Canonical Quantization of Free Scalar Fields: particles & Antiparticles

QFT

WS 2025

Lorenzo Tancredi

it's finally time to discuss how to quantize our field theories. We will start with a quick discussion of the method of CANONICAL QUANTIZATION applied to the case of FREE SCALAR & SPINOR FIELDS.

After that, we will describe an alternative quantization procedure based on Dirac / Feynman's PATH INTEGRAL:

> 1] introduce interactions
>
> 2] study the quantization of GAUGE THEORIES

CANONICAL QUANTIZATION

the idea is very simple: we leverage principles of Quantum Mechanics, with the identification

$$\text{coord.} \quad q^i(t) \longrightarrow \phi(\underbrace{t,\vec{x}}) = \phi(x) \quad \text{field}$$

$$\text{momentum} \quad p^i(t) \longrightarrow \Pi(\underbrace{t,\vec{x}}_{\text{LABELS}}) = \Pi(x) \quad \text{conjugated momentum}$$

Now, in QM we PROMOTE $p^i, q^i \to$ operators

then in SCHRÖDINGER PICTURE [evolution in wave function]

we IMPOSE COMMUTATION RELATIONS

$$[q^i, p^j] = i\,\delta^{ij} \qquad i,j = 1,2,3 \text{ SPACE COMPONENTS}$$

In HEISENBERG PICTURE [evolution in operators]

we impose commutation relations at EQUAL TIME !

the LABELS "i", "j" become $(\vec{x}, \vec{y})$ and therefore

$\delta_{ij} \sim \delta^{(3)}(\vec{x}-\vec{y})$ DIRAC DELTA FUNCTION

Principles of Canonical Quantization: [reasonable "guess"]

- $\phi(t,\vec{x})$, $\pi(t,\vec{x})$ become OPERATORS
- We IMPOSE $[\phi(t,\vec{x}), \pi(t,\vec{x})] = i\,\delta^{(3)}(\vec{x}-\vec{y})$

AT EQUAL TIME $\quad [\phi(t,\vec{x}), \phi(t,\vec{y})] = [\pi(t,\vec{x}), \pi(t,\vec{y})] = 0$

if $\phi(t,\vec{x})$ is a FREE REAL SCALAR FIELD, we have seen we can write for general solution of K.G.

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3(2E_p)} \left[a(p)\, e^{-ip\cdot x} + a(p)^*\, e^{ip\cdot x} \right]_{p_0 = E_p}$$

$\Downarrow$ upgrade to OPERATOR

remember they are only function of $\vec{p}$: $p_0 \equiv E_{\vec{p}}$!

means $a(p)$ OPERATOR

$a^*(p) \rightarrow a^\dagger(p)$

$$\phi(t,\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \left[a(p)\, e^{-ip\cdot x} + a^\dagger(p)\, e^{ip\cdot x} \right]$$

the conjugate momentum is $\Pi(t,\vec{x}) = \partial_0 \phi(t,\vec{x})$

$$\Pi(t,\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \left[-iE_p\, a(p)\, e^{-ip\cdot x} + iE_p\, a^\dagger(p)\, e^{ip\cdot x} \right]$$

$$= -i \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} E_p \left(a(p)\, e^{-ip\cdot x} - a^\dagger(p)\, e^{ip\cdot x} \right)$$

We can now impose $[\phi(t,\vec{x}),\pi(t,\vec{y})]=i\delta^{(3)}(\vec{x}-\vec{y})$

$$\pi(t,\vec{y})\,\phi(t,\vec{x})=\int\frac{d^3\bar{p}\,d^3\bar{q}}{(2\pi)^6\,2E_p\,2E_q}(-iE_q)\left[a(q)\,e^{-iq\cdot y}-a^\dagger(q)\,e^{iq\cdot y}\right]\times\left[a(p)\,e^{-ip\cdot x}+a^\dagger(p)\,e^{ip\cdot x}\right]$$

$$=\int\frac{d^3\bar{p}\,d^3\bar{q}}{(2\pi)^6\,2E_p\,2E_q}(-iE_q)\Big[a(q)a(p)\,e^{-iq\cdot y-ip\cdot x}+a(q)a^\dagger(p)\,e^{-iq\cdot y+ip\cdot x}-a^\dagger(q)a(p)\,e^{iq\cdot y-ip\cdot x}-a^\dagger(q)a^\dagger(p)\,e^{iq\cdot y+ip\cdot x}\Big]$$

$$\phi(t,\vec{x})\pi(t,\vec{y})=\int[\quad]\Big(a(p)a(q)\,e^{-iq\cdot y-ip\cdot x}+a^\dagger(p)a(q)\,e^{ip\cdot x-iq\cdot y}-a(p)a^\dagger(q)\,e^{-ip\cdot x+iq\cdot y}-a^\dagger(p)a^\dagger(q)\,e^{ip\cdot x+iq\cdot y}\Big)$$

$\Rightarrow$ ASSUME $[a(p),a(q)]=[a^\dagger(p),a^\dagger(q)]=0$ then

we are left with

$$[\phi(t,\vec{x}),\pi(t,\vec{y})] = \int \frac{d^3p\, d^3q}{(2\pi)^6\, 2E_p 2E_q} (-iE_q) \Big([a^\dagger(p), a(q)]\, e^{-iq\cdot y + ip\cdot x}$$

$$- [a(p), a^\dagger(q)]\, e^{-ip\cdot x + iq\cdot y} \Big)$$

then it's clear that by further imposing:

$$[a(p), a^\dagger(q)] = 2E_p (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$$ we got

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \int \frac{d^3\vec{q}}{\cancel{(2\pi)^3 2E_q}} (-iE_q)\, \cancel{(2\pi)^3 2E_q} \times$$

$$\times \left(-\delta^3(\vec{p}-\vec{q})\, e^{+i\vec{p}(\vec{x}-\vec{y})} - \delta^{(3)}(\vec{p}-\vec{q})\, e^{-i\vec{p}(\vec{x}-\vec{y})} \right)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{i\, E_p}{2E_p} \Big[e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + \underbrace{e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}}_{\vec{p}\to-\vec{p}} \Big]$$

$$= i \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$$= i\, \delta^{(3)}(\vec{x}-\vec{y})$$ EXACTLY WHAT WE WANTED !

so equal-time commutation relations can be translated into commutation relations for the operators $a(p), a^\dagger(p)$

$$\begin{cases} [a(p), a(q)] = [a^\dagger(p), a^\dagger(q)] = 0 \\ [a(p), a^\dagger(q)] = 2E_p (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \end{cases}$$

to remove $2E_p$ from this relation, it is usual to redefine the operators as follows

$$a(p) = \sqrt{2E_p}\, a_p \quad ; \quad a^\dagger(p) = \sqrt{2E_p}\, a_p^\dagger$$

$$\phi(t,\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x} \right)$$

$$[a_p, a_q^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$$ "more standard"

but integration measure is not explicitly Lorentz covariant!

$\Rightarrow$ remember from exercises

$$\int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} = \int \frac{d^4p}{(2\pi)^4} (2\pi)\, \delta(p^2-m^2)\, \theta(p^0)$$

Covariant under $SO^+(1,3)$ $\Rightarrow$ does not change sign of $p^0 = E_p$!

By discretizing the system we can recover the usual commutation relations of an HARMONIC OSCILLATOR

put system in volume $V = L^3$

Impose periodic boundary conditions on fields

one gets QUANTIZED MOMENTA

$$p^i = \left(\frac{2\pi}{L}\right) n^i \qquad n^i = 0, \pm 1, \pm 2, \dots$$

$$\int d^3\vec{p} \longrightarrow \left(\frac{2\pi}{L}\right)^3 \sum_{\vec{n}}$$

so, in turn,

$$\int d^3\vec{q}\ \delta^{(3)}(\vec{p}-\vec{q}) = 1 \quad \Rightarrow \quad \delta^{(3)}(\vec{p}-\vec{q}) \to \left(\frac{L}{2\pi}\right)^3 \delta_{\vec{p},\vec{q}}$$

$$\underbrace{(2\pi)^3\ \delta^{(3)}(\vec{p}=0)}_{\text{infinite volume}} \to V$$

infinite volume → finite "regularized" volume V

$$\Rightarrow \quad [a_p, a_q^\dagger] = (2\pi)^3\ \delta^{(3)}(\vec{p}-\vec{q}) \to V\ \delta_{\vec{p},\vec{q}}$$

up to normalization, commutation relations for an INFINITE SET of independent HARMONIC OSCILLATORS

We can then verify the remaining commutation rel.

$$[\phi(t,\vec{x}), \phi(t,\vec{y})] = [\pi(t,\vec{x}), \pi(t,\vec{y})] = 0$$

it's useful to start from the more general case of commutation relations at DIFFERENT TIMES

$$[\phi(x), \phi(y)] = \dots = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \left(\underbrace{e^{-ip(x-y)} - e^{ip(x-y)}} \right)$$

these are 4-momenta
Cannot send $p \to -p$
changes also E_p !

$$= \Delta(x-y)$$

$$= \Delta_+(x-y) - \Delta_+(y-x)$$

where
$$\boxed{\Delta_+(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip\cdot x}}$$

by fixing $x_0 = y_0$ (EQUAL TIME) it's easy to see

$$\Delta(x-y)\Big|_{x^0=y^0} = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \left(e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \right)$$

$\vec{p} \to -\vec{p}$

$$= 0 \;!$$

and similarly $[\pi(t,\vec{x}), \pi(t,\vec{y})] = 0$

Note that since $\Delta(x-y)\Big|_{x^0=y^0} = 0$, this

implies that it remains true $\forall\ |x-y|^2<0$,

SPACE-LIKE separated points by Lorentz covariance

$\Rightarrow$ $\Delta(x-y)$ Lorentz Invariant, can be computed in

any frame, and if $|(x-y)|^2<0 \Rightarrow \exists$ frame

with $x^0-y^0=0$!

Let's compute also the HAMILTONIAN

$$H = \int d^3\vec{x}\,\left(\Pi(t,\vec{x})\,\dot{\phi}(t,\vec{x}) - \mathcal{L}(t,\vec{x})\right)$$

$$= \frac{1}{2}\int d^3\vec{x}\,\left(\Pi^2 + (\nabla\phi)^2 + m^2\phi^2\right)$$

$$= \ldots = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\, E_p\left[a^\dagger(p)\,a(p) + a(p)\,a^\dagger(p)\right]$$

similar manipulations as before

using commutation relations

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} E_p \underbrace{a^\dagger(p)\, a(p)}_{\text{number operator}} + \underbrace{\int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \frac{1}{2}(2\pi)^3 \delta^3(\vec{0})}_{}$$

INFINITE ZERO-POINT ENERGY $\Rightarrow$ infinite number of "harmonic oscillators"

$$\boxed{N = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} a^\dagger(p)\, a(p)}$$

To make these considerations precise, we need to build the FOCK SPACE $\Rightarrow$ Hilbert Space on which operators $a(p)$, $a^\dagger(p)$ act !

We start with a VACUUM STATE $|\Omega\rangle$ defined, as in Q.M., such that

$$\left.\begin{aligned} &a(p)\,|\Omega\rangle = \sqrt{2E_p}\; a_p\,|\Omega\rangle = 0 \\ &a(p) \text{ is ANNIHILATION OPERATOR} \end{aligned}\right\} \quad \underline{\underline{\langle\Omega|\Omega\rangle = 1}}$$

The existence of this state follows the usual construction for HARMONIC OSCILLATORS in Q.M.

Now consider the state $a^\dagger(p)|\Omega\rangle = |p\rangle$

We will prove that this is a state of mom. p^μ and therefore we identify it with our one-particle states discussed in <u>LECTURE 4</u>

To see this, let us consider the 3-momentum as conserved CHARGE from Noether theorem.

Remember that

$$\Theta^0_j = \frac{\partial \mathcal{L}}{\partial \dot\phi}\partial_j\phi = \partial_0\phi\,\partial_j\phi \quad \left(\begin{matrix}\text{FOR}\\ \text{Real}\\ \text{scalar}\\ \mathcal{L}\end{matrix}\ !\right)$$

$$P_j = \int d^3\vec{x}\,(\partial_0\phi\,\partial_j\phi)$$

$$\vec{P} = -\int d^3\vec{x}\,(\partial_0\phi\,\vec{\nabla}\phi) = -\int d^3\vec{x}\,\pi\,\vec{\nabla}\phi$$

Again, let us substitute expressions for fields in terms of $a(p)$, $a^\dagger(p)$ and use commutation rel to find

$$\vec{P} = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \frac{1}{2}\vec{p}\Big[a(p)a(-p) + a^\dagger(p)a(p) + a(p)a^\dagger(p) + a^\dagger(p)a^\dagger(-p)\Big]$$

Note that $\vec{p}\,a(p)a(-p)$ & $\vec{p}\,a^\dagger(p)a^\dagger(-p)$ are both ODD under $\vec{p} \to -\vec{p}$ $\Rightarrow$ integrate to zero!

Similarly, zero-point contribution here integrates to zero

$$\vec{p}\,[a(p), a^\dagger(p)] \sim \vec{p}\cdot\delta^{(3)}(0) \quad \text{ODD } \vec{p} \to -\vec{p}$$

$$\boxed{\vec{P} = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\,\vec{p}\,a^\dagger(p)a(p)}$$

now compute

$$\vec{P}\,|p\rangle = \int \frac{d^3q}{(2\pi)^3\, 2E_q}\, \vec{q}\; a^\dagger(q)\, \underbrace{a(q)\, \overbrace{a^\dagger(p)|\Omega\rangle}^{|p\rangle}}_{a^\dagger(p)\,a(q) + 2E_q(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})}$$

$a(q)|\Omega\rangle = 0$ since VACUUM !

$$\vec{P}\,|p\rangle = \int d^3\vec{q}\;\delta^{(3)}(\vec{p}-\vec{q})\;\vec{q}\;a^\dagger(q)|\Omega\rangle$$

$= \vec{p}\,|p\rangle$ the state has momentum $\vec{p}$!

Similarly we find that

$H|p\rangle = E_p|p\rangle$ eigenstate of momentum operator & Energy

$E_p = \sqrt{\vec{p}^2 + m^2}$

$\Rightarrow |p\rangle$ describes a one-particle state with momentum $\vec{p}$ and mass m $\Rightarrow E_p = \sqrt{\vec{p}^2 + m^2}$

$\Rightarrow \langle p_1 | p_2 \rangle = 2E_{p_1} (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$ normalization from $a^\dagger(p)$

From here we can define MULTIPARTICLE STATES

1. acting with multiple $a^\dagger(p_i)$
2. constructing tensor product of $|p_i\rangle$

In fact, we can give two equivalent definitions:

$$\underbrace{|p_1 \dots p_N\rangle}_{\text{some might be equal}} = \left[\prod_{i=1}^{N} \frac{1}{\sqrt{n_i(p)!}} \right] a^\dagger(p_1) a^\dagger(p_2) \dots a^\dagger(p_N) |\Omega\rangle$$

$n_i(p)$: # times p_i appears in set $\{p_1, \dots, p_N\}$

here we see BOSONIC NATURE $\Rightarrow$ symmetric under swapping two momenta! $\Rightarrow N |p_1 \dots p_N\rangle = n |p_1 \dots p_N\rangle$

number op!

Similarly one can show the equivalent definition

$$|p_1 \dots p_N\rangle = \frac{1}{\sqrt{N!}} \left[\prod_{i=1}^{N} \frac{1}{\sqrt{n_i(p)!}} \right] \sum_{\sigma \in S_N} P_\sigma \left(|p_1\rangle \otimes \dots \otimes |p_N\rangle \right)$$

S_N permutations of $\{p_1, \dots, p_N\}$

Scalar field has only 1 degree of freedom $\forall\, x$

$\Rightarrow$ SPIN 0 particles, BOSONS

Calling $\mathcal{H}$ = single-particle Hilbert space

$\mathcal{H}_N^S = \mathcal{H} \otimes \mathcal{H} \otimes \dots \otimes \mathcal{H}$ product N-times symmetrized

$\Rightarrow$ then FOCK SPACE on which ϕ, $a(p)$, $a^\dagger(p)$ act

$$F^S = \bigoplus_{N=0}^{\infty} \mathcal{H}_N^S$$ where $\mathcal{H}_0 \sim |\Omega\rangle$ VACUUM

What about the zero-point (infinite) energy ?

$\Rightarrow$ As long as GRAVITY NEGLECTED, the only thing that matters are ENERGY DIFFERENCES

We formalize this through concept of NORMAL ORDERING

$\forall$ operator built out of $a(p), a^\dagger(p)$ we define

$:\mathcal{O}: \; = \; \mathcal{O}$ with all creation operators on left of destruction operators

$$\Rightarrow \quad :a(p)\,a^\dagger(p): \; = \; a^\dagger(p)\,a(p)$$

DEFINE

$$H = \frac{1}{2}\int d^3\vec{x} \; :\pi^2 + (\nabla\phi)^2 + m^2\phi^2:$$

$$= \int \frac{d^3p}{(2\pi)^3\, 2E_p}\, E_p \, a^\dagger(p)\, a(p)$$

no zero-point energy !

Notice $[H, N] = 0$ # particles CONSERVED in FREE theory !

COMPLEX SCALAR FIELD

It's easy to generalize these considerations to a complex scalar field. In this case we start from

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$$

Free theory

$\phi^* \longrightarrow \phi^\dagger$

promoted to operator

then we write for Fourier Expansion

$$\phi(t, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \left(a(p) e^{-ip\cdot x} + b^\dagger(p) e^{ip\cdot x} \right)$$

$$\phi^\dagger(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \left(a^\dagger(p) e^{ip\cdot x} + b(p) e^{-ip\cdot x} \right)$$

And impose $[a(\vec{p}), a^\dagger(\vec{q})] = [b(\vec{p}), b^\dagger(\vec{q})] = (2\pi)^3 2E_p \delta^{(3)}(\vec{p}-\vec{q})$

and all other commutation relations are zero

With this, we can define FOCK SPACE starting from

$|\Omega\rangle$ VACUUM ; $a(p)|\Omega\rangle = b(p)|\Omega\rangle = 0$

then generate all space with $\underbrace{a^\dagger(p) \;\&\; b^\dagger(p)}_{\text{create two different quanta of } \phi}$

Using normal ordering, we find as before

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} E_p \left(a^\dagger(p) a(p) + b^\dagger(p) b(p) \right)$$

$$P^i = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} p^i \left(a^\dagger(p) a(p) + b^\dagger(p) b(p) \right)$$

$\Rightarrow$ quanta of a complex scalar field have same mass "m" by treating the field classically, we have used Noether theorem to demonstrate that there is a symmetry U(1) global $\phi \to e^{i\alpha}\phi$

$\Rightarrow$ conserved charge

$$Q_{U(1)} = i\int d^3\vec{x} :\phi^{\dagger}\overleftrightarrow{\partial_0}\phi:$$

normal ordering

Substituting expressions in terms of $a(p)$ & $b(p)$ we can prove that

$$Q_{U(1)} = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \underbrace{\left[a^{\dagger}(p)a(p) - b^{\dagger}(p)b(p)\right]}$$

AFTER NORMAL ORDERING!

$a^{\dagger}(p)\,a(p)$ counts quanta of type a

$b^{\dagger}(p)\,b(p)$ counts quanta of type b

$\Rightarrow$ one can easily see that

$$Q_{U(1)}\, a^\dagger(p)|\Omega\rangle = (+1)\, a^\dagger(p)|\Omega\rangle$$

$$Q_{U(1)}\, b^\dagger(p)|\Omega\rangle = (-1)\, b^\dagger(p)|\Omega\rangle$$

$\Rightarrow$ $a^\dagger(p)|\Omega\rangle$ particle with $\vec{p}$, m & charge $+1$

$b^\dagger(p)|\Omega\rangle$ particle with $\vec{p}$, m & charge -1

Conventional who we call ± 1, $\begin{Bmatrix} \pm j^\mu \\ \pm Q_{U(1)} \end{Bmatrix}$

We call $a^\dagger(p)|\Omega\rangle$ the PARTICLE

$b^\dagger(p)|\Omega\rangle$ the ANTIPARTICLE

Real. Scalar Field has $a(p) = b(p)$ $\Rightarrow$ the particle is its own ANTIPARTICLE, $\underbrace{\text{neutral under } U(1)}_{\text{NO CHARGE}}$!

What about our discussion of Poincaré & single particle states?

Under Poincaré $|p\rangle \to U(\Lambda, a)|p\rangle$

$\phi(x)$ = operator that creates particles & antiparticles

$$\phi(x)|\Omega\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{ip\cdot x} a^\dagger(p)|\Omega\rangle$$

↑ vacuum

Under Lorentz + translations we have <u>FOR SPIN 0</u>:

$U(\Lambda, 0)|p\rangle = |\Lambda^{-1}p\rangle$ ← trivial Wigner rep!

$U(1, a)|p\rangle = e^{-ip\cdot a}|p\rangle$

PRECISE NOW

PASSIVE VIEW POINT!

$U(\Lambda, a)|\Omega\rangle = |\Omega\rangle$ vacuum is invariant!

We use $U(\Lambda, a) = U(\Lambda)U(a)$ TRANSLATION ACTS FIRST!

$$a^\dagger(p)|\Omega\rangle \longrightarrow U(\Lambda,a)\,a^\dagger(p)|\Omega\rangle$$

$$= U(\Lambda,a)\,a^\dagger(p)\,U^{-1}(\Lambda,a)\,\underbrace{U(\Lambda,a)|\Omega\rangle}_{|\Omega\rangle}$$

$$= U(\Lambda,a)\,a^\dagger(p)\,U^{-1}(\Lambda,a)\,|\Omega\rangle$$

$$\overset{!}{\equiv} e^{-ia\cdot p}\,|\Lambda_p^{-1}\rangle$$

impose transforms like this!

$$= e^{-ia\cdot p}\,a^\dagger(\Lambda_p^{-1})\,|\Omega\rangle$$

$\boxed{e^{ia\cdot\Lambda p} \text{ if } U(a)\,U(\Lambda)=U(\Lambda,a)}$

which implies then for creation (SCALAR) operators

$$U(\Lambda,a)\,a^\dagger(p)\,U^{-1}(\Lambda,a) = e^{-ia\cdot p}\,a^\dagger(\Lambda_p^{-1})$$

and in turn

$$U(\Lambda,a)\,a(p)\,U^{-1}(\Lambda,a) = e^{+ia\cdot p}\,a(\Lambda_p^{-1})$$

finally, for the field itself we can then write

$$U(\Lambda,a)\,\phi(x)\,U^{-1}(\Lambda,a) = \int \frac{d^3p}{(2\pi)^3\, 2E_p}\left[e^{-ip\cdot x}e^{+ip\cdot a}\, a(\Lambda^{-1}p) + e^{ip\cdot x}e^{-ip\cdot a}\, a^{\dagger}(\Lambda^{-1}p)\right]$$

$$= \int \frac{d^3p}{(2\pi)^3\, 2E_p}\left[e^{-ip(x-a)}\, a(\Lambda^{-1}p) + e^{ip(x-a)}\, a^{\dagger}(\Lambda^{-1}p)\right]$$

now do $p \to \Lambda p$ and use invariance of measure plus $(\Lambda p)\cdot x = p\cdot(\Lambda^{-1}x)\ \forall x$ to write

$$= \int \frac{d^3p}{(2\pi)^3\, 2E_p}\left[e^{-ip(\Lambda^{-1}x-\Lambda^{-1}a)}\, a(p) + e^{ip(\Lambda^{-1}x-\Lambda^{-1}a)}\, a^{\dagger}(p)\right]$$

$$= \phi(\Lambda^{-1}(x-a))$$

$$U(\Lambda, a)\,\phi(x)\,U^{-1}(\Lambda, a) = \phi\left(\Lambda^{-1}(x-a)\right)$$

$$x \to \Lambda\, y + a$$

$$U(\Lambda, a)\,\phi(\Lambda y + a)\,U^{-1}(\Lambda, a) = \phi(y)$$

$$U(\Lambda, a)\,\phi(y')\,U^{-1}(\Lambda, a) = \phi(y)$$

$$\phi'(y') = \phi(y)$$

transformation rule we started from when discussing CLASSICAL SCALAR FIELD !

Consistent with passive view point $\Rightarrow$ remember that different references use different conventions !

FROM FIELDS TO WAVE FUNCTIONS :

Go back to real field $\phi(x)$ for simplicity.

clearly $\quad \phi(x)|\Omega\rangle = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2E_p} e^{ip\cdot x} |p\rangle$

$\uparrow$

$a^\dagger(p)|\Omega\rangle$

in non-relativistic case $\frac{1}{2E_p} \sim \frac{1}{2m}$ "constant"

So it very much looks like $|x\rangle$ in QM

expressed in basis $|p\rangle$

$\Rightarrow$ Led to interpret $\phi(x)$ as operator that

creates a particle at position "x"

$$\langle\Omega|\phi(x)|p\rangle = \langle\Omega| \int \frac{d^3p'}{(2\pi)^3 2E_{p'}} \left(a_{p'} e^{-ip'\cdot x} + a^\dagger_{p'} e^{ip'\cdot x} \right) a^\dagger_p |\Omega\rangle$$

$= e^{ip\cdot x}$ SINGLE PARTICLE WAVE FUNCTION OF STATE $|p\rangle$