t}\right]^{3/2} \right]$$

$$\sim \frac{C}{\Delta t^{3/2}} e^{-im\Delta t}$$

oscillates and decays as power law when $\underline{\Delta t \to \infty}$

Finite for every $\Delta t \neq \infty$

FREE COMPLEX SCALAR FIELD

Very little changes for complex fields

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

Since we effectively have TWO FIELDS, we must also generalize the generating functional

$$Z_0[J, J^*] = |N|^2 \int [\mathcal{D}\phi][\mathcal{D}\phi^*] \, e^{i\int d^4x \, [\mathcal{L}_0 + J\phi + J^*\phi^*]}$$

By using $\phi = \frac{(\phi_1 + i\phi_2)}{\sqrt{2}}$ with $\phi_i^* = \phi_i$

we can reduce the calculation to the one for the real scalar field and prove

$$Z_0[J, J^*] = \exp\left\{ - \int d^4x \, d^4y \, J(x) \Delta_F(x-y) J^*(y) \right\}$$

no $\frac{1}{2}$ factor ; $\Delta_F(x-y)$ SAME AS REAL FIELD

We can generate the free Green functions differentiating with respect to $\frac{\delta}{\delta J}$ and $\frac{\delta}{\delta J^*}$ as before

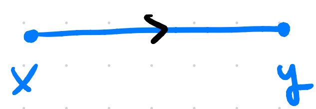
$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = 0$$
$$\langle\Omega|T\{\phi^\dagger(x)\phi^\dagger(y)\}|\Omega\rangle = 0$$

same field gives zero!

(FOR OPERATORS use $\phi^\dagger$ instead of ϕ^*)

$$\langle\Omega|T\{\phi(x)\phi^\dagger(y)\}|\Omega\rangle = \Delta_F(x-y)$$

$$\overbrace{\phi(x)\,\phi^\dagger(y)} =$$



arrow here indicates direction from $\phi \to \phi^\dagger$

U(1) CHARGE FLOW

Direction of arrow is conventional!

INTERACTING SCALAR THEORIES

Let's consider now for simplicity a real scalar field

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{INT}$$

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \; ; \quad \mathcal{L}_{INT} = \alpha \phi^3 + \beta \phi^4 + \dots$$

In principle, we proceed in the same way

- Define $$Z[J] = \frac{\int [\mathcal{D}\phi] \exp\left\{ i \int d^4x \, \mathcal{L} + J\phi \right\}}{\int [\mathcal{D}\phi] \exp\left\{ i \int d^4x \, \mathcal{L} \right\}}$$

- $$\langle \Omega | T\{\phi(x_1) \dots \phi(x_n)\} | \Omega \rangle = G(x_1, \dots, x_n) = \prod_{j=1}^{n} \left(\frac{1}{i} \frac{\delta}{\delta J(x_j)} \right) Z[J] \Big|_{J=0}$$

ϕ is now INTERACTING FIELD!

the problem is that, at variance with free theory, we do not know how to compute $Z[J]$ exactly $\Rightarrow$ 1] LATTICE attempts this by Discretizing SPACE TIME & simulating the result on LARGE COMPUTERS

Euclidean $t \to i\tau$, NO REAL TIME!

2] Assume interactions are "WEAK" and attempt PERTURBATIVE EXPANSION

We follow 2] and expand the interaction action

$$\exp\left[i\int d^4x\, \mathcal{L}_{INT}\right] = \sum_{n=0}^{\infty} \frac{1}{n!}\left(i\int d^4x\, \mathcal{L}_{INT}\right)^n$$

this is now a polynomial in $\phi(x)$!

Exactly like Green functions of Free theory!

so we get

$$Z[J] = |N|^2 \underbrace{\int [D\phi]\, e^{i\int d^4x(\mathcal{L}_0 + J\phi)}}_{Z_0[J]} \cdot$$

$Z_0[J=0] = 1$

$$\cdot \sum_{n=0}^{\infty} \frac{1}{n!}\, i \int d^4x_1 \dots d^4x_n\, \mathcal{L}_{int}(\phi(x_1)) \dots \mathcal{L}_{int}(\phi(x_n))$$

$$= |N|^2 \sum_{n=0}^{\infty} \frac{1}{n!} \left[i \int d^4x\, \mathcal{L}_{INT}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right]^n Z_0[J]$$

we had to exchange sum & integral

Functional derivatives act on $Z_0[J]$ pulling down appropriate powers of ϕ

Formally, we can resum exponential and write

$$Z[J] = |N|^2 \exp\left[i \int d^4x\, \mathcal{L}_{INT}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] Z_0[J]$$

this part give INTERACTION VERTICES !

PROPAGATORS

By using the fact that $\mathcal{L}_{INT}$ is small, we can expand and truncate the expansion at the order we want $\Rightarrow$ WHAT DOES EACH ORDER LOOK LIKE?

$\langle \Omega | T\{\phi(x_1) \dots \phi(x_N)\} | \Omega \rangle \Rightarrow$ at order n will give

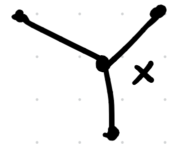
$$\Rightarrow \int [D\phi]\, e^{i\int d^4x (\mathcal{L}_0 + J\phi)} \frac{1}{n!} \left[i \int d^4z\, \mathcal{L}_{INT} \right]^n \phi(x_1) \dots \phi(x_N)$$

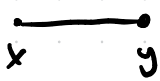
Each occurrence of $\mathcal{L}_{INT}$ is a polynomial in ϕ so we expect same structure as before:

$$= \int dz_1 \dots dz_n \left[\text{sum of all pairwise contractions of fields in } \underbrace{\mathcal{L}_{INT}(z_1) \dots \mathcal{L}_{INT}(z_n)\, \phi(x_1) \dots \phi(x_N)}_{\text{Wick products encountered before!}} \right]$$

Let's look at an explicit example

EXAMPLE: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3$

Graphically, $\mathcal{L}_{INT} = -\frac{g}{3!}\phi^3(x) \rightarrow$  $\left(-\frac{ig}{3!}\right)$ in exp!

$\Delta_F(x-y) =$  Scalar propagator

We can now follow same exercise we did in FREE CASE and compute the result in terms of DIAGRAMS

Assume $g \ll 1$ SMALL COUPLING

try to compute $\langle\Omega|T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle$

in full theory EXPANDING TO $O(g^2)$ INCLUDED

$\Rightarrow$ two occurrences of $\mathcal{L}_{INT}$!

- $O(g^0)$: there is only  $= \Delta_F(x_1 - x_2)$

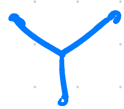
no occurrences of $\mathcal{L}_{INT}$!

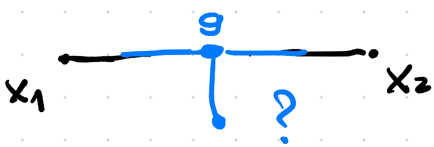
• $O(g)$: 1 occurrence of $\mathcal{L}_{INT}$ means we have product of 5 free fields

$$\sim \phi(x_1)\,\phi(x_2)\,\phi^3(z)$$

We have seen that all odd-point free Green functions are zero because we always have one J left!

so nothing contributes

$\Rightarrow$ graphically, no way to attach one single  to two-point function



• $O(g^2)$: is finally $\neq 0$ $\Rightarrow$ because I have 8 $\phi(x_i)$

$$\Rightarrow \phi(x_1)\,\phi(x_2)\,\phi^3(z_1)\,\phi^3(z_2)$$

need to consider all PAIRWISE CONTRACTIONS !

$$(1) \quad \phi(x_1)\,\phi(x_2)\;\phi(z_1)\,\phi(z_1)\,\phi(z_1)\;\phi(z_2)\,\phi(z_2)\,\phi(z_2)$$

what does it give?

$$\left[\frac{1}{2}\cdot 3\cdot 3\right]\left(-\frac{ig}{3!}\right)^2 \underbrace{\Delta_F(x_1-x_2)}_{\mathrm{I}}\int \underbrace{d^4z_1\, d^4z_2}_{\text{integrations left!}}\ \overset{\mathrm{II}}{\Delta_F(z_1-z_2)}\ \underbrace{\Delta_F(0)^2}_{\mathrm{III},\,\mathrm{IV}}$$

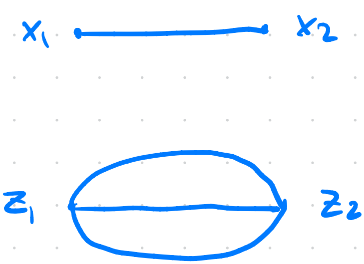
↑ numerical factor because :

$\frac{1}{2}$: expansion of exponential give $\frac{1}{2}$ at $O(g^2)$

$3\cdot 3$: symmetry factor => I get same diagram in many different ways

$$\left.\begin{array}{l} 1 \text{ for (I) contraction} \\ 3\cdot 3 \text{ for (II) contraction} \end{array}\right\} \text{ then III \& IV FIXED!}$$

(2) $\underbrace{\phi(x_1)\,\phi(x_2)}_{\mathrm{I}}\;\underbrace{\phi(z_1)\,\underbrace{\phi(z_1)\,\overbrace{\phi(z_1)\;\phi(z_2)}^{\mathrm{IV}}\,\phi(z_2)}_{\mathrm{III}}\,\phi(z_2)}_{\mathrm{II}}$



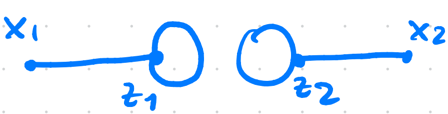
$$\frac{1}{2}\cdot\overbrace{3\cdot 2\cdot 1}^{\text{symmetry factor}} \leftarrow \text{LAST PAIR FIXED}$$

second z_1 fixed, 2 possibilities for z_2

First z_1 Fixed, 3 possibilities for z_2

$$= \frac{1}{2}(3\cdot 2)\left(-\frac{ig}{3!}\right)^2 \Delta_F(x_1-x_2)\int d^4z_1\, d^4z_2\, \Delta_F(z_1-z_2)^3$$

(3) $\underbrace{\phi(x_1)\,\phi(x_2)\;\phi(z_1)}\,\underbrace{\phi(z_1)\,\phi(z_1)}\;\phi(z_2)\,\underbrace{\phi(z_2)\,\phi(z_2)}$



$$\frac{1}{2}\left[3\cdot 3\cdot 2\right] \quad \text{symmetry factor}$$

swap $z_1 \leftrightarrow z_2$!

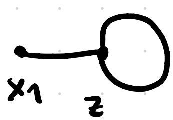
$$= \frac{1}{2}(3\cdot 3\cdot 2)\left(-\frac{ig}{3!}\right)^2 \int d^4z_1\, d^4z_2\; \Delta_F(x_1-z_1)\,\Delta_F(x_2-z_2)\,\Delta_F(0)^2$$

$$= G^{(1)}(x_1)\, G^{(1)}(x_2)$$

one-point Green function, in fact:

$$\phi(x_1)\,\phi(z)\,\phi(z)\,\phi(z) =$$ symmetry factor $\Rightarrow \underline{\underline{3}}$

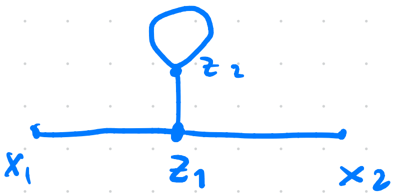
$$G^{(1)}(x_1) = \left[\frac{-ig}{3!}\right] 3 \int d^4z\; \Delta_F(x_1-z)\,\Delta_F(0)$$

$=$ 

and $\frac{1}{3!}\cdot 3 = \frac{1}{2}$; $\Rightarrow G(x_1)G(x_2) \sim \underline{\underline{\frac{1}{4}}}$

while two-point: $\frac{1}{\cancel{2}}\,\cancel{3}\cdot\cancel{3}\cdot\cancel{2}\cdot\frac{1}{\cancel{3}\cdot 2}\,\frac{1}{\cancel{3}\cdot 2} = \underline{\underline{\frac{1}{4}}}$ the same!

(4) $\phi(x_1)\,\phi(x_2)\,\phi(z_1)\,\phi(z_1)\,\phi(z_1)\,\phi(z_2)\,\phi(z_2)\,\phi(z_2)$



$x_1 \to z_1$

$$= \frac{1}{2} \cdot \underbrace{2 \cdot 3 \cdot 2 \cdot 3}_{\text{symmetry factor}}$$

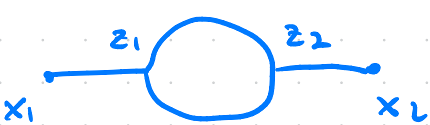
$z_1 \to z_2$

$x_2 \to z_1$

swap $z_1 \leftrightarrow z_2$

$$= \frac{1}{2}\left(2 \cdot 3 \cdot 2 \cdot 3\right)\left(-\frac{ig}{3!}\right)^2 \int d^4z_1\, d^4z_2\, \Delta_F(x_1 - z_1)\,\Delta_F(x_2 - z_1)\,\Delta_F(z_1 - z_2)\,\Delta_F(0)$$

(5) $\phi(x_1)\,\phi(x_2)\,\phi(z_1)\,\phi(z_1)\,\phi(z_1)\,\phi(z_2)\,\phi(z_2)\,\phi(z_2)$



$x_2 \to z_2$

$$= \frac{1}{2} \cdot 2 \cdot 3 \cdot 3 \cdot 2$$

$z_1 \to z_2$

$z_1 \leftrightarrow z_2$ SWAP

$x_1 \to z_1$

which then gives

$$= \frac{1}{2}\, 2\cdot 3\cdot 3\cdot 2 \left(-\frac{ig}{3!}\right)^2 \int d^4z_1 d^4z_2 \; \Delta_F(x_1-z_1)\,\Delta_F(x_2-z_2)\,\Delta_F(z_1-z_2)^2$$

we call (1), (2) DISCONNECTED VACUUM GRAPHS (not connected to x_1, x_2)

(3) is product of lower points

(4), (5) CONNECTED DIAGRAMS

Notice now that (1), (2) (vacuum) do not contribute because we still need to account for DENOMINATOR!

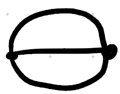
$$Z[J] = \frac{\int [D\phi] \exp\left\{ i \int d^4x \left(\mathcal{L} + J\phi\right)\right\}}{\int [D\phi] \exp\left\{ i \int d^4x \left(\mathcal{L}\right)\right\}}$$

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{INT}$

Denominator is J independent, but still must be expanded in $\mathcal{L}_{INT}$!

$$\int [\mathcal{D}\phi] \exp\left\{ i \int d^4x \left(\mathcal{L}_0 + \mathcal{L}_{INT} \right) \right\} =$$ no fields ϕ only VACUUM DIAGS.

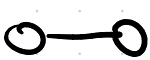
$$= \int [\mathcal{D}\phi] \exp\left\{ i \int d^4x\, \mathcal{L}_0 \right\} \left[1 + \mathcal{L}_{INT} + \frac{1}{2} \mathcal{L}_{INT}^2 + \cdots \right]$$

$= \underbrace{1}_{O(g^0)} +$  $+$  $+ O(g^4)$

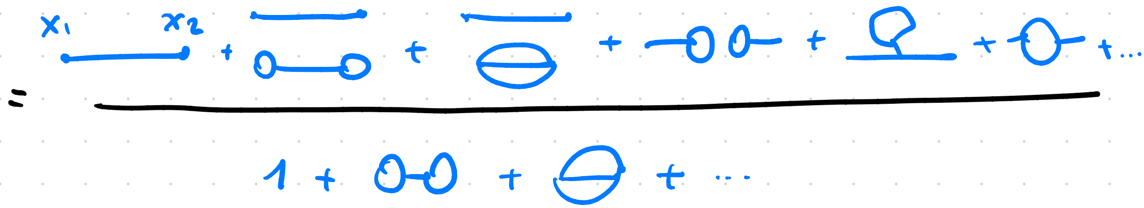
$O(g^2)$

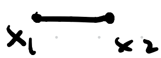
also for vacuum, no $O(g)$ because $\mathcal{L}_{INT} \propto \phi^3$ no contractions of odd numbers of fields !

same symmetry factors as with $x_1\ x_2$

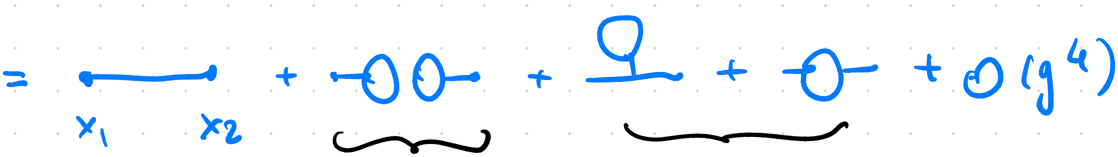
 $\sim$  etc then

$$\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)} Z[J]\Big|_{J=0} =$$



expanding to $O(g^2)$ you see that denominator times 

reproduces disconnected vacuum diags. with opposite sign!



DISCONNETED $G^{(1)}(x_1)\, G^{(1)}(x_2)$

Connected part

$= G^{(2)}(x_1, x_2)$ two-point Green function to order $O(g^2)$ in terms of DIAGRAMS

$\Rightarrow$ Every diagram is a graphical tool to depict a mathematical expression made of:

- products of propagators
- integration over internal vertex points
- vertex "couplings" $\left(-ig/3!\right)$
- symmetry factors $\Rightarrow$ ACCOUNT FOR THE FACT THAT DIAGRAMS ARE IDENTICAL IF WE ONLY SWAP INTERNAL (INTEGRATED) POINTS !

We can generalize this by defining

FEYNMAN RULES in COORDINATE SPACE

Consider $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{INT}$

$$\mathcal{L}_{INT} = g_{i_1 \dots i_k} \phi_{i_1} \dots \phi_{i_k}$$ polynomial interaction

We can obtain the perturbative expansion for an n-point GREEN FUNCTION

$$\langle \Omega | T \{ \phi_1(x_1) \dots \phi_n(x_n) \} | \Omega \rangle$$

as follows:

1. Determine Propagators $= \phi_i(x)\phi_j(y) = \Delta_F^{ij}(x-y)$

   if different types of fields ϕ_i, there could be different PROPAGATORS!

   From Inverting Kinetic term

   $\Rightarrow$ FREE LAGRANGIAN

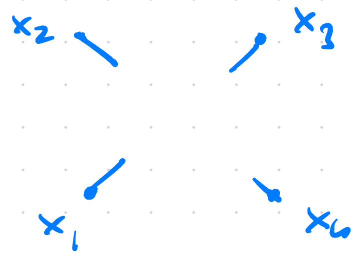
   need kinetic mixing $\partial_\mu \phi^i \partial^\mu \phi^j$ to mix them

   typically, it can be DIAGONALIZED by Field ROTATION!

2. Determine Vertices $= i g_{i_1 \dots i_k}$

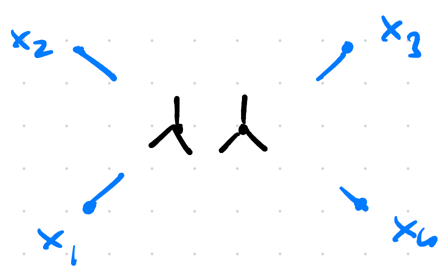
2. Draw a point for each x_i and a line starting from it

ex: $\langle \Omega | T(\phi\phi\phi\phi) | \Omega \rangle$



3. For $O(g^m)$ draw m internal vertex points with each k lines depending on type of interaction

ex ϕ^3 and $O(g^2)$



4. Connect lines in all possible ways

5. Exclude disconnected VACUUM subdiagrams

6. Add the appropriate symmetry factor by counting all possible ways of making a contraction

7. Integrate over $\int d^4 z_i$ for all internal points