Introduction & Motivation

QFT 1

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QUANTUM FIELD THEORY

What is Quantum Field theory? Undoubtedly one of the main achievements of the 20th century!

QFT is a <u>framework</u>. It was (mainly) developed to combine the two main theories developed in the first decades of the 20th century

Special Relativity

$\Downarrow$

deals with phenomena at "high Energies"

$\Rightarrow$ $v \sim c$

c speed of light

$E_k \sim mc^2$

Quantum Mechanics

$\Downarrow$

Deals with microscopic phenomena

$\lambda_{DB} \sim h/p$

h Planck constant

p particle momentum

the two theories become relevant TOGETHER when we deal with small scales and high energies

⟹ Typically, PARTICLE PHYSICS

QFT is the only consistent framework we know to account for these phenomena

it is the BACKBONE of the STANDARD MODEL OF PARTICLE PHYSICS

It describes very succesfully FUNDAMENTAL HIGH E. PHYSICS in $D=4$ SPACE-TIME DIMENSIONS

But the reach of QFT goes beyond that!

⟹ For example, crucial formalism also in Condensed Matter physics, lower dimensional and not necessarily relativistic!

⇓

In general, right language to deal with COLLECTIVE PHENOMENA

In this course we will mainly focus on RELATIVISTIC QFTs in $D = 4$ SPACE-TIME DIMENSIONS.

- understand fundamental physics at highest energies and smallest scales
- we will see how Lorentz invariance puts very strong constraints on the types of QFTs that make sense!

QFT formalism has demonstrated to be extremely powerful to describe interactions characterized by small coupling constant $\Rightarrow$ PERTURBATION THEORY

One of most impressive examples of this is the so-called $g-2$ of the electron, which can be computed to very high precision in QUANTUM ELECTRODYNAMICS $\Rightarrow$ coupling $\alpha = \frac{e^2}{4\pi\hbar c} \sim \frac{1}{137}$

the theory says that the electron has a magnetic moment

$$\mu = g \frac{|e|\hbar}{4 m_e c} \qquad g = \text{gyromagnetic ratio}$$

MAXWELL $g = 1$ (wrong !)

DIRAC $g = 2$ (almost right ...)

QED $$\left(\frac{g-2}{2}\right)_{TH} = 0.001\ 159\ 652\ 140\ \overbrace{(5)(4)(27)}^{\text{errors}}$$

EXPERIMENT $$\left(\frac{g-2}{2}\right)_{EX} = 0.001\ 159\ 652\ 187\ (4)$$

which is one of the best and most precisely known quantity in science !

$\Rightarrow$ this course will give you the basis to understand how such an agreement can be achieved and, equally importantly, what are the limitations of this formalism !

Before diving into QFT, let us take some time to recap the main DIFFICULTIES that researchers one century ago had to overcome to create a consistent relativistic, quantum theory.

Remember main "Postulates" of QM & SR

QM:
- state of system $|\psi;t\rangle$ is a vector in a Hilbert Space
- Observables are hermitian operators
- their measurement yields an eigenvalue
- the time-evolution of states is governed by Schrödinger equation $i\hbar\,\partial_t|\psi;t\rangle = \underset{\uparrow}{H}|\psi,t\rangle$

Hamiltonian

SR:
- physics description must be equivalent in all inertial frames

$$x^\mu = (ct, \vec{x}) \longrightarrow y^\mu = (c\tau, \vec{y})$$

$$y^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \qquad \Lambda^\mu{}_\nu \text{ Lorentz transf.}$$

Where I use the "mostly minus" convention for the 4-dimensional Minkowski metric

$$x_\mu = g_{\mu\nu} x^\nu \; ; \quad x^\mu = g^{\mu\nu} x_\nu$$

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \; ; \quad g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$$

"IDENTITY"

$x^\mu x_\mu = (ct)^2 - \vec{x}^2$ norm of a Minkowski vector

note that with this $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$; $\Box = \partial_\mu \partial^\mu = \partial_0^2 - \vec{\nabla}^2$

$$\Rightarrow p^\mu = +i\hbar \partial^\mu \underset{\hbar = 1}{\Rightarrow} p^i = i\partial^i = -i\partial_i = -i\frac{\partial}{\partial x^i} = -i\nabla^i$$

We'll say much more about Lorentz transformations in next lecture, for now just remember

$$x^\mu x_\mu = x^\mu x^\nu g_{\mu\nu} \xrightarrow{\Lambda} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma g_{\mu\nu}$$

invariance $\quad g_{\rho\sigma} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g_{\mu\nu}$

So how Do we get a Quantum, relativistic theory ?

A first naive (but natural) attempt would be to try to promote $\mathcal{H}$ in Schrödinger Eq. to a relativistic form. For a free particle

$$\mathcal{H} = \frac{p^2}{2m} \longrightarrow \sqrt{m^2c^4 + \vec{p}^2c^2} = mc^2 + \frac{p^2}{2m} + O\left(\frac{p^4}{m^2}\right)$$

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{x},t) = \sqrt{m^2c^4 + \vec{p}^2c^2}\ \psi(\vec{x},t)$$

various issues

1. Not explicitly relativistic invariant as it treats t, $\vec{x}$ very differently

2. Not local $\Rightarrow$ $\sqrt{-\hbar^2c^2\nabla^2 + m^2c^4}$ produces an infinite number of derivatives acting on $\psi(\vec{x},t)$

⋮

A second attempt to ameliorate this was to try and square this equation

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(\vec{x},t) = \left(-\hbar^2 c^2 \vec{\nabla}^2 + m^2 c^4\right) \psi(\vec{x},t)$$

using $\partial_\mu \partial^\mu = \partial^2 = \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$

we can write it as

$$\left(\Box + \frac{m^2 c^2}{\hbar^2}\right) \psi(\vec{x},t) = 0$$

which is manifestly Lorentz invariant since

$$\Box = \partial^\mu \partial_\mu \xrightarrow{\Lambda} \partial_\mu \partial^\mu \quad !$$

this equation is called Klein Gordon equation (KG)
we will re-encounter it soon with a different interpretation.

the problem with interpreting this equation as a relativistic wave equation is that, contrary to Schrödinger Equation, KG is second order in time $\Rightarrow$ this has various bad consequences, in particular, a probabilistic interpretation is difficult.

Remember that in Non-Relativistic Schrödinger eq. we have PROBABILITY DENSITY $\rho = |\psi|^2$

such that $$\frac{\partial}{\partial t}|\psi|^2 - \frac{i\hbar}{2m}\vec{\nabla}\cdot\underbrace{(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)}_{\mathrm{Im}[\psi^*\vec{\nabla}\psi]} = 0$$

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{J} = 0 \quad \text{Conservation law}$$

In KG. $$\rho = N\,\mathrm{Im}\left[\psi^*\frac{\partial}{\partial t}\psi\right]$$

$$\vec{J} = N c^2\,\mathrm{Im}\left[\psi^*\vec{\nabla}\psi\right]$$

$\Rightarrow$ ρ IS NOT POSITIVE DEFINITE !

you can try with a plane-wave solution

$$\psi(\vec{x},t) = e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}$$

$$\rho = \frac{E}{mc^2} \rightarrow$$ but $E = \pm\sqrt{p^2c^2+m^2c^4}$

there are negative energy solutions ! two issues are connected

Dirac was the first to realize that the presence of this derivative $\frac{\partial}{\partial t}\psi$ in ρ is due to having a 2nd order eq in time !

$\Longrightarrow$ He enforced a First-Order differential Eq.

At the price of allowing ψ to become a "vector" $\Rightarrow$ multidimensional

$$i\hbar\frac{\partial}{\partial t}\psi = \mathcal{H}\cdot\psi$$

$\mathcal{H}$ is now MATRIX

$\mathcal{H}\left(\frac{\partial}{\partial x_i}\right)$ also first order to be relativistic invariant

Dirac showed that a solution exists consistent with SR. In modern notation

$$(i\hbar\gamma^\mu \partial_\mu - mc)\psi = 0$$

$\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ are 4 DIRAC γ-matrices

such that $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}$

and $(\gamma^0)^2 = \mathbb{1}$; $(\gamma^i)^2 = -\mathbb{1}$; $tr(\gamma^\mu) = 0$

one finds that γ^μ must be AT LEAST 4x4

Again, we'll say much more about these γ^μ matrices later, for now the important point is that ψ is 4-dimensional $\Rightarrow$ 2-vector natural to represent an electron of spin $\frac{1}{2}$

the remaining two solutions had some issue of solutions of KG $\Rightarrow$ negative energy states!

In fact, acting on a momentum eigenstate

$$\psi_p(x) = u(p)\, e^{-\frac{i}{\hbar} p_\mu x^\mu}$$ $u(p)$ 4-dim "spinor"

Dirac Eq $\quad (p^\mu \gamma^\mu - mc)\, u(p) = 0$

$$\gamma^0 (p_0 \gamma^0 - \vec{p} \cdot \vec{\gamma} - mc)\, u(p) =$$

$$= (p^0 - p^i \gamma^0 \gamma^i - mc\gamma^0)\, u(p) = 0$$

$$p^0 = \frac{E}{c} \Rightarrow \underset{\uparrow}{E} = c\, \gamma^0 \gamma^i \cdot p^i + mc\gamma^0$$

HAMILTONIAN H

$$\mathrm{Tr}(c\gamma^0\gamma^i p^i + mc\gamma^0) = 4mc\, \mathrm{Tr}[\gamma^0]$$
$$+ c p^i\, \mathrm{Tr}[\gamma^0 \gamma^i] = 0$$

$$\mathrm{Tr}[\mathcal{H}] = 0 \qquad \mathcal{H} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4 \end{bmatrix}$$

$$\sum_i \lambda_i = 0$$ sum eigenvalues is zero

$$E(p),\ E(p),\ -E(p),\ -E(p)$$

$$E(p) = \sqrt{p^2c^2 + m^2c^4}$$

this is a big problem, especially for electrons that can interact with photons, so one could imagine that a positive energy electron could emit photons until its energy becomes negative down to infinity $\Rightarrow$ a real Catastrophe !

Dirac proposed an ingenious, but PARTIAL, solution $\Rightarrow$ using PAULI EXCLUSION PRINCIPLE

we can assume that all negative energy states are FILLED $\Rightarrow$ HOLE THEORY if we excite one negative energy electron out of this SEA of negative electrons we would be able to "see the hole" as a positive charged particle behaving like an electron (except the charge)

⇒ PREDICTION of POSITRON discovered by ANDERSON in 1932 (5 years after Dirac's prediction in 1927)

⇒ CORRECT (largely) PREDICTION of HYDROGEN Energy levels

This was revolutionary but:

1] what about bosons ?

2] to work, we are shifting naturally from a single particle → infinite number of particles !

We know today that a relativistic theory cannot be consistent if it does not allow for "change in number of particles"

$$p_1 + p_2 \rightarrow \sum_{j=1}^{n} q_j$$

n grows depending on energy of p_1 & p_2 !

$$\underline{E = mc^2}$$

Even in QM (non rel)

$$\delta E = \langle 0 | \delta V | 0 \rangle$$

$$+ \sum_{n=0}^{\infty} \frac{|\langle 0 | \delta V | n \rangle|^2}{E_0 - E_n} \leftarrow$$

contribution from all intermediate states

What is "strange" is that DIRAC's theory worked so well for Hydrogen $\Rightarrow$ even without considering pair production etc

this was a fluke and kind of "distracted" physicists from the correct solution $\Rightarrow$

One way to think about why a consistent theory of SR + QM was so elusive is to realize that what we have been trying to do with KG & DIRAC is by construction not treating equally SPACE & TIME

$\Rightarrow$ in QM "All observables are represented by Hermitian Operators"

is not entirely true, or better

$$\begin{cases} \vec{x} \Rightarrow \hat{x} \text{ OPERATOR but} \\ t \text{ time remains a } \underline{\text{PARAMETER}}! \end{cases}$$

treated very differently $\Rightarrow$ hint to new formalism?

there are at least <u>two</u> <u>possibilities</u> :

1. PROMOTE t to an operator like $\hat{x}$

2. DEMOTE $\hat{x}$ to a simple parameter

1. Promoting $t \Rightarrow \hat{T}$ operator

we need a new parameter to parametrize "world-line" of a particle. We can use for example PROPER TIME τ, or any $f(\tau)$

$f(\tau)$ monotonic

$X^\mu(\tau)$ where $\hat{X}^\mu$ operator !

$\Rightarrow$ WORLD-LINE FORMALISM

We can even consider more parameters

$X^\mu(\tau, \sigma, \dots)$ if only τ, σ 2-dim

WORLD-SHEET

this is a starting point to construct STRING THEORY

$\Rightarrow$ reparametrization invariance renders formalism complicated, we will not follow this path here ! $\Rightarrow$ EQUIVALENT TO 2. $\longrightarrow$

2. Demote position to a LABEL

It means that in this theory, fundamental objects must be OPERATORS which are FUNCTIONS OF SPACE-TIME

$\varphi(x^\mu) = \varphi_H(\vec{x}, t)$ operator in Heisenberg picture evolves with t

$\varphi_S(\vec{x})$ in Schrödinger picture only functions of SPACE $\vec{x}$

$$\varphi_H(\vec{x}, t) = e^{iHt/\hbar} \varphi_S(\vec{x}, 0) e^{-iHt/\hbar}$$

$\uparrow$

Democratic role of SPACE & TIME as LABELS

$\varphi(\vec{x}, t)$ called QUANTUM FIELD

& theory is QUANTUM FIELD THEORY

NOTATION AND UNITS

Across the course we will almost exclusively work in so-called NATURAL UNITS $\hbar = c = 1$

$\hbar$ & c are UNIVERSAL CONSTANTS

$c = 299\,792\,458$ m/s with NO ERROR

because it defines the meter, after "second" defined by cesium-133 transition

→ putting $c = 1$ means redefining the <u>unit of LENGTH</u>

velocity then in units of $c \Rightarrow 0 \leq v < 1$

$$\hbar = \frac{6{,}62607015 \cdot 10^{-34}}{2\pi} \; J \cdot s \quad \Rightarrow [\text{energy}] \times [\text{time}]$$

so putting $\hbar = 1$ we fix the <u>ENERGY UNIT</u>

In natural units, then

[velocity] = pure number

[energy] = [momentum] = [mass] $\quad E = \sqrt{m^2 + p^2}$

[length] = [mass]$^{-1}$

$\Rightarrow \frac{\hbar}{mc}$ is a length

All physical quantities have then MASS dim!

$\alpha = \frac{e^2}{4\pi\hbar c} \sim \frac{1}{137}$ pure number $\Rightarrow$ e is also pure number in natural units!

It's useful to remember that

$\hbar c$ = [energy] x [length] in ordinary units

$\hbar c \simeq 200$ MeV $\cdot$ fm $\quad \begin{cases} 1 \text{ MeV} = 10^6 \text{ eV} \\ 1 \text{ fm} = 10^{-13} \text{ cm} \end{cases}$

remember H energy $E \sim 13.6$ eV

1 fm $\sim$ PROTON RADIUS

$\Rightarrow$ In natural units 200 MeV $\sim (1\text{ fm})^{-1}$

$m_e \simeq 511$ keV ~ 0.5 MeV

$\sim 7.8 \cdot 10^{20}\ s^{-1}$

$\sim 2.6 \cdot 10^{10}$ cm^{-1}

Compton wave length (reduced $\frac{1}{2\pi}$)

$\lambda_c = \frac{\hbar}{mc}$ electron $\sim 3{,}86 \cdot 10^{-13}$ m

proton $\sim 2.10 \cdot 10^{-16}$ m

$= 0.2$ fm

In Natural units

0.511 MeV electro

$\lambda_c = \frac{1}{m} \Rightarrow$ inverse λ_c is 1 GeV proton