Path Integral Quantization 1: From QM to QFT

QFT
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We are finally in the position of quantizing INTERACTING THEORIES. We will deal with two problems

1. What happens when we add an INTERACTION term
2. How do we make contact with OBSERVABLES that can be measured in EXPERIMENTS

Our goal will be to compute matrix elements

$$\propto \langle \Omega | \phi(x_1) \dots \phi(x_n) | \Omega \rangle \Rightarrow \text{GREEN FUNCTIONS}$$

to do that, we will introduce a new formalism:

THE PATH INTEGRAL

We will do this first for QUANTUM MECHANICS and then generalize it to QFT. This will provide us a powerful technique to

- Understand how to quantize interacting theories
- Quantize gauge theories like QED $\Rightarrow U(1)$
- Generalize everything to NON ABELIAN theories

the path integral representation of QM was introduced by Feynman (1942) based on previous work by Dirac. We will use this formulation to compute GREEN FUNCTIONS and then we will see how to relate them to the QFT version of SCATTERING AMPLITUDES

the IDEA is that we "forget" about commutation relations, and instead put at the CENTER the notion of a PROPAGATOR: $K(q_f, t_f ; q_i, t_i)$ which provides the TIME EVOLUTION of the wave function $\psi(q,t)$ following Huygen's principle

IN 1 DIMENSION:

$$\psi(q_f, t_f) = \int \underbrace{K(q_f t_f ; q_i t_i)}_{\text{TRANSITION probability}} \psi(q_i, t_i) \, dq_i$$

probability amplitude for q_f at t_f

probability ampl for q_i at t_i

System defined by Q_n , P_n operators

$Q_n \to$ coordinates, $P_n \to$ momenta (N dimensions)

$[Q_n, P_m] = i\hbar\,\delta_{n,m}$ and zero otherwise

$|q\rangle$ eigenstates of Q_n ; $Q_n|q\rangle = q_n|q\rangle$

$|p\rangle$ eigenstates of P_n ; $P_n|p\rangle = p_n|p\rangle$

$\Longrightarrow$ ALL in SCHRÖDINGER PICTURE

$$\boxed{\begin{array}{ll} \langle q'|q\rangle = \prod_n \delta(q_n'-q_n)\,; & \mathbb{1} = \int \prod_n dq_n\,|q\rangle\langle q| \\ \langle p'|p\rangle = (2\pi)^N \prod_n \delta(p_n'-p_n)\,; & \mathbb{1} = \int \left(\prod_n \frac{dp_n}{2\pi}\right)|p\rangle\langle p| \\ \langle q|p\rangle = e^{ip\cdot q/\hbar} & \underbrace{|q;t\rangle_S = e^{-iHt/\hbar}|q\rangle}_{\text{EVOLUTION}} \end{array}}$$

in HEISENBERG PICTURE $Q_n(t)$; $P_n(t)$

$$Q_n(t) = e^{iHt/\hbar}\,Q_n\,e^{-iHt/\hbar}\,; \quad |q;t\rangle_H = e^{+iHt/\hbar}|q;t\rangle_S$$

States in Heisenberg picture fulfill the same orthonormability & completeness relations as in Schrödinger Picture

Now we consider EVERYTHING in HEISENBERG PICTURE since

$$ {}_H\langle q\, t \,|\, \psi \rangle_H = {}_S\langle q\, t \,|\, e^{-iHt}\, e^{+iHt} \,|\psi\rangle_S $$

expectation values don't change in two pictures!

NOTE: $|q, t\rangle_H$ might look like evolved state @ t

it's NOT! We are in Heisenberg pictures, states don't evolve with time

$\Rightarrow$ it's EIGENVECTOR of $Q_n(t)$ with EIGENVALUE q_n

Its time "dependence" is $e^{iHt}|q\rangle$

opposite of Schrödinger evolution!

then consider the WAVE FUNCTION.

$$\psi(q_f,t_f) = {}_H\langle q_f; t_f | \psi\rangle_H = \int \prod_n dq_{i,n} \langle q_f t_f | q_i t_i\rangle \langle q_i; t | \psi\rangle_H$$

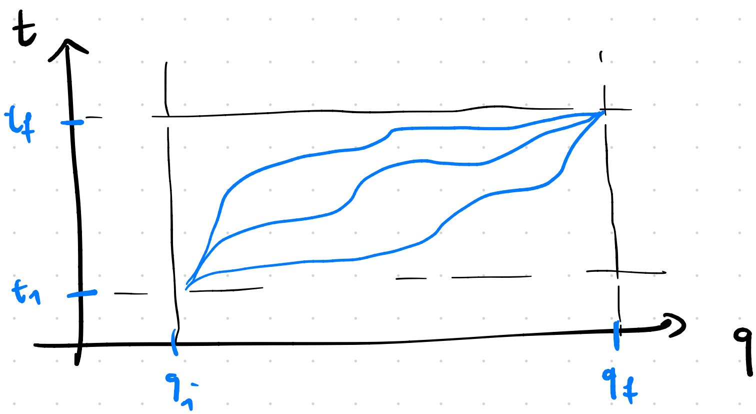
or if we are in 1 dimension:

$$\psi(q_f,t_f) = \int dq_i \, \langle q_f; t_f | q_i; t_i\rangle \langle q_i; t_1 | \psi\rangle_H$$

$$= \int dq_i \, \underbrace{\langle q_f t_f | q_i t_i\rangle}_{\text{"K"}} \, \psi(q_i, t_1) \qquad (*)$$

"K" ⇒ propagator contains all dynamics and SUBSTITUTES Schrödinger Eq!

We want now an expression for K ⇒ we will express it as a PATH INTEGRAL



sum over all possible PATHS

to compute this, notice first that

$${}_H\langle q_f\, t_f | q_i\, t_i \rangle_H = {}_S\langle q_f | e^{-iH(t_f - t_i)/\hbar} | q_i \rangle_S$$

↑ STATES in SCHRÖDING.

$\mathcal{H} = H(Q_n, P_m)$ chosen such that all Q_n are always on LEFT of P_m!

With this, consider infinitesimal $\Delta t \to dt$

$$\langle q', t+dt | q, t \rangle = \langle q' | e^{-iHdt/\hbar} | q \rangle$$

insert "$|p\rangle\langle p|$" $= \mathbb{1}$

forces all p to RIGHT!

$$= \int \prod_n \frac{dp_n}{2\pi} \underbrace{\langle q' | e^{-iH(Q_n, P_n)dt}}_{} | p \rangle \langle p | q \rangle \qquad (\hbar = 1)$$

$e^{-iH(q_n', p_n)dt}$ becomes a number

BECAUSE $dt \ll 1$

$e^{-iHdt} \sim 1 - iHdt$

LINEAR only!

$$= \int \prod_n \frac{dp_n}{(2\pi)} e^{-iH(q_n', p_n)dt} \quad \underbrace{\langle q'|p\rangle\langle p|q\rangle}_{e^{i\sum_n (q_n'-q_n)p_n}}$$

$$= \int \prod_n \frac{dp_n}{(2\pi)} e^{-iH(q_n', p_n)dt + i\sum_n (q_n'-q_n)p_n}$$

this is true for infinitesimal dt

Now go back to full propagator, and split

$\Delta t = t_f - t_i$ into $N+1$ intervals $dt = t_{j+1} - t_j$

$$= \frac{t_f - t_i}{N+1}$$

then

$$\langle q_f t_f | q_i t_i\rangle = \int \prod_n dq_{1n} \dots \prod_n dq_{Nn} \langle q_f t_f | q_N t_N\rangle \dots$$

$$\cdot \langle q_1 t_1 | q_i ; t_i\rangle$$

and using infinitesimal formula above

$$= \int \prod_{k=1}^{N} \left[\prod_n dq_{kn} \right] \prod_{k=0}^{N} \left[\prod_n \frac{dp_{kn}}{2\pi} \right] \cdot$$

$$\cdot \exp\left\{ i \sum_{k=1}^{N+1} \left[\sum_n (q_{k,n} - q_{k-1,n}) p_{k-1,n} - H(q_{k,n}, p_{k-1,n})\, dt \right] \right\}$$

$$q_0 = q_i \,; \quad p_0 = p_i$$

$$q_{N+1} = q_f \,; \quad p_{N+1} = p_f$$

now we take limit $N \to \infty$, $dt \to 0$, then variables become continuous functions of time

$$q_{kn} \to q_n(t) \qquad p_{kn} \to p_n(t)$$

$$q_{kn} - q_{k-1,n} \to \dot{q}_n(t)\, dt$$

$$\sum_{k=1}^{N+1} dt \to \int_{t_i}^{t_f} dt$$

what about integrals in dq_{kn}, dp_{kn} ?

$$\int \prod_{k=1}^{N} \left[\prod_n dq_{kn}\right] \prod_{k=0}^{N} \left[\prod_n \frac{dp_{kn}}{2\pi}\right] \longrightarrow$$

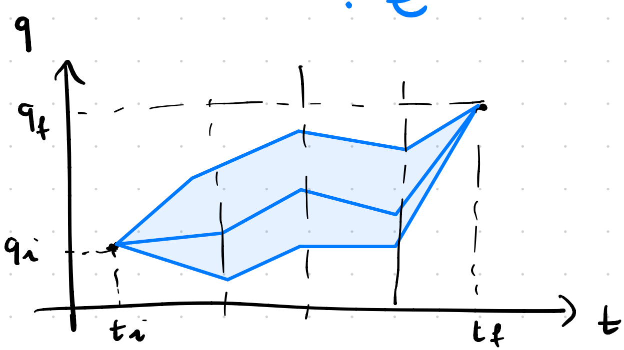
FORMAL DEFINITION

$$\to \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}[q_n(t)]\, \mathcal{D}[p_n(t)]$$

integral over all FUNCTIONS p_n, q_n

$\Rightarrow$ PATH INTEGRAL with fixed initial and final points

$$\langle q_f; t_f | q_i; t_i \rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}[q_n(t)]\, \mathcal{D}[p_n(t)] \cdot e^{i\int_{t_i}^{t_f} dt \left(\sum_n \dot{q}_n(t) p_n(t) - H(q_n(t), p_n(t))\right)}$$



$\forall\, t_k$ we integrate over all $q_n \Rightarrow$ all trajectories !

this is the HAMILTONIAN FORM of the PATH INTEGRAL formulation of Quantum Mechanics

We can generalize to FIELDS with usual RECIPE:

SCALAR FIELD (1 real field for simplicity)

$|q; t\rangle \to |\varphi; t\rangle;$ $\quad \hat{\phi}_H(x)\,|\phi; t\rangle_H = \phi(\vec{x})\,|\phi, t\rangle_H$

$$H = \int d^3\vec{x}\ \mathcal{H}(\phi(x), \pi(x))$$ HAMILTONIAN density

now we can compute quantum transition amplitude from $\phi_i(\vec{x})$ at t_i to $\phi_f(\vec{x})$ at t_f

$$\langle \phi_f t_f | \phi_i t_i \rangle = \int_{\substack{\phi(t_i, \vec{x}) = \phi_i(\vec{x}) \\ \phi(t_f, \vec{x}) = \phi_f(\vec{x})}} \mathcal{D}[\phi(x)]\,\mathcal{D}[\pi(x)]$$

$$\cdot \exp\left\{ i \int_{t_i}^{t_f} dt \int d^3\vec{x}\ \left(\pi(x)\dot{\phi}(x) - \mathcal{H}(\phi, \pi)\right)\right\}$$

where in this formula, as in QM case, we are summing only over CLASSICAL FIELD CONFIGURATIONS

$\Rightarrow$ every ϕ, π in path integral is a function; not an operator !

Now going back to QM case, we can use PATH INTEGRAL also to represent MATRIX ELEMENTS between $\langle q_f t_f | \ldots | q_i t_i \rangle$ of (TIME ORDERED) PRODUCTS OF OPERATORS

↑ technical requirement to be discussed –

Consider operator $O(\underbrace{P_n(t), Q_n(t)}) = O(t)$

Assume opposite convention to $\mathcal{H}$

i.e. all P_n LEFT of Q_n !

And consider the following matrix element

$$\langle q_f\, t_f |\, O_A(t_A)\; O_B(t_B) \;\ldots\; | q_i\, t_i \rangle \qquad \underline{t_f > t_A > t_B > \ldots > t_i}^{\text{TIME ORDERING}}$$

now divide $[t_f, t_i]$ interval into small $N+1$ "dt"

and insert completeness relations. We can easily do this if $t_f > t_A > t_B > \ldots > t_i$ (TIME ORDERING)

$$\langle q_f\, t_f |\, T\left\{ O_A(t_A)\; O_B(t_B) \;\ldots \right\} | q_i\, t_i \rangle$$

$$\Rightarrow \boxed{T\{\phi(x)\phi(y)\} = \theta(x_0 - y_0)\,\phi(x)\phi(y) + \theta(y_0 - x_0)\,\phi(y)\phi(x)}$$

then we must consider matrix element of single operator over infinitesimal time dt

$$\langle q';\, t + dt |\, O_A(P_n(t_A), Q_n(t_A))\, | q, t \rangle =$$

$$= \int \prod_n \frac{dp_n}{2\pi} \langle q' |\, e^{-iH(Q_n, P_n)dt} | p \rangle \underbrace{\langle p |\, O_A(P_n, Q_n)\, | q \rangle}_{\text{All } P_n \text{ to LEFT !}}$$

$$= \int \prod_n \frac{dp_n}{2\pi} \, e^{-iH(q_n', p_n)\, dt + i\sum (q_n' - q_n) p_n} \; \underbrace{O_A(p_n, q_n)}_{\text{no more operator!}}$$

Repeating this for all operators, and by inserting each operator at the "proper" place

$$\left[\text{if} \quad t_k < t_A < t_{k+1} \;\Rightarrow\; \langle q_{k+1} t_{k+1} | O_A(t_A) | q_k t_k \rangle \quad \text{etc} \right]$$

gives the natural generalization to matrix elements:

$$\langle q_f t_f | T\{ O_A(t_A) O_B(t_B) \ldots \} | q_i t_i \rangle =$$

$$\int \mathcal{D}[q_n(t)] \, \mathcal{D}[p_n(t)] \; e^{i \int_{t_i}^{t_f} dt \left[\sum_n \dot{q}_n(t) p_n(t) - H(q_n(t), p_n(t)) \right]}$$

$$\cdot \, O_A(p_n(t_A), q_n(t_A)) \, O_B(p_n(t_B), q_n(t_B)) \ldots$$

which in turn we can further generalize to <u>FIELDS</u>

When dealing with fields, we would like to consider matrix elements among VACUUM STATES $\langle\Omega|\dots|\Omega\rangle$

$\Rightarrow$ what we called a Green function

Inserting a complete set of states $|\phi_f, t_f\rangle$ and $|\phi_i, t_i\rangle$ and sending $t_{\substack{i\\f}} \to \mp\infty$ we cover ALL TIMES

$$\langle\Omega|T\{O_A(t_A)\,O_B(t_B)\dots\}|\Omega\rangle =$$

$$= \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \int \prod_{\vec{x}} d\phi_i(\vec{x}) \prod_{\vec{x}} d\phi_j(\vec{x}) \, \langle\Omega|\phi_f, t_f\rangle \cdot \underbrace{\langle\phi_f t_f|T\{O_A(t_A)\,O_B(t_B)\dots\}|\phi_i, t_i\rangle}\langle\phi_i t_i|\Omega\rangle$$

use generalization to fields of previous formula

$$= \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \int \mathcal{D}[\phi(x)]\, \mathcal{D}[\pi(x)]\, \langle \Omega | \phi_f, t_f \rangle \langle \phi_i, t_i | \Omega \rangle$$

$$\cdot\ e^{i \int d^3x \int_{t_i}^{t_f} dt\, \left[\pi(x) \dot{\phi}(x) - \mathcal{H}(\phi(x), \pi(x)) \right]}$$

$$\cdot\ O_A(t_A)\, O_B(t_B) \dots$$

note we have included in $\underline{\mathcal{D}[\phi(x)]\mathcal{D}[\pi(x)]}$ $\left(= [\mathcal{D}\phi][\mathcal{D}\pi]\right)$ also the integral over $d\phi_i\ d\phi_j$ $\Rightarrow$ we are integrating over ALL FIELDS CONFIGURATIONS without any constraints anymore !

As a next step, we take $t_i \to -\infty$, $t_f \to +\infty$ explicitly

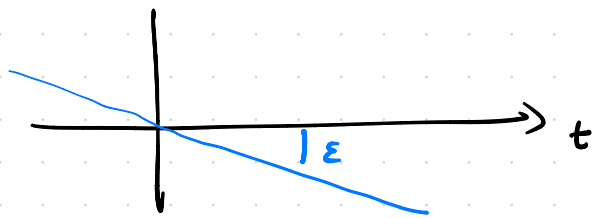
We ASSUME that at such times, $\phi(x)$ behaves as a FREE FIELD $\Rightarrow$ like in scattering theory in QM we will say more about this when we study SCATTERING.

For now, you can think about it as in QM

then $\langle \phi ; \mp\infty | \Omega \rangle$ = wave function of vacuum state in coordinate represent.

this limit is delicate. One can show that

We need to take limit as $T \to \infty(1 - i\varepsilon)$



tilting the axis along which we go to ∞

$$\langle \Omega | \phi ; +\infty \rangle \langle \phi ; -\infty | \Omega \rangle =$$

$$= |N|^2 \exp\left\{ \frac{1}{2} \varepsilon \int d^4x \, [\dots] \right\}$$

same effect of $= |N|^2 \exp\left\{ \frac{1}{2} \varepsilon \int d^4x \, \phi^2(x) \right\}$

this is like adding to Hamiltonian

$$\mathcal{H} \to \mathcal{H} - \frac{i\varepsilon}{2}\phi^2$$

ε PRESCRIPTION on the mass

$$\mathcal{H} \stackrel{!}{=} \frac{1}{2}\left[(\partial_0\phi)^2 + (\vec{\nabla}\phi)^2 + (m^2 - i\varepsilon)\phi^2\right]$$

I can try to argue "handwavingly" about this:

consider $\langle\Omega|\phi;t\rangle$ in SCHRÖDINGER PICTURE

$$|\phi;t\rangle_s = e^{-iHt}|\phi\rangle_s$$

$${}_s\langle\phi;t| = {}_s\langle\phi|\, e^{+iHt}$$

matrix element does not change

evolve them back to some $t = t_0$

and put $t_f = T$ $t_i = -T$; $T \to \infty$ then

$$\langle\Omega|\phi_f;T\rangle = \langle\Omega|\, e^{-iHT}|\phi_f\rangle$$

$$\langle\phi_i;-T|\Omega\rangle = \langle\phi_i|\, e^{-iHT}|\Omega\rangle$$

now let me decompose initial and final state in terms of eigenstates of HAMILTONIAN

$$|\phi_i\rangle = \sum_n \langle n|\phi_i\rangle \, |n\rangle \quad \text{such that}$$

$$\langle \Omega|\phi_f, T\rangle = \sum_n \langle \Omega|n\rangle \langle n|\phi_f\rangle \, e^{-iE_nT}$$

I cannot just send $T \to \infty$; instead give it a small im part $T \to \infty(1-i\varepsilon)$ with $\varepsilon > 0$

$$e^{-iE_nT} \longrightarrow e^{-iE_n(\infty - i\varepsilon\cdot\infty)}$$

$$\sim e^{-iE_n\infty} \, e^{-\varepsilon\infty E_n} \to 0$$

$\Uparrow$

as long as $\varepsilon > 0$

the larger E_n, the FASTER $\to 0$ so this limit projects out only contribution from VACUUM [as long as $|\phi_f\rangle$ $|\phi_i\rangle$ have overlap with $|\Omega\rangle$]

$|0\rangle = |\Omega\rangle$ and we can write

$$\lim_{T\to\infty(1-i\varepsilon)} \langle\Omega|\phi_f T\rangle \longrightarrow \underbrace{\langle\Omega|\Omega\rangle}_{1}\cdot \mathcal{N}_f$$

phases and overlap $\langle\Omega|\phi_f\rangle$

$$\lim_{T\to\infty(1-i\infty)} \langle\phi_i, -T|\Omega\rangle \to \underbrace{\langle\Omega|\Omega\rangle}_{1}\cdot \mathcal{N}_i$$

so in this way we get

$$\langle\Omega|T\{O_A(t_A)\,O_B(t_B)\ldots\}|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)}$$

$$\mathcal{N}\int \mathcal{D}[\phi(x)]\,\mathcal{D}[\Pi(x)]\cdot O_A(t_A)\,O_B(t_B)\cdots$$

$$\cdot\, e^{i\int_{-T}^{T} d^4x\,\left[\Pi(x)\dot{\phi}(x) - \mathcal{H}(\phi(x),\Pi(x))\right]}$$

We can easily fix the constant $\mathcal{N}$ by imposing that our path integral representation is consistent with $\langle\Omega|\Omega\rangle = 1$ (vacuum normalized to 1)

Indeed, repeating the same argument without any operators $O_1(t_1)$ etc we get

$$\langle\Omega|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \mathcal{N} \int [D\phi][D\pi]\, e^{i\int_{-T}^{T} d^4x\,(\pi\dot{\phi}-\mathcal{H})}$$

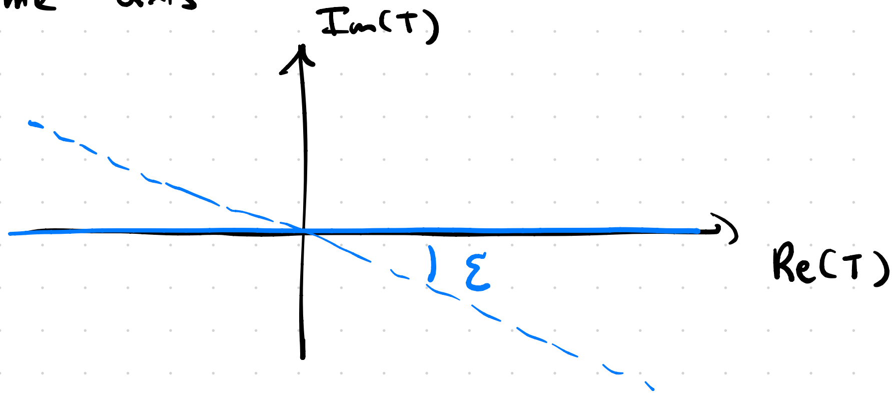
$$= 1$$

$$\Rightarrow \mathcal{N} = \frac{1}{\lim\limits_{T\to\infty(1-i\varepsilon)} \int [D\phi][D\pi]\, e^{i\int_{-T}^{T} d^4x\,(\pi\dot{\phi}-\mathcal{H})}}$$

As already anticipated:

$\lim\limits_{T\to\infty(1-i\varepsilon)}$? $\rightarrow$ $\lim\limits_{T\to\infty e^{-i\varepsilon}}$

we are taking the limit rotating a bit the Time axis



One can relatively easily show that this is equivalent to having added an appropriate imaginary part to the ORIGINAL HAMILTONIAN. Loosely:

$$e^{-iHT} \to e^{-iHTe^{-i\varepsilon}} = e^{-iH(1-i\varepsilon)T}$$

same effect as

$$= e^{-i[H-i\tilde{\varepsilon}]T}$$

$\varepsilon H \to \tilde{\varepsilon} > 0$ if energy positive

↳ IMPORTANT: this regulates also $\int_{-T}^{T} [\pi\dot{\phi} - \mathcal{H}]\, dt$!

this is WHAT MATTERS!

this effect in $\mathcal{H}$ can be obtained by adding

$$\mathcal{H} \to \mathcal{H} - \frac{i\varepsilon}{2}\phi^2$$ with some $\varepsilon > 0$

↑ scalar field

this term is a convenient implementation of our $T \to T(1-i\varepsilon)$ rule, as it can be seen as a simple addition of $-i\varepsilon$ to the m^2 appearing in the HAMILTONIAN. For free field

$m^2 \to m^2 - i\varepsilon$

$$\mathcal{H} = \frac{1}{2}\left[(\partial_0\phi)^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right]$$

indeed $m^2 \to m^2 - i\varepsilon$ corresponds $\mathcal{H} - \frac{i\varepsilon}{2}\phi^2$

remember: THIS IS JUST A PRESCRIPTION TO MAKE SENSE OF $T \to \infty$ LIMIT!

So our final formula reads in compact notation

$$\boxed{\langle\Omega| T\{ O_A(t_A)\, O_B(t_B) \cdots \} |\Omega\rangle = \frac{\int [D\phi][D\pi] \left(O_A(t_A)\, O_B(t_B) \cdots \right) e^{i\int d^4x \left[\pi \dot{\phi} - \bar{\mathcal{H}} \right]}}{\int [D\phi][D\pi]\, e^{i\int d^4x \left[\pi \dot{\phi} - \bar{\mathcal{H}} \right]}}}$$

where $\bar{\mathcal{H}} = \mathcal{H} - \frac{i\varepsilon}{2}\phi^2$

Now we might think $\pi\dot{\phi} - \mathcal{H} \sim \mathcal{L}$ LAGRANGIAN

this is delicate, because this is strictly true

only if $\pi = \dfrac{\partial \mathcal{L}}{\partial[\partial_0\phi]}$ $\Rightarrow$ in our formula

instead π is a FREE integration variable !

So in general that is NOT the LAGRANGIAN