Canonical Quantization of Free Spinor Field

+ comments on spin 1

(Double Lecture)

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As the next step, we consider the generalization of our previous discussion to FREE SPIN 1/2 DIRAC FIELDS

Recall the free Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \; ; \quad \Pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger$$

so $\psi^\dagger$ plays the role of the conjugate momentum

Following our previous discussion, we could be tempted to

- promote $\psi, \psi^\dagger$ to OPERATORS
- impose the following commutation relations

$$[\psi_a(t,\vec{x}), \psi^\dagger_b(t,\vec{y})] = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

SPINOR INDICES

no "i" because $\Pi = i\psi^\dagger$!

By expanding $\psi, \psi^\dagger$ in Fourier modes:

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \sum_{s=1,2} \Big[\underbrace{a_s(p) u_s(p) e^{-ip\cdot x} + b_s^\dagger(p) v_s(p) e^{ip\cdot x}}_{\text{creation \& annihilation operators}} \Big]$$

$$\bar{\psi}(x) = \int \frac{d^3\bar{p}}{(2\pi)^3 2E_p} \sum_{s=1,2} \Big[b_s(p) \bar{v}_s(p) e^{-ip\cdot x} + a_s^\dagger(p) \bar{u}_s(p) e^{ip\cdot x} \Big]$$

with the u_s, v_s computed in lecture 8.

We can then derive the corresponding commutation relations for the $a_s(p)$, $b_s(p)$ etc:

$$[a_s(p), a_r^\dagger(q)] = [b_s(p), b_r^\dagger(q)] = 2E_p (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \delta_{rs}$$

all remaining ones being zero

the problem with this, is that this not consistent with <u>RELATIVISTIC CAUSALITY</u> !

two manifestations of this fact :

1] Compute the HAMILTONIAN

$$H = \int d^3\vec{x} :\mathcal{H}: \quad \text{(normal ordering)}$$

$$\mathcal{H} = \Pi \partial_0 \psi - \mathcal{L} = i\psi^\dagger \partial_0 \psi - \bar{\psi}(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\psi$$

$$= \bar{\psi}(-i\gamma^i\partial_i + m)\psi$$

By substituting Fourier expansions & using commutation relations and SPIN SUMS for u_s, v_s we get

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \sum_{s=1,2} \left[E_p \left(a_s^\dagger(p) a_s(p) - b_s^\dagger(p) b_s(p) \right) \right]$$

minus sign is problematic

$\Rightarrow$ if we attempt same particle interpretation as for scalar field, $b_s^\dagger(p)$ create particle of NEGATIVE E_p !

2] Equivalently, compute commutator of two fields

at SPACE-LIKE separations

$$[\psi(x), \bar{\psi}(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \Big[u_s(p)\bar{u}_s(p)\, e^{-ip\cdot(x-y)} - v_s(p)\bar{v}_s(p)\, e^{ip\cdot(x-y)} \Big]$$

Use completeness relation for SPINORS $\sum u\bar{u} = \not{p} + m$, $\sum v\bar{v} = \not{p} - m$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \left((\not{p}+m)\, e^{-ip(x-y)} + (-\not{p}+m)\, e^{ip\cdot(x-y)} \right)$$

$$= (i\not{\partial}_x + m) \int \frac{d^3p}{(2\pi)^3 2E_p} \left(e^{-ip(x-y)} + e^{ip(x-y)} \right)$$

$$= (i\not{\partial}_x + m) \left[\Delta_+(x-y) + \Delta_-(x-y) \right] \neq 0 \; !$$

these are two manifestations of the <u>SPIN-STATISTICS</u>

<u>THEOREM</u> $\Rightarrow$ integer spins should commute <u>BOSONS</u>

half-integer spins should anticommute

$\Rightarrow$ FERMIONS !

the solution is therefore to impose ANTICOMMUTATION RELATIONS at equal time

$$\{\psi_a(t,\vec{x}),\psi_b^\dagger(t,\vec{y})\} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

$$\{a,b\} = ab + ba$$

spinor indices!

which translates into $\{\ \}$ relations for $a(p), b(p)$:

$$\{a_r(p), a_s^\dagger(q)\} = \{b_r(p), b_s^\dagger(q)\} = 2E_p(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})\,\delta_{rs}$$

with the other ANTICOMMUTATORS being zero.

With these relations we get

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \sum_{s=1,2} \left[E_p \left(a_s^\dagger(p) a_s(p) + b_s^\dagger(p) b_s(p) \right) \right]$$

positive energy from both $a^\dagger$ & $b^\dagger$!

Similarly, with these anticommutation relations we get

$$\{\psi_\alpha(x), \overline{\psi}_\beta(y)\} = (i\gamma^\mu\partial_\mu + m)_{\alpha\beta} \underbrace{[\Delta_+(x-y) - \Delta_+(y-x)]}_{}$$

$= 0$ for space-like separations!

We can then generate Hilbert space as for scalar case

$a_s(p)|\Omega\rangle = b_s(p)|\Omega\rangle = 0$ defines VACUUM

$a_s^\dagger(p)|\Omega\rangle$, $b_s^\dagger(p)|\Omega\rangle$ give particle & antiparticle of mom. $\vec{p}$ and SPIN $s = \pm\frac{1}{2}$

Acting multiple times I can generate multiparticle states & whole FOCK SPACE note that since $a_s^\dagger(p)$ & $b_s^\dagger(p)$ all anticommute

states are ANTISYMMETRIC => Fermi Statistics

Also $\underbrace{a_s^\dagger(p)\, a_s^\dagger(p)}\,|\Omega\rangle = 0$!

if same p & s ! PAULI EXCLUSION Principle

Let us also compute the momentum operator

$$\vec{P} = \int d^3\vec{x}\ \psi^\dagger\left(-i\vec{\nabla}\right)\psi = \dots$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \sum_{s=1,2} \left[\vec{p}\left(a_s^\dagger(p)\,a_s(p) + b_s^\dagger(p)\,b_s(p)\right)\right]$$

both $b_s^\dagger(p)$ & $a_s^\dagger(p)$ have E_p & $\vec{p}$

we call $a_s^\dagger(p)$ <u>FERMIONS</u>

$b_s^\dagger(p)$ <u>ANTIFERMIONS</u>

Finally, we can compute the ANGULAR MOMENTUM $\Rightarrow$

Noether Charge associated to SPATIAL ROTATIONS

Remember for full Lorentz group, we have two pieces: "ORBITAL + SPIN" (Lecture 4, 5)

$$J^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} G_{i,a} - \Theta^\mu_{\ \nu} Y^\nu_a$$

$$\begin{cases} \delta X^\mu = \varepsilon^a Y^\mu_a(x) \\ \Delta\phi_i = \varepsilon^a G_{i,a}(\phi, \partial\phi) \\ \Rightarrow \varepsilon^a = \omega^{\rho\sigma} \, ; \, Y^\mu_a = \left[\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho\right] \end{cases}$$

Orbital part comes from $G_{i,a} = 0$ [scalar field!]

$$J^\mu_{\rho\sigma} = -\left[T^\mu_{\ \rho} x_\sigma - T^\mu_{\ \sigma} x_\rho\right]$$

Lecture 5

for DIRAC we have $\Delta\phi_i = \Delta\psi = (1-\Lambda_D)\psi \neq 0$

$$\Lambda_D = e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \Rightarrow \boxed{\Lambda_{\substack{L\\R}} = e^{(-i\vartheta \mp \eta)\cdot\frac{\vec{\sigma}}{2}}}$$

$$(1-\Lambda_D) \simeq -\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}$$

$$= -\frac{i\vartheta}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} - \frac{\eta}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$$

then remember $\mathcal{L}_D = \bar{\psi}(i\not{\partial} - m)\psi$ depends only on $\partial_\mu\psi$ and not on $\partial_\mu\bar{\psi}$, so we find

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \frac{\partial}{\partial(\partial_\mu\psi)}\left[\bar{\psi}(i\not{\partial} - m)\psi\right] = \bar{\psi}i\gamma^\mu$$

such that the SPIN contribution is : (removing $\omega_{\rho\sigma}$)

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\left[-i\Sigma_{\rho\sigma}\right] = \bar{\psi}\gamma^\mu\Sigma_{\rho\sigma}\psi$$

[removed $\frac{1}{2}$ for double counting $\omega_{\mu\nu}$]

$$\Rightarrow\ j^\mu_{\rho\sigma} = \bar{\psi}\gamma^\mu \Sigma_{\rho\sigma}\psi \underbrace{-\, \partial^\mu_\nu Y^\nu_\alpha}_{-\left[T^\mu_\rho x_\sigma - T^\mu_\sigma x_\rho\right]}$$

ORBITAL PART

And we have $\boxed{\partial_\mu j^\mu_{\rho\sigma} = 0}$

From here, the conserved charge is

$$M_{\rho\sigma} = \int d^3\vec{x}\, j^0_{\rho\sigma} =$$

$$= \int d^3\vec{x} \left[\bar{\psi}\gamma^0 \Sigma_{\rho\sigma}\psi - \left(T^0_\rho x_\sigma - T^0_\sigma x_\rho\right)\right]$$

Let's now focus on SO(3) rotations $\Rightarrow M_{ij}$

And look at the spin part

$$M^{S}_{ij} = \int d^3\vec{x}\ \bar{\psi}\gamma^0 \Sigma_{ij}\psi$$

$$\Sigma_{ij} = \frac{i}{4}[\gamma_i, \gamma_j] \ ; \quad [\gamma_i, \gamma_j] = \gamma_i\gamma_j - \gamma_j\gamma_i$$

$$\gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} \qquad \gamma_i\gamma_j = \begin{bmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{bmatrix}$$

$$\Rightarrow [\gamma_i, \gamma_j] = -\begin{pmatrix} [\sigma_i, \sigma_j] & 0 \\ 0 & [\sigma_i, \sigma_j] \end{pmatrix}$$

$$= -2i\,\varepsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

so finally using $\bar{\psi}\gamma^0 = \psi^\dagger$

$$M^{S}_{ij} = \int d^3\vec{x}\ \frac{i}{4}(-2i)\,\varepsilon_{ijk}\ \psi^\dagger \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}\psi$$

$$M_{ij}^{s} = \frac{1}{2}\varepsilon_{ijk}\int d^3\vec{x}\, \psi^\dagger \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \psi$$

$$M_s^{ij} = \frac{1}{2}\varepsilon^{ijk}\int d^3\vec{x}\, \psi^\dagger \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \psi$$

Define as usual vector component $\quad S^l = \frac{1}{2}\varepsilon^{ijl} M_s^{ij}$

use $\varepsilon^{ijl}\varepsilon^{ijk} = 2\delta^{lk}$ to write

$$S^l = \frac{1}{2}\int d^3\vec{x}\, \psi^\dagger \begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix} \psi$$

if you had added also the ORBITAL PART :

$$J^l = \int d^3x\, \psi^\dagger \Big[\underbrace{\big(\vec{x} \times (-i\vec{\nabla})\big)^l}_{\text{ORBITAL}} + \underbrace{\frac{1}{2}\begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix}}_{\text{SPIN}} \Big] \psi$$

one can use this expression to prove that particles created by $a_s^+(p)$ & $b_s^+(p)$ have SPIN $\frac{1}{2}$.

Now let us consider the TRANSFORMATION LAW of one-particle states & fields under Poincaré.

In scalar case we had for $U(\Lambda, a)$:

$$U(\Lambda, 0)|p\rangle = |\Lambda^{-1}p\rangle$$
$$U(\mathbb{1}, a)|p\rangle = e^{-ip\cdot a}|p\rangle$$

now transformation law is richer, in particular

$$U(\Lambda, 0)|p, s\rangle = \# \sum_{s'} D_{ss'}(W_{\Lambda^{-1}p})\,|\Lambda^{-1}p, s'\rangle$$

normalization $\equiv 1$

see Lecture 4

$\sigma, \sigma' \to s, s'$

$\Lambda \to \Lambda^{-1}$

WIGNER ROTATION

$\boxed{\text{PASSIVE VIEW POINT}}$

and similarly we impose that for FULL POINCARE'

$$U(\Lambda, a) |p,s\rangle = e^{-ip\cdot a} \sum_{s'} D_{ss'}(W_{\Lambda^{-1},p}) |\Lambda^{-1}p, s'\rangle$$

then

$$|p,s\rangle = a_s^\dagger(p)|\Omega\rangle$$ and

$$U(\Lambda,a)|p,s\rangle = U(\Lambda,a)\, a_s^\dagger(p)\, U^{-1}(\Lambda,a)\, \underbrace{U(\Lambda,a)\, |\Omega\rangle}_{|\Omega\rangle}$$

vacuum invariant

$$= U(\Lambda,a)\, a_s^\dagger(p)\, U^{-1}(\Lambda,a)\, |\Omega\rangle$$

now impose $\stackrel{!}{=} e^{-ip\cdot a} \sum_{s'} D_{ss'}(W_{\Lambda^{-1},p}) |\Lambda^{-1}p, s'\rangle$

$$= e^{-ip\cdot a} \sum_{s'} D_{ss'}(W_{\Lambda^{-1},p})\, a_{s'}^\dagger(\Lambda^{-1}p)\, |\Omega\rangle$$

$$\Rightarrow \quad U(\Lambda,a)\, a_s^\dagger(p)\, U^{-1}(\Lambda,a) =$$

$$= e^{-ip\cdot a} \sum_{s'} D_{ss'}(W_{\Lambda^{-1}p})\, a_{s'}^\dagger(\Lambda^{-1}p)$$

similarly we get

$$U(\Lambda,a)\, a_s(p)\, U^{-1}(\Lambda,a) =$$

$$= e^{+ip\cdot a} \sum_{s'} \overbrace{D^{*}_{ss'}}^{\text{complex conjugated matrix}}(W_{\Lambda^{-1}p})\, a_{s'}(\Lambda^{-1}p)$$

and the very same laws apply to $b_s(p)$ & $b_s^\dagger(p)$!

Finally, we need transformation laws for $u(p,s)$
$v(p,s)$

u, v are SPINORS and one can prove that

$$u(p,s) \xrightarrow{\Lambda^{-1}} \sum_{s'} u(\Lambda^{-1}p, s')\, D_{s,s'}(W_{\Lambda^{-1}p})$$

$$v(p,s) \xrightarrow{\Lambda^{-1}} \sum_{s'} u(\Lambda^{-1}p, s')\, D_{s,s'}(W_{\Lambda^{-1}p})$$

with these, we can finally put everything together and derive transformation law for ψ

$$U(\Lambda,a)\,\psi(x)\,U^{-1}(\Lambda,a) = \Lambda_D\,\psi(\Lambda^{-1}(x-a))$$

$$\Rightarrow \underbrace{U(\Lambda,a)\,\psi(\Lambda x+a)\,U^{-1}(\Lambda,a) = \Lambda_D\,\psi(x)}_{\text{EXACTLY what we had in Lecture 4}} \quad !$$

writing $U\psi(x')U^{-1} = \underbrace{\psi'(x')}_{\text{TOTAL TRANSF OF FIELD}}$

EXERCISE

U(1) CHARGE

Finally, we can also compute the charge associated to the U(1) symmetry $\psi \to e^{i\alpha}\psi$

$$j^\mu = \bar{\psi}\gamma^\mu\psi \Rightarrow Q = \int d^3x\, \bar{\psi}\gamma^0\psi$$

$$= \int d^3x\, \psi^\dagger\psi$$

with usual NORMAL ORDERING we then get

$$Q = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \sum_s \Big[\underbrace{a_s^\dagger(p)\, a_s(p)}_{\text{fermions with charge } \underline{\underline{+1}}} - \underbrace{b_s^\dagger(p)\, b_s(p)}_{\text{antifermions with charge } \underline{\underline{-1}}} \Big]$$

As a last topic for what concerns the FREE Dirac Theory, we consider DISCRETE SYMMETRIES

$\Rightarrow$ how do one-particle states & fields transform under PARITY, TIME REVERSAL, CHARGE CONJUGATION

P T C

PARITY TRANSFORMATIONS

Under Parity $\vec{p} \to -\vec{p}$ while $\vec{L}, \vec{J}, \vec{S}$ do not change because the angular momentum is a pseudovector!

so consider a generic one-particle state $|\vec{p}, s; a\rangle$

extra labels if present

we need them for $\psi, \bar{\psi}$ & ϕ, ϕ^* to distinguish particles $a_s^\dagger$ from anti particles $b_s^\dagger$

then we must have $P|\vec{p}, s; a\rangle = \eta_a |-\vec{p}, s; a\rangle$

PHASE

to be consistent with Poincaré we should call it $U(P)$

UNITARY OPERATOR (Wigner theorem)

$$\underbrace{}$$

physics is in RAYS in Hilbert space!

η_a called INTRINSIC PARITY

now $\mathcal{P} \cdot \mathcal{P} = \mathbb{1}$ so we might think this is enough

to conclude $\eta_a^2 = 1$ $\Rightarrow$ situation is more delicate,

fermionic operators appear always in PAIRS in $\underbrace{\text{observables}}_{\bar{\psi}\psi \text{ etc}}$

$\Rightarrow$ $\eta_a^2 = \pm 1$ to be precise.

this is because OBSERVABLES must COMMUTE at

space-like separations

$$[A(x), B(y)] = 0$$
$$\text{if } (x-y)^2 < 0$$

↳ MICROCAUSALITY

$\psi, \bar{\psi}$ ANTICOMMUTE $\Rightarrow$ if A, B anticommuted,

meaning one would FLIP the SIGN of the other!

$$A(x)\, B(y)\, |\psi\rangle = -\, B(y)\, A(x)\, |\psi\rangle \quad !$$

$\Rightarrow$ always EVEN # of ψ & $\bar{\psi}$!

However, one can demonstrate that for SPIN $\frac{1}{2}$ WEYL & DIRAC we can always restrict $\eta_a^2 = +1$

[see Weinberg]

Note that this is NOT TRUE for MAJORANA !

given $\mathcal{P}|\vec{p}, s; a\rangle = \eta_a |-\vec{p}, s; a\rangle$

we must have

$$\mathcal{P}\, a_s^\dagger(p)|\Omega\rangle =$$

$$= \mathcal{P}\, a_s^\dagger(p)\, \mathcal{P}^{-1} \underbrace{\mathcal{P}|\Omega\rangle}_{|\Omega\rangle}$$

$$= \mathcal{P}\, a_s^\dagger(p)\, \mathcal{P}^{-1} |\Omega\rangle$$

$$= \eta_a\, a_s^\dagger(-\vec{p})|\Omega\rangle$$

$$\Rightarrow \mathcal{P}\, a_s^\dagger(p)\, \mathcal{P}^{-1} = \eta_a\, a_s^\dagger(-\vec{p})$$

$$\mathcal{P}\, b_s^\dagger(p)\, \mathcal{P}^{-1} = \eta_b\, b_s^\dagger(-\vec{p})$$

and since $\mathcal{P} \cdot \mathcal{P} = \mathbb{1} \Rightarrow \mathcal{P} = \mathcal{P}^{-1}$

$$\mathcal{P} \, a_s^{\dagger}(\vec{p}) \, \mathcal{P} = \eta_a \, a_s^{\dagger}(-\vec{p})$$

$$\mathcal{P} \, b_s^{\dagger}(\vec{p}) \, \mathcal{P} = \eta_b \, b_s^{\dagger}(-\vec{p})$$

recall $\mathcal{P}$ UNITARY

$\mathcal{P}\mathcal{P}^{\dagger} = \mathbb{1}$

$\mathcal{P}^{\dagger} = \mathcal{P}^{-1} = \mathcal{P}$

special for Parity

so by taking hermitian conjugate we get also

$$\mathcal{P} \, a_s(\vec{p}) \, \mathcal{P} = \eta_a \, a_s(-\vec{p})$$

$$\mathcal{P} \, b_s(\vec{p}) \, \mathcal{P} = \eta_b \, b_s(-\vec{p})$$

using $\eta_{a,b} = \pm 1$

REAL !

(not true for Majorana!)

Let's now apply this to a FERMION FIELD

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 \, 2E_p} \sum_{s=1,2} \Big[a_s(\vec{p}) \, u_s(\vec{p}) \, e^{-ip\cdot x} + b_s^{\dagger}(\vec{p}) \, v_s(\vec{p}) \, e^{ip\cdot x} \Big]$$

then

$$P\psi(x)P = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \sum_s \Big[\eta_a\, a_s(-\vec{p})\, u_s(\vec{p})\, e^{-ip\cdot x}$$

$$+ \eta_b\, b_s^\dagger(-\vec{p})\, v_s(\vec{p})\, e^{ip\cdot x} \Big]$$

do now $\vec{p} \to -\vec{p}$, where $p_0 = E_p$ does not change

$$\Rightarrow\ e^{-ip\cdot x} \to e^{-iE_p t - i\vec{p}\cdot\vec{x}} = e^{-ip\cdot x'}$$

where $x' = (t, -\vec{x})$; and similarly:

$$e^{ip\cdot x} \mapsto e^{ip\cdot x'}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \sum_s \Big[\eta_a\, a_s(\vec{p})\, u_s(-\vec{p})\, e^{-ip\cdot x'} + \eta_b\, b_s^\dagger(\vec{p})\, v_s(-\vec{p})\, e^{ip\cdot x'} \Big]$$

what about $u_s(-\vec{p})$ & $v_s(-\vec{p})$?

From their explicit representation one can show that

$$u_s(-\vec{p}) = \gamma^0 u_s(\vec{p}) \quad ; \quad v_s(-\vec{p}) = -\gamma^0 v_s(\vec{p})$$

in fact $u_s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi_s \\ \sqrt{p\cdot\bar{\sigma}}\ \xi_s \end{pmatrix}$; $v_s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi_s \\ -\sqrt{p\cdot\bar{\sigma}}\ \xi_s \end{pmatrix}$

$$u_s(-\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\ \xi_s \\ \sqrt{p\cdot\sigma}\ \xi_s \end{pmatrix} = \gamma_0\, u_s(\vec{p}) \qquad \text{etc}$$

$\Longrightarrow$ $v_s(\vec{p})$ change sign !

so finally

$$P\psi(x)P = \gamma^0 \int \frac{d^3\bar{p}}{(2\pi)^3 2E_p} \sum_s \Big[\eta_a\, a_s^\dagger(\vec{p})\, u_s(\vec{p})\, e^{-ip\cdot x'} - \eta_b\, b_s^\dagger(\vec{p})\, v_s(\vec{p})\, e^{ip\cdot x'} \Big]$$

if we want this to be proportional to ψ [it means ψ is representation of PARITY OPERATOR]

$\Rightarrow \quad \eta_a = -\eta_b$ only way

$\mathcal{P}\psi(x)\mathcal{P} = \eta_a \gamma^0 \psi(t, -\vec{x})$ same CLASSICAL ACTION up to phase η_a

Note that by CONSISTENCY particles and antiparticles in FERMIONIC CASE must have opposite intrinsic parity !

this comes from different sign in $v_s(\vec{p})$ versus $u_s(\vec{p})$

if we had done same thing for complex scalar field there would be no u_s, v_s

$\Rightarrow \quad \eta_a = +\eta_b$ SAME INTRINSIC PARITY FOR SPIN 0 !

CHARGE CONJUGATION

Talking about CLASSICAL FIELDS, we defined

$\psi^c = -i\gamma^2\psi^*$ Charge Conjugation of Dirac Field

this operation on quantum states reverses PARTICLES and ANTIPARTICLES. Let's demonstrate it.

Define $C\, a_s(\vec{p})\, C = \eta_c\, b_s(\vec{p})$

$C\, b_s(\vec{p})\, C = \eta_c\, a_s(\vec{p})$

as for P

$C = U(C)$

UNITARY

Assumed $C \cdot C = \mathbb{1} \rightarrow C = C^{-1} = C^\dagger$

then $C \cdot C\, a_s(\vec{p})\, C \cdot C = \eta_c^2\, a_s(\vec{p})$

$\eta_c = \pm 1$

And same equations apply for $a_s^\dagger(\vec{p})$

$b_s^\dagger(\vec{p})$

then

$$C\psi(x)C = \eta_c \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \sum_s \Big[b_s(\vec{p})\, u_s(\vec{p})\, e^{-ip\cdot x}$$

$$+ a_s^\dagger(\vec{p})\, v_s(\vec{p})\, e^{ip\cdot x} \Big]$$

now we notice that (explicitly)

$$u_s(\vec{p}) = -i\gamma^2 (v_s(\vec{p}))^*$$

$$v_s(\vec{p}) = -i\gamma^2 (u_s(\vec{p}))^*$$

$$= \eta_c (-i\gamma^2) \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \sum_s \Big[b_s(\vec{p})\, (v_s(\vec{p}))^*\, e^{-ip\cdot x}$$

$$+ a_s^\dagger(\vec{p})\, (u_s(\vec{p}))^*\, e^{ip\cdot x} \Big]$$

$$= \eta_c [-i\gamma^2] \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \sum_s \Big[\left(b_s^\dagger(p)\, v_s(\vec{p})\, e^{ip\cdot x}\right)^*$$

$$+ \left(a_s(\vec{p})\, u_s(\vec{p})\, e^{-ip\cdot x}\right)^* \Big]$$

$$C\,\psi(x)\,C = \underbrace{\eta_c}_{\text{Quantum phase}}\,\big(\underbrace{-i\gamma^2\psi^*}_{\text{CLASSICAL TRANSFORMATION}}\big)$$

- corresponds to choosing $C = i\gamma^2\gamma^0$

- It's also easy to see that $C\bar{\psi}\gamma^\mu\psi C = -\bar{\psi}\gamma^\mu\psi$

  current changes sign!

  $\Rightarrow$ CONSERVED CHARGE changes sign too ! $\Leftarrow$

- One can also show explicitly that

  Charge conjugation swaps HELICITIES $\left(\pm\frac{1}{2}\right)$

TIME REVERSAL

We have seen in ex. 2 that Time Reversal must be implemented by ANTIUNITARY OPERATOR

$$\begin{cases} \langle \psi | \phi \rangle \to \langle U\psi | U\phi \rangle = \langle \psi | \phi \rangle^* \\ \qquad U c|\phi\rangle = c^* U|\phi\rangle \quad c \in \mathbb{C} \end{cases}$$

If we call the Time-reversal ANTIUNITARY OPERATOR T

we require that $T\psi(t,\vec{x})T$ satisfies Time-Reversed DIRAC EQUATION

this requires $T a_s(\vec{p}) T = a_{-s}(-\vec{p}) = \left(a_2(-\vec{p}), -a_1(-\vec{p})\right)$

$$T b_s(\vec{p}) T = b_{-s}(\vec{p}) = \left(b_2(-\vec{p}), -b_1(-\vec{p})\right)$$

T changes momentum & flips SPIN

applying this on Dirac Field we get

$$T\,\psi(t,\vec{x})\,T^{\dagger} = \underbrace{+\gamma^1\gamma^3\,\psi(-t,\vec{x})}_{}$$

SATISFIES DIRAC EQ. with $t \to -t$

(in principle, up to η_T PHASE)

Note finally that $\forall$ term that we can write in $\mathcal{L}$ which fulfils:

1. LOCALITY
2. LORENTZ INVARIANCE
3. HERMITICITY
4. POSITIVE ENERGY
5. SPIN-STATISTICS

will ALWAYS be invariant under CPT

Manifestation of CPT theorem !

Can be proved in AXIOMATIC QFT (Wightman Axioms)

What about spin 1 massless fields?

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Describes 4 classical Fields A^0, A^1, A^2, A^3 $\Rightarrow$ we could try to quantize each as a BOSONIC scalar field

We have seen that using A^μ to describe the theory introduces extra unphysical degrees of freedom

$\Rightarrow$ A^μ has 4 components versus 2 independent polarizations of an electromagnetic wave

$\Downarrow$

Manifestation of gauge redundancy of description

One way to resolve the problem is to FIX the gauge $\Rightarrow$ radiation gauge $A_0 = 0$, $\vec{\nabla} \cdot \vec{A} = 0$

reduces problem to 2 degrees of freedom

Price to pay $\Rightarrow$ BREAK Lorentz Invariance!

We will discuss how to solve this problem preserving Lorentz invariance with PATH INTEGRAL.

Here, we will stress instead the CRUCIAL connection between Lorentz Invariance and
Gauge invariance

to write a consistent field theory of a spin 1 field $\Rightarrow$ we will see that if we insist on Lorentz invariance, we must have gauge invariance!

Let's start from Classical Field in Fourier basis

$$A_\mu(x) = \sum_{\lambda=\pm1} \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \left[\varepsilon_\mu^\lambda(p)\, a^\lambda(p)\, e^{-ip\cdot x} + \varepsilon_\mu^\lambda(p)^*\, a_\lambda^*(p)\, e^{ip\cdot x} \right]$$

And now promote field to a bosonic operator

$A_\mu(x)$ operator $\Rightarrow$ $a_\lambda(p)$, $a_\lambda^\dagger(p)$ operators if we do everything consistently, the field should transform under Lorentz Transformations as

$$U(\Lambda,0)\, A^\mu(x)\, U^{-1}(\Lambda,0) = \Lambda^\mu{}_\nu\, A^\nu(\Lambda^{-1}\cdot x)$$

or using ACTIVE view point we expect

$$U(\Lambda,0)\, A^\mu(x)\, U^{-1}(\Lambda,0) = (\Lambda^{-1})^\mu{}_\nu\, A^\nu(\Lambda\cdot x)$$

Unfortunately, in case of a massless spin 1 field things are less trivial.

In fact, take expansion in Fourier modes in radiation gauge $\Rightarrow$ $A^0 = 0$, this cannot be preserved by equation above !

Let's look at what happens more in detail as for scalar field, the creation and destruction operators transform as they must from our definition of one-particle states

$$U(\Lambda,0)\, a_\lambda(p)\, U^{-1}(\Lambda,0) = e^{+i\lambda\theta(\Lambda,p)}\, a_\lambda(\Lambda p)$$

Repr. of little group for massless spin 1 $\left\{ D_{\lambda\lambda'}(W(\Lambda,p)) = e^{i\lambda\theta}\delta_{\lambda\lambda'} \right.$

IRREPS are ORTHOGONAL in helicity basis!

Similarly

$$U(\Lambda,0)\, a^\dagger_\lambda(p)\, U^{-1}(\Lambda,0) = e^{-i\lambda\theta(\Lambda,p)}\, a^\dagger_\lambda(\Lambda p)$$

both for $\lambda = \pm 1$, two helicity states

which give

$$U(\Lambda,0) A^\mu U^{-1}(\Lambda,0) = \sum_{\lambda=\pm1}\int \frac{d^3\bar{p}}{(2\pi)^3 2E_p}\left(e^{-ip\cdot x} e^{i\lambda\theta} \varepsilon^\mu_\lambda(p)\, a_\lambda(\Lambda p) + h.c.\right)$$

now what can we say about the pol. vector?

$$\Lambda^\mu{}_\nu \, \varepsilon^\nu_\lambda(p) = \; ?$$

remember pol vectors are rather special vectors $\Rightarrow$ there are only 2 independent ones

FOR EXAMPLE if $p^\mu = (E, 0, 0, E)$ then

$\varepsilon^\mu_1(p) = (0, 1, 0, 0)$; $\varepsilon^\mu_2(p) = (0, 0, 1, 0)$

[or any linear combination thereof]

Lorentz transf will in general transform ε^μ

nontrivially $\Rightarrow$ $p^\mu \to \Lambda^\mu{}_\nu p^\nu$; $\varepsilon^\mu_i \to \varepsilon^\mu_{\pm 1} + \alpha \, p^\mu$!

So it will reintroduce a LONGITUDINAL POLARIZATION that we deemed as UNPHYSICAL.

We can see this in general going back to Little group of massless momentum

Fix $k^\mu = (k, 0, 0, k)$ & $p^\mu = [L(p)]^\mu{}_\nu k^\nu$

and let me then define $\tilde{\varepsilon}^\mu_\lambda(k)$ and

$$\varepsilon^\mu_\lambda(p, \lambda) = [L(p)]^\mu{}_\nu \, \tilde{\varepsilon}^\nu_\lambda(k)$$

Now given a generic Lorentz transform Λ, define

$W(\Lambda, p) = L^{-1}(\Lambda p)\, \Lambda \, L(p) \in$ Little Group of k^μ

Since k^μ is massless, we know $W \in ISO(2)$

translations + rotations

The culprit is the "translation part"

Explicitly $k^\mu = (k, 0, 0, k)$

$$\underbrace{\tilde{\varepsilon}_1^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0) \qquad \tilde{\varepsilon}_2^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0)}_{\text{helicity basis } \tilde{\varepsilon}^\mu_{1,2}(k)\,!}$$

then $W(\Lambda, p) = T(a, b)\, R_z(\theta)$

$T(a,b)$: translation that preserves k^μ

$R_z(\theta)$: rotation along z axis

$T(a,b)\, k = k \quad \Rightarrow T = e^{-ia\tau_a - ib\tau_b} = e^{-i\tau(a,b)}$

with:

$$\tau(a,b) = i \begin{bmatrix} 0 & -a & -b & 0 \\ -a & 0 & 0 & a \\ -b & 0 & 0 & b \\ 0 & -a & -b & 0 \end{bmatrix}$$

$\tau_a \cdot k = (0, 0, 0, 0)$

$\tau_b \cdot k = 0$

NOT HERMITIAN

FIN. DIM. REP of LORENTZ is NOT UNITARY

$$\tau(a,b) = a \underbrace{\begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}}_{\tau_a} + b \underbrace{\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{\tau_b}$$

now try to act with these generators on $\tilde{\varepsilon}^\mu_{1,2}$

$$\Rightarrow (\tau_a)^\mu{}_\nu \tilde{\varepsilon}^\nu_1 = -\frac{i}{\sqrt{2}}(1,0,0,1) \propto k^\mu$$

$$(\tau_b)^\mu{}_\nu \tilde{\varepsilon}^\nu_1 = +\frac{1}{\sqrt{2}}(1,0,0,1) \propto k^\mu$$

and similarly for $\tilde{\varepsilon}^\mu_2$

so we find

$$\left[\tau(a,b)\cdot\tilde{\varepsilon}_1\right]^\mu = \left[(a\tau_a + b\tau_b)\tilde{\varepsilon}_1\right]^\mu = i\left(\frac{a+ib}{\sqrt{2}}\right)\frac{k^\mu}{k}$$

alternatively, you can easily exponentiate these matrices and find the FINITE transformation

$$T = e^{-i\tau(a,b)} = \begin{bmatrix} 1+\frac{a^2+b^2}{2} & -a & -b & -\frac{a^2+b^2}{2} \\ -a & 1 & 0 & a \\ -b & 0 & 1 & b \\ -\frac{a^2+b^2}{2} & -a & -b & 1-\frac{a^2+b^2}{2} \end{bmatrix}$$

Such that $(T)^\mu{}_\nu \tilde{\varepsilon}^\nu_1 = \tilde{\varepsilon}^\mu_1 - \frac{1}{\sqrt{2}}(a+ib)\frac{k^\mu}{k}$

$$(T)^\mu{}_\nu \tilde{\varepsilon}^\nu_2 = \tilde{\varepsilon}^\mu_2 - \frac{1}{\sqrt{2}}(a-ib)\frac{k^\mu}{k}$$

So we can say that

$$(W(\Lambda,p))^{\mu}{}_{\nu}\,\tilde{\varepsilon}^{\nu}_{1,2}(k) = \overbrace{\left[L^{-1}(\Lambda p)\,\Lambda\,L(p)\right]^{\mu}{}_{\nu}}^{A}\,\tilde{\varepsilon}^{\nu}_{1,2}$$

$$= \left[T(a,b)\,R_z(\theta)\right]^{\mu}{}_{\nu}\,\tilde{\varepsilon}^{\nu}_{1,2}(k) \qquad \text{rotation acts first}$$

$$= e^{\pm i\theta}\left[\tilde{\varepsilon}^{\nu}_{1,2}(k) - \frac{1}{\sqrt{2}}(a \pm ib)\frac{k^{\nu}}{k}\right]$$

rotates $\varepsilon^{\mu}_{1,2}$ by $\pm\theta$ which corresponds just to action on components by $e^{i\theta}$! [signs opposite!! IMPORTANT!]

From here, multiplying by $L(\Lambda p)$ we can read off

$$\overbrace{[\Lambda]^{\mu}{}_{\nu}}^{A}\,\big(\underbrace{L(p)\,\varepsilon_{1,2}}_{\varepsilon^{\nu}_{1,2}(p)}\big)^{\nu} = \overbrace{e^{\pm i\theta}\,\big[\underbrace{L(\Lambda p)\,\varepsilon_{1,2}}_{\varepsilon^{\mu}_{1,2}(\Lambda p)}\big]^{\mu} - \frac{1}{\sqrt{2}}(a \pm ib)\frac{(\Lambda p)^{\mu}}{k}\,e^{\pm i\theta}}^{B}$$

so we can use this to invert for $\varepsilon^\mu_{1,2}(p)$

$$\varepsilon^\mu_{1,2}(p) = e^{\pm i\vartheta}\left[(\Lambda^{-1})^\mu{}_\nu\,(\varepsilon_{1,2}(\Lambda p))^\nu - \frac{1}{\sqrt{2}}(a\pm ib)\frac{p^\mu}{k}\right]$$

now let's go back to field transformation:

$$U(\Lambda,0)\,A^\mu\,U^{-1}(\Lambda,0) = \sum_{\lambda=\pm1}\int\frac{d^3\bar{p}}{(2\pi)^3 2E_p}\left(e^{-ip\cdot x}\,e^{i\lambda\vartheta}\,\varepsilon^\mu_\lambda(p)\,a_\lambda(\Lambda p)\right.$$
$$\left. + \text{h.c.}\right)$$

$e^{i\lambda\vartheta}\,\varepsilon^\mu_\lambda(p) \Rightarrow e^{\mp i\vartheta}\,\varepsilon_{1,2}(p)$ so use eq above:

$\parallel$

$$= (\Lambda^{-1})^\mu{}_\nu\,[\varepsilon_{1,2}(\Lambda p)]^\nu - \underbrace{\frac{1}{2}(a\pm ib)\frac{p^\mu}{k}}_{c_\pm}$$

$$= (\Lambda^{-1})^\mu{}_\nu \sum_{\lambda=\pm1}\int\frac{d^3\bar{p}}{(2\pi)^3 2E_p}\left(e^{-ip\cdot x}\,\varepsilon^\nu_\lambda(\Lambda p)\,a_\lambda(\Lambda p) + \text{h.c}\right)$$

$+\,\partial_\mu f_\Lambda(x) \Rightarrow$ because extra $p^\mu \to \partial_\mu$ acting on exponentials !

$p \to \Lambda^{-1} p$ then

$(\Lambda^{-1} \cdot p) \cdot x = p \cdot \Lambda \cdot x$ so we can write

$$= (\Lambda^{-1})^{\mu}{}_{\nu} \sum_{\lambda = \pm 1} \int \frac{d^3 p}{(2\pi)^3 2E_p} \left[e^{-ip(\Lambda x)} \varepsilon_{\lambda}^{\nu}(p)\, a_{\lambda}(p) + h.c. \right]$$

$+ \partial_{\mu} f_{\Lambda}(x)$ and finally

$$U(\Lambda,0)\, A^{\mu}\, U^{-1}(\Lambda,0) = (\Lambda^{-1})^{\mu}{}_{\nu}\, A^{\nu}(\Lambda x) + \underbrace{\partial_{\mu} f_{\Lambda}(x)}$$

Extra Term Like <u>GAUGE TRANSF!</u>

so we see that a Lorentz transformation actually generates an extra term $\propto \partial_{\mu} f$!

this means that a term like $A^{\mu} A_{\mu}$ would not even really be Lorentz invariant in quantum theory !

$\Rightarrow$ we MUST build a GAUGE INVARIANT $\mathcal{L}$ to guarantee that it is also LORENTZ INVARIANT !

$$\Rightarrow U(\Lambda,0)\, F^{\mu\nu}\, U^{-1}(\Lambda,0) = (\Lambda^{-1})^{\mu}_{\ \rho}\, (\Lambda^{-1})^{\nu}_{\ \sigma}\, F^{\rho\sigma}$$

the field-strength tensor is instead <u>TRULY LORENTZ INVARIANT</u>

Similarly, we can couple A^μ directly to some 4-vector J_μ only if $\partial_\mu J^\mu = 0$

$$\Rightarrow \quad A^\mu J_\mu \rightarrow A^\mu J_\mu + (\partial^\mu f) J_\mu$$

$$= A^\mu J_\mu + \underbrace{\partial^\mu (f \cdot J_\mu)}_{\substack{\text{boundary term} \\ = 0}} + \underbrace{f \cdot \partial^\mu J_\mu}_{\substack{= 0 \\ \text{Conservation equation}}}$$

$\Rightarrow$ So in Quantum theory

$$A^\mu \cdot (\phi^* \partial_\mu \phi - \phi\, \partial_\mu \phi^*) \quad \text{or} \quad A^\mu \bar{\Psi} \gamma_\mu \Psi \qquad \underline{\underline{\text{GOOD!}}}$$

Scalar QED $\qquad\qquad$ Spinor QED