Classical Field theory :

Spin 1 fields

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In the previous lecture we have seen how imposing invariance of SCALAR or SPINOR theory under a LOCAL GAUGE TRANSFORMATIONS requires the introduction of a VECTOR FIELD A^μ

$\Rightarrow$ we have already hinted to the fact that this is exactly the Electromagnetic Field : SPIN 1

this provides one more justification to study Field Theories for fields of higher (integer) spin

We stress once more that all our considerations here are CLASSICAL $\Rightarrow$ we have not discussed Field Quantization yet, which will LATER allow us to build a Hilbert space and make a connection with one-particle states !

VECTOR FIELD

As we know, the E.M. field is described by a four-vector A^μ; let's see that this is exactly the one defined in previous lectures

E.M. field $\Rightarrow$ gauge invariance $A_\mu \to A_\mu + \partial_\mu f$

Observables must be GAUGE INVARIANT $\Rightarrow$ $\vec{E}, \vec{B}$

electric & magnetic fields

From Electrodynamics we know how to build them from vector potential A^μ:

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ Field strength tensor

GAUGE INVARIANT !

$\Downarrow$

$F^{0i} = -E^i$; $F^{ij} = -\varepsilon^{ijk} B^k$

We can then build Lorentz + Gauge invariant $\mathcal{L}$

$\mathcal{L} = c\, F^{\mu\nu} F_{\mu\nu}$ ⇐ simplest term! not unique!

how do we fix it?

* First, derive Eq. of motion

$$\cancel{\frac{\partial \mathcal{L}}{\partial A_\nu}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0$$

$$\mathcal{L} = c\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

$$\rightarrow \quad 4c\,\boxed{\partial_\mu F^{\mu\nu} = 0}$$

Free MAXWELL EQUATIONS (in Vacuum!)

EXERCISE

c does not influence them!

Let's compute the Hamiltonian ⇒ energy density

we start from the Energy-momentum tensor

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L}$$

$$= 4c\, F^{\mu\rho} \partial^\nu A_\rho - c\, g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

DOES NOT SEEM GAUGE INVARIANT

$A_\mu \to A_\mu + \partial_\mu f$ gives

$$\Theta^{\mu\nu} \to \Theta^{\mu\nu} + 4c\, F^{\mu\rho} \partial^\nu \partial_\rho f$$

$$= \Theta^{\mu\nu} + \partial_\rho \left[4c\, F^{\mu\rho} \partial^\nu f \right] + \cancel{4c\, \partial^\nu f\, \partial_\rho F^{\mu\rho}}$$

$\overset{\shortparallel}{0}$ Equation of motion!

gauge invariance breaking term is TOTAL DERIVATIVE

$\Rightarrow$ This means that conserved charges are gauge inv!

$P^\nu = \int d^3\vec{x}\, \Theta^{0\nu} \Rightarrow$ under gauge transf.

$$= P^\nu + 4c\int d^3\vec{x}\, \partial_\rho\left(F^{0\rho}\partial^\nu f\right)$$

$$= P^\nu + 4c\int d^3\vec{x}\, \underbrace{\partial_i\left(F^{0i}\partial^\nu f\right)}_{=0 \text{ at infinity !}} \quad \text{using } F^{00} = 0$$

So conserved charges are gauge invariant !

We can make $\Theta^{\mu\nu}$ explicitly gauge invariant using the fact that, as discussed in Lecture 5

$$\Theta^{\mu\nu} + \partial_\rho f^{\rho\mu\nu} = T^{\mu\nu}$$

Describes the same conserved charges if $f^{\rho\mu\nu} = -f^{\mu\rho\nu}$

↑ fulfilled !

$\Rightarrow$ choose $f^{\rho\mu\nu} = 4c\, F^{\mu\rho} A^\nu$

$$T^{\mu\nu} = \Theta^{\mu\nu} - 4c\,\partial_\rho\left(F^{\mu\rho}A^\nu\right)$$

$$= 4c\, F^{\mu\rho}\left[\partial^\nu A_\rho - \partial_\rho A^\nu\right] - c\, g^{\mu\nu} F^{\rho\sigma}F_{\rho\sigma}$$

I used $\partial_\rho F^{\mu\rho} = 0$ EoM !

$$\Rightarrow T^{\mu\nu} = 4c\, F^{\mu\rho} F^{\nu}{}_{\rho} - c\, g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

Constant c still not fixed, but in this form

$T^{\mu\nu}$ is explicitly GAUGE INVARIANT (& symmetric!)

Now let's compute the energy $\mathcal{H}$

$$T^{0}{}_{0} = 4c F^{0\rho} F_{0\rho} - c F^{\rho\sigma} F_{\rho\sigma}$$

$$= -4c \vec{E}^2 - 2c (\vec{B}^2 - \vec{E}^2)$$

$$= -2c \left[\vec{E}^2 + \vec{B}^2 \right]$$

$\Rightarrow$ To have usual E.M. energy density

$$\mathcal{H} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \Longrightarrow c = -\frac{1}{4}$$

$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ FIXES LAGRANGIAN!

With this normalization, the spatial momentum then becomes

$$P^i = \int d^3\vec{x}\, T^{0i} = \int d^3\vec{x}\, (\vec{E}\times\vec{B})^i$$ Poynting vector!

Is $F^{\mu\nu}F_{\mu\nu}$ the only Lorentz + Gauge invariant term we can build?

1] consider $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$

still gauge & Lorentz invariant

but it changes sign under PARITY

due to $\varepsilon^{\mu\nu\rho\sigma}$!

With this, you should first notice that

$$\partial_\mu \tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\left[\partial_\rho A_\sigma - \partial_\sigma A_\rho\right]$$

rename σ, ρ + swap in ε

$$= \underbrace{\varepsilon^{\mu\nu\rho\sigma}}_{\text{Antisymm}}\underbrace{\partial_\mu\partial_\rho}_{\text{Symm}} A_\sigma = 0$$

$\Rightarrow$ as long as A^μ normal, well defined, function,

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$ SECOND PAIR OF MAXWELL EQ. IN VACUUM !

2] With $\tilde{F}^{\mu\nu}$ we could build $\tilde{\mathcal{L}} = a F^{\mu\nu} \tilde{F}_{\mu\nu}$

[what about $\tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu}$?]

One can prove that

$$F_{\mu\nu} \tilde{F}^{\mu\nu} = \partial_\mu K^\mu$$ TOTAL DERIVATIVE

$$K^\mu = \varepsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma$$

Which means that

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + a F^{\mu\nu} \tilde{F}_{\mu\nu}$$ describes the

same equations of motion $\Rightarrow$ we usually neglect it!

pay attention to TOPOLOGICAL EFFECTS !

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Should we consider more complicated terms?

For example $(\partial_\rho F^{\mu\nu})(\partial^\rho F_{\mu\nu})$ and similar ...

this is an example of a higher-dimensional operators $\Rightarrow$ they have higher MASS DIMENSION:

Remember Action is dimensionless $S = \int d^4x \, \mathcal{L}$

$\Rightarrow [\mathcal{L}] = 4$ because $[d^4x] = -4$

in natural units!

this implies $[F^{\mu\nu}] = 2$

so $[F^{\mu\nu} F_{\mu\nu}] = 4$ (coupling dimensionless as $(\partial_\mu \phi)\partial^\mu \phi$ or ϕ^4)

but $[(\partial_\rho F^{\mu\nu})(\partial^\rho F_{\mu\nu})] = 6 \Rightarrow$ "suppressed" by mass scale Λ

From Electrodynamics we know E.M. field has 2 degrees of freedom $\Rightarrow$ { SPIN 1 massless particle, PARITY EVEN, $h = \pm 1$ }

[see classification of Poincaré irreps!]

A^μ has 4 degrees of freedom, this is at the basis of all difficulties with QUANTIZATION OF GAUGE FIELDS that we will study LATER.

As long as we limit ourselves to classical fields, we know from electrodynamics that this is solved by FIXING THE GAUGE. A possibility to see the physical degrees of freedom is to use the RADIATION GAUGE, defined as

$$A_0 = 0 \; ; \quad \bar{\nabla} \cdot \bar{A} = 0 \quad \rightsquigarrow \text{ NOT LORENTZ INVARIANT!}$$

notice that this implies Lorentz gauge $\partial_\mu A^\mu = 0$ but it is STRONGER!

in Lorentz gauge, Equations of motion become

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \underbrace{\partial_\mu \partial^\nu A^\mu}_{\substack{= \\ 0 \\ \text{in Lorentz gauge}}} = \underline{\Box A^\nu = 0}$$

Recall Klein-Gordon Eq $(\Box + m^2)\phi = 0$

$\Rightarrow$ here we get 4 K.G.-like equations
for fields with $m=0$
massless dispersion relation $E_p = |\vec{p}|$!

Only after quantization we will be able to connect these with PHOTONS. For now, we can look for solutions expanded in Fourier Modes, as before

$$A_\mu(x) = \sum_{\lambda=\pm 1} \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \left[\varepsilon^\lambda_\mu(p)\, a^\lambda(p)\, e^{-ip\cdot x} + h.c. \right]$$

$\uparrow$ info on POLARIZATIONS

$\uparrow$ Like for ϕ

the Equation of motion then requires

$\Box A_\mu = 0 \quad \Rightarrow \quad p^2 = 0$ massless particles

moreover, Lorentz gauge gives $\partial_\mu A^\mu = 0$

$\Rightarrow \quad \varepsilon_\mu^\lambda(p) \cdot p = 0$ which reduces polarizations from $4 \to 3$

Finally Radiation gauge gives $\begin{cases} \varepsilon_0^\lambda(p) = 0 \\ \vec{p} \cdot \vec{\varepsilon}^\lambda = 0 \end{cases}$

$\Rightarrow$ two TRANSVERSE POLARIZATIONS, as we know from classical Electrodynamics !

Still talking about CLASSICAL FIELDS we can ask ourselves what change if we want to describe a MASSIVE SPIN 1 field $\Rightarrow$ it must have 3 polarizations !

MASSIVE VECTOR FIELD (FREE THEORY)

It should be rather natural to guess that the right Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$$

In fact, Euler-Lagrange Equations change to become

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0$$

$$\hookrightarrow m^2 A^\nu + \partial_\mu F^{\mu\nu} = 0$$

PROCA-MAXWELL Free Equation

We still have an EXTRA DEGREE OF FREEDOM

$\underline{3}$ polarizations $\Rightarrow$ A^μ has $\underline{\underline{4}}$ degrees of freedom!

$\Rightarrow$ A^μ is irreducible repr. of Lorentz (vector!)

but it is REDUCIBLE wrt Little group of massive particle remember little group is $SO(3) \Rightarrow 2j+1$ states for particle with spin j !

How does a Lorentz vector $V^\mu = (V^0, \vec{V})$ transform under $SO(3)$?

$\Rightarrow$ $V^0 \to V^0$ does not change
this component has "spin 0" or transforms under trivial irrep of $SO(3)$!

$\vec{V} \to R \cdot \vec{V}$ irrep of spin 1 !

In group theory language we say that

$$V^\mu \in \underset{\substack{\uparrow \\ \text{label by} \\ j}}{0 \oplus 1} \quad \text{or} \quad \underset{\substack{\uparrow \\ \text{label by} \\ \text{DIMENSION of} \\ SO(3) \text{ irrep !}}}{1 \oplus 3}$$

Interestingly, at variance with massless case, we do not NEED to impose any extra condition by hand $\Rightarrow$ the extra condition is already implicit in the equations of motion

$$\partial_\nu \left[m^2 A^\nu + \partial_\mu F^{\mu\nu} = 0 \right]$$

$$\Rightarrow \quad m^2 \partial_\nu A^\nu + \cancel{\underbrace{\partial_\nu \partial_\mu}_{\text{symm}} F^{\mu\nu}} = 0$$

antisymmetric

so as long as $m^2 \neq 0$ this implies $\boxed{\partial_\mu A^\nu = 0}$

this looks like Lorenz Gauge, but it's not a gauge choice, there is NO GAUGE FREEDOM in the massive case !

$A^\mu \to A^\mu + \partial^\mu f$ Spoils MASS TERM $A^\mu A_\mu$!

In conclusion, classical theory of free MASSIVE spin 1 field is described by

$$\begin{cases} \partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \\ \partial_\mu A^\mu = 0 \end{cases}$$

which together imply $(\Box + m^2) A^\nu = 0$

Klein Gordon Equation for each A^ν !

Writing again

$$A_\mu = \sum_{\lambda=1}^{3} \int \frac{d^3p}{(2\pi)^3 2E_p} \left[\varepsilon_\mu^\lambda(p)\, a_\lambda(p)\, e^{-ip\cdot x} + h.c. \right]$$

we have now massive dispersion relation $p^2 = m^2$

$E_p = \sqrt{\vec{p}^2 + m^2}$

Later on we will see how under QUANTIZATION many subtleties emerge taking $m \to 0$ Limit !

In building a gauge invariant theory for the scalar & spinor field, we have seen that it requires a minimal interaction with a massless spin 1 field (m=0 to preserve GAUGE INV!)

$$\mathcal{L} = D_\mu \phi D^\mu \phi^* - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$ SCALAR ELECTRODYNAMICS

$$\mathcal{L} = \bar{\psi} (i \not{D} - m) \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$ SPINOR ELECTRODYNAMICS

Up to here, everything was entirely CLASSICAL, but we have already learnt a lot about fields, symmetries and how to build INTERACTION TERMS

Starting from the next lecture, we will begin our study of QUANTIZED FIELD THEORIES