Path Integral Quantization 2: Lagrangian formulation & the scalar propagator

In previous lecture we have obtained an expression for the expectation value of a string of TIME-ORDERED operators as a PATH INTEGRAL.

$\Rightarrow$ we "defined" a strange type of integral over $[\mathcal{D}\phi][\mathcal{D}\pi]$ $\Rightarrow$ ALL FIELDS CONF.

We integrate (and we will differentiate) over FUNCTIONS instead of variables

Quantities like our PATH INTEGRAL are called FUNCTIONAL INTEGRALS $\Rightarrow$ it depends on the value of the functions $\phi(x)$ & $\pi(x)$ FOR ALL x^μ!

We will always assume that functions are smooth $f \in C^\infty(\mathcal{M})$ over some manifold $\mathcal{M}$.

Then a functional is $F[f]$; $F: C^\infty(\mathcal{M}) \to \mathbb{C}$

$C^\infty(M)$ might be too restrictive, but we will ignore this aspect here

$\Rightarrow$ to be more precise, $f \in S'(M)$

space of TEMPERED DISTRIBUTIONS

In any case, as long as we can perform standard operations on these functions, we can use them to define $\begin{cases} \text{FUNCTIONAL DIFFERENTIATION and} \\ \text{FUNCTIONAL INTEGRATION} \end{cases}$

FUNCTIONAL DIFFERENTIATION :

"Intuitive" definition $(x, y \in \mathbb{R}^d)$

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\varepsilon \to 0} \left(\frac{F[f(x) + \varepsilon \delta^{(d)}(x-y)] - F[f(x)]}{\varepsilon} \right)$$

such that

- $$\frac{\delta f(x)}{\delta f(y)} = \delta^{(d)}(x-y)$$

- if $F[f] = \int_{\mathbb{R}^d} f(x)\, d^dx$ then

$$\frac{\delta F[f]}{\delta f(y)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \int \left(f(x) + \varepsilon\, \delta^{(d)}(x-y) \right) d^dx - \int f(x)\, d^dx \right\}$$

$$= \int d^dx\; \delta^{(d)}(x-y) = 1$$

- if $F_x[f] = \int d^dy\; G(x,y)\, f(y)$

$$\frac{\delta F_x[f]}{\delta f(z)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \int d^dy\; G(x,y) \left[f(y) + \varepsilon\, \delta^{(d)}(y-z) \right] - \int d^dy\; G(x,y)\, f(y) \right\} = G(x,z)$$

From these we see that an equivalent definition is also

$$\delta F[f] = \int_M d^d x \, \frac{\delta F}{\delta f(x)} \, \delta f(x)$$

which generalizes $dF = \frac{\partial f}{\partial x_i} dx_i$

- $F[f] = \int d^d x \, f^n(x)$

$$\frac{\delta F[f]}{\delta f(y)} = \int d^d x \; n \, f^{(n-1)}(x) \, \delta^{(d)}(x-y)$$

$$= n \, f^{(n-1)}(y)$$

- $F[f] = \int d^d x \, (\partial_\mu f)(\partial^\mu f)$ { can be seen integrating by parts ! }

$$\frac{\delta F[f]}{\delta f(y)} = -2 \, \partial_\mu \partial^\mu f(y) = -2 \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y_\mu} f(y)$$

FUNCTIONAL INTEGRATION

Functional integration is what we used to define our PATH INTEGRAL

$$\int [\mathcal{D}f]\, F[f] \overset{\text{def}}{=} \lim_{N\to\infty} \left[\prod_{i=1}^{N} \int df_i \; F[f_1, \dots, f_N] \right]$$

where $f_i = f(x_i)$ on DISCRETIZED SPACE-TIME (LATTICE!)

we will be interested in performing explicitly GAUSSIAN-LIKE integrals $\Rightarrow$ we would like to generalize

$$\int_{-\infty}^{+\infty} dx \; e^{-ax^2/2} = \left[\frac{2\pi}{a} \right]^{1/2}$$

to functionals! (with $a > 0$)

Let's start with usual generalization to $\mathbb{R}^n$

$\Rightarrow$ product of n such integrals

$$\prod_{i=1}^{n} \left[\int_{-\infty}^{+\infty} dx_i \, e^{-\frac{a_i x_i^2}{2}} \right] = \int d^n x \, e^{-\frac{1}{2} x^T A x}$$

where $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$

$$\begin{cases} A = n \times n \text{ diagonal matrix with } a_i \text{ on DIAGONAL} \Rightarrow a_i \text{ eigenvalues !} \\ \det A = \prod_{i=1}^{n} a_i \end{cases}$$

then clearly I can use this to write

$$\int d^n x \, e^{-\frac{x^T A x}{2}} = \frac{(2\pi)^{n/2}}{\left[\prod_{i=1}^{n} a_i^{1/2}\right]} = \frac{[2\pi]^{n/2}}{\sqrt{\det A}} = \left[\det \frac{A}{2\pi}\right]^{-\frac{1}{2}}$$

We can generalize this in order:

1] to quadratic forms $Q(x) = \frac{1}{2} x^T A x + b \cdot x + c$

by completing the square:

$$Q(x) = Q(\bar{x}) + \frac{1}{2} [x - \bar{x}]^T \cdot A \cdot [x - \bar{x}] \qquad \bar{x} = -A^{-1} \cdot b$$

in fact $\partial_i Q = \frac{1}{2} [Ax]_i + \frac{1}{2} (A^T x)_i + b_i = 0$

$$= [Ax]_i + b_i = 0 \qquad (A^T = A \,!)$$

$$x = -A^{-1} \cdot b \quad \text{minimum} \,!$$

$Q(\bar{x}) = -\frac{1}{2} b^T A^{-1} b + c$ such that

$$\int d^n x \; e^{-Q(x)} = e^{\left[\frac{1}{2} b^T A^{-1} b - c\right]} \left(\det\left[\frac{A}{2\pi}\right]\right)^{-\frac{1}{2}}$$

after shifting $x - \bar{x} \equiv x$

2] Similarly we can generalize this to HERMITIAN (complex) matrices, noticing that

$$\int_{\mathbb{R}^2} e^{-a(x^2+y^2)/2}\, dx\, dy = \frac{2\pi}{a}$$

$\hookrightarrow$ $z = x + iy \Rightarrow \int e^{-a z^* z} \frac{dz^*}{\sqrt{2\pi i}} \frac{dz}{\sqrt{2\pi i}} = \frac{1}{a}$

and then if A is POSITIVE DEF. HERMITIAN MATRIX

$$\int e^{-z^\dagger A z} \frac{d^n z}{[2\pi i]^{n/2}} \frac{d^n z^*}{[2\pi i]^{n/2}} = \frac{1}{[\det A]}$$

All these formulas are rigorously correct. With some faith, we can generalize them to FUNCTIONALS

$$\int [\mathcal{D}\phi] \, e^{-\frac{1}{2}\int d^4x \, \phi(x) A \phi(x)} = \left[\det \frac{A}{2\pi}\right]^{-1/2}$$

NOTE :

this "2π" would be multiplied out an infinite # of times

normalization will go away !

$\det A \rightarrow \prod_n \lambda_n$ product of eigenvalues

$$\int [\mathcal{D}\phi][\mathcal{D}\phi^*] \, e^{-\int \phi^*(x) A \phi(x) \, d^4x} = \left[\det \frac{A}{2\pi i}\right]^{-1}$$

the $(2\pi i)$ or (2π) factors will go away

$$\langle \Omega | O_A(t_A) O_B(t_B) \ldots | \Omega \rangle = \frac{\int [\mathcal{D}\phi][\mathcal{D}\pi] \, e^{S} \, O_A O_B}{\int [\mathcal{D}\phi][\mathcal{D}\pi] \, e^{S}}$$

RATIO MEASURES!

Now that we have rules to perform integrals, let us see if we can use them.

First question is, when is it the case that in PATH INTEGRAL

$$e^{i\int d^4x\,(\pi\dot{\phi}-\mathcal{H})} \longrightarrow e^{i\int d^4x\,\mathcal{L}}$$

$\uparrow$ LAGRANGIAN

remember $i\varepsilon$ hidden in $\mathcal{H}$: $m^2 \to m^2 - i\varepsilon$

to do that, we must be able to integrate out EXPLICITLY $[\mathcal{D}\pi(x)]$ $\Rightarrow$ we can easily do it if it's a Gaussian Integral!

$$\int [\mathcal{D}\pi(x)]\, e^{i\int d^4x\,(\pi(x)\dot{\phi}(x)-\mathcal{H}(x))}$$

And $\Theta_A(t_A)\,\Theta_B(t_B)$ DON'T DEPEND on $\pi(x)$!

Let's ASSUME
- $\mathcal{H}$ quadratic form in π
- $\Theta_A(t_A)$ don't depend on π

then, following previous notation we write quadratic part

$$\sim -\frac{1}{2} \int d^4x \, d^4y \;\; i \, A[\phi(x)] \, \Pi(x) \Pi(y)$$

with $A[\phi(x)] \propto \delta^{(4)}(x-y)$

If this is the case, we have

$$\int [\mathcal{D}\Pi(x)] \, e^{i \int d^4x \left(\Pi(x) \dot{\phi}(x) - \mathcal{H}(x) \right)} =$$

$$= \left[\det \left(2\pi i \, A[\phi(x)] \right) \right]^{-\frac{1}{2}} \exp\left[i \int d^4x \left(\bar{\Pi} \dot{\phi} - \mathcal{H}(\phi, \bar{\Pi}) \right) \right]$$

where we also used the usual normalization

$$\int \frac{dp}{2\pi} \to \int \frac{[\mathcal{D}\Pi(x)]}{2\pi}$$

which adds an extra $\frac{1}{2\pi}$ for $\mathcal{D}\Pi$ vs $\mathcal{D}\phi$!

$\bar{\Pi}(x)$ is, as for finite-dimensional case, the stationary point of the quadratic form

$$\frac{\delta}{\delta \Pi(y)} \int d^4x \left[\Pi(x)\dot{\phi}(x) - \mathcal{H}(\phi, \Pi) \right] =$$

$$\dot{\phi}(y) - \frac{\partial \mathcal{H}}{\partial \Pi(y)} = 0$$

which fixes $\bar{\Pi}(x)$ as a function of x^μ

we recognize $\dot{\phi}(y) = \dfrac{\partial \mathcal{H}}{\partial \bar{\Pi}(y)}$ as one of the HAMILTON EQUATIONS

which implies $\bar{\Pi}(x) = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$ is the canonical conjugate momentum!

$\Rightarrow$ the effect of the GAUSSIAN integral over $\Pi(x)$ is that of performing a LEGENDRE TRANSFORM:

$$\int [\mathcal{D}\pi(x)] \, e^{i\int d^4x \, (\pi(x)\dot{\phi}(x) - \mathcal{H}(x))}$$

$$= \left[\det\left(2\pi i \, A[\phi(x)]\right)\right]^{-\frac{1}{2}} \exp\left[i\int d^4x \, \mathcal{L}\left(\phi(x), \partial_\mu\phi(x)\right)\right]$$

such that we can write

$$\langle\Omega| T\{ O_A(t_A) O_B(t_B) \dots \} |\Omega\rangle =$$

$$= \frac{\int [\mathcal{D}\phi] \left[\det(\cancel{2\pi i} \, A(\phi))\right]^{-1/2} O_A O_B \dots e^{i\int d^4x \, \mathcal{L}}}{\int [\mathcal{D}\phi] \left[\det(\cancel{2\pi i} \, A(\phi))\right]^{-1/2} e^{i\int d^4x \, \mathcal{L}}}$$

$(2\pi i)$ - factor cancels out

$\det[A(\phi)]$ cancels ONLY if $A(\phi(x))$ does not depend on field $\phi(x)$!

this is typical : FREE SCALAR THEORY + INT

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + \mathcal{L}_{INT}$$

$\hookrightarrow \propto \phi^n \quad n > 2$

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2 - \mathcal{L}_{INT}$$

$A[\phi(x)] = \delta^{(4)}(x-y)$ and then $\det A$ = number

it could happen $A[\phi(x)]$ not just $\delta^{(4)}$!

then we will use [AQFT ⚠]

$$[\det A]^{-\frac{1}{2}} = \left(e^{\operatorname{tr} \ln A}\right)^{-\frac{1}{2}} = e^{-\frac{1}{2}\underbrace{\operatorname{tr} \ln A}}$$

this can be reabsorbed into $\mathcal{L}$!

We will not need this in QFT 1, so we remove it! but keep it in mind !

The LAGRANGE form of the path integral generates the intuitive picture:

$$\langle q_f ; t_f | q_i ; t_i \rangle \propto \int \mathcal{D}[q(t)]\, e^{i\int_{t_i}^{t_f} \mathcal{L}(q,\dot{q})}$$

$$= \int \mathcal{D}[q(t)]\, e^{i \underbrace{S[q(t)]}_{\text{ACTION}}}$$

$\Rightarrow$ transition amplitude can be calculated "summing" over all trajectories, WEIGHED by the phase factor e^{iS}

Putting back factors $\hbar$ we get

$$\langle q_f\, t_f | q_i\, t_i \rangle \propto \int \mathcal{D}[q(t)]\, e^{\frac{i}{\hbar} S[q(t)]}$$

CLASSICAL LIMIT : $\hbar \to 0$ integral dominated by $\delta S = 0$ $\Rightarrow$ method of STATIONARY PHASE !

Now how do we compute this? If $\mathcal{L}$ is Free Lagrangian $\mathcal{L}_0$, we should be able to do these integrals since $\mathcal{L}_0$ is QUADRATIC in ϕ and we end up again with GAUSSIAN INTEGRALS

$\Rightarrow$ we will see how to do this in a moment.

if instead $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{INT}$ typically integral will be IMPOSSIBLE $\Rightarrow$ one solution is PERTURBATION THEORY as long as $\mathcal{L}_{INT}$ is "small"

Take finite-dimensional example

$$\int_{-\infty}^{+\infty} dq\, e^{\overbrace{-\frac{1}{2}m^2q^2}^{\mathcal{L}_0}\, \overbrace{-\lambda q^4}^{\mathcal{L}_{INT}}} = \int_{-\infty}^{+\infty} dq\, e^{-\frac{m^2}{2}q^2}\left[1 - \lambda q^4 + \frac{\lambda^2}{2}q^8 + \dots\right]$$

we can compute all ints one by one, or use a Trick:

DEFINE the generating functional

$$Z[J] = \int_{-\infty}^{+\infty} dq \; e^{-\frac{m^2}{2}q^2 - \lambda q^4 + Jq}$$ with

$$Z_0[J] = \int_{-\infty}^{+\infty} dq \; e^{-\frac{m^2}{2}q^2 + Jq} = \sqrt{\frac{2\pi}{m^2}} \; e^{+\frac{J^2}{2m^2}}$$

the FREE GENERATING FUNCTIONAL

then it's immediate to see that

$$\int_{-\infty}^{+\infty} dq \; e^{-\frac{m^2}{2}q^2} [q]^n = \left[\frac{d^n}{dJ^n} \int_{-\infty}^{+\infty} dq \; e^{-\frac{m^2}{2}q^2 + Jq}\right]_{J=0}$$

which implies

$$Z[J] = e^{-\lambda\left[\frac{d}{dJ}\right]^4} Z_0[J]$$

$$= \sqrt{\frac{2\pi}{m^2}} \; e^{-\lambda\left[\frac{d}{dJ}\right]^4} e^{\frac{J^2}{2m^2}}$$

$\Rightarrow$ this allows us to obtain correlators in interacting theory as sums of correlators in free theory with "source J"

GREEN FUNCTIONS OF FREE THEORY

We do the same in QFT. Consider a Scalar theory with

$$\mathcal{L} = \underbrace{\mathcal{L}_0}_{\text{Free}} + \underbrace{\mathcal{L}_{INT}}_{\text{interaction}}$$

where $\mathcal{L}_0 = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{m^2}{2}\phi^2$

Define the GENERATING FUNCTIONAL for Free Green functions

$$Z_0[J] = |N|^2 \int [D\phi] \exp\left\{ i \int d^4x \left(\mathcal{L}_0 + J(x)\phi(x) \right) \right\}$$

then using:

$$\frac{1}{i} \frac{\delta}{\delta J(y)} e^{i\int d^4x\, J(x)\phi(x)} = \phi(y)\, e^{i\int d^4x\, J(x)\phi(x)}$$

it's easy to see that

$$\left[\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right]\cdots\left[\frac{1}{i}\frac{\delta}{\delta J(x_n)}\right] Z_0[J]\Big|_{J=0} =$$

$$= |N|^2 \int [D\phi]\, \phi(x_1)\cdots\phi(x_n)\, e^{iS_0[\phi]}$$

$$= \frac{\int [D\phi]\, \phi(x_1)\cdots\phi(x_n)\, e^{iS_0[\phi]}}{\int [D\phi]\, e^{iS_0[\phi]}}$$

$$= \langle \Omega | T\{\phi(x_1)\cdots\phi(x_n)\} | \Omega \rangle$$

$$= G_0(x_1, \dots, x_n)$$ n-point (FREE) Green Function

We defined $S_0[\phi] = \int d^4x\, \mathcal{L}_0$ Free Action!

So if we can get $Z_0[J]$, we are done

$\Rightarrow$ we can obtain all green functions by differentiation

Indeed, $Z_0[J]$ can be evaluated explicitly since it's a GAUSSIAN INTEGRAL : to make it manifest let's write it as :

$$Z_0[J] = |N|^2 \int [\mathcal{D}\phi] \exp\left\{ -\frac{1}{2} \int d^4x\, d^4y\, \phi(x)\, D(x,y)\, \phi(y) - \int d^4x \left(-iJ(x)\right) \phi(x) \right\}$$

where $D(x,y) = i\, \delta^4(x-y) \left(\frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y_\mu} + m^2 \right) = i\, \delta^4(x-y) \left(\Box_y + m^2 \right)$

$m^2 \to m^2 - i\varepsilon$ remember!

Using now formula for GAUSSIAN INTEGRAL on page 7 with $A \to D(x,y)$ $B = -iJ(x)$ we get

$$= \underbrace{|N|^2 \det\left[\frac{D(x,y)}{2\pi}\right]^{-\frac{1}{2}}}_{\text{this is nothing but } Z_0[J=0]\cdot|N|^2 = 1} \exp\left\{ \frac{1}{2} \int d^4x\, d^4y \left(-iJ(x)\right) D^{-1}(x,y) \left(-iJ(y)\right) \right\}$$

this is so because $\frac{1}{|N|^2} = \frac{1}{Z_0[J=0]}$! check def !

then we are left with

$$Z_0[J] = \exp\left\{-\frac{1}{2}\int d^4x\, d^4y\; J(x)\, \Delta_F(x-y)\, J(y)\right\}$$

with $\Delta_F(\underbrace{x-y}) =$ inverse of $D(x,y)$

we anticipated that it only depends on $x-y$
not on x & y separately !

$\Delta_F(x-y)$ is called FEYNMAN PROPAGATOR

Let's compute it explicitly : D^{-1} defined as

$$\int d^4z\; i\,\delta^{(4)}(x-z)\,(\Box_z + m^2)\,D^{-1}(z,y) = \delta^4(x-y)$$

$$(\Box_x + m^2)\, i\,\Delta_F(x-y) = \delta^{(4)}(x-y)$$

Fourier transforming

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \tilde{\Delta}_F(p)\, e^{-ip\cdot(x-y)}$$

$$\delta^4(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}$$ we get

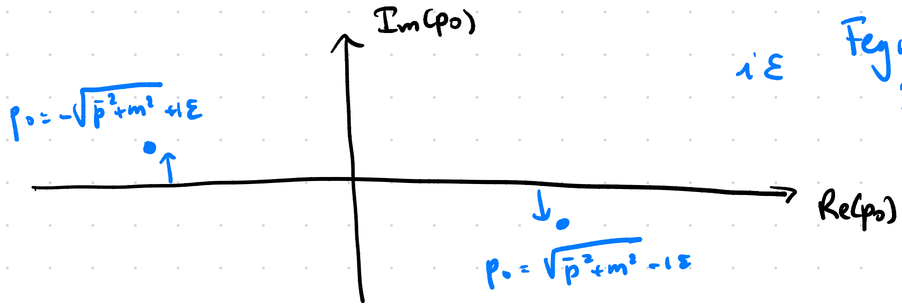
$$i[-p^2+m^2]\tilde{\Delta}_F(p) = 1 \Rightarrow \tilde{\Delta}_F(p) = \frac{i}{p^2-m^2+i\varepsilon}$$

re-instating the $i\varepsilon$ coming from $T \to \infty(1-i\varepsilon)$!

so

$$\boxed{\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2-m^2+i\varepsilon}}$$

$i\varepsilon$ regulates singularity at $p^2=m^2$!



$i\varepsilon$ Feynman Prescription