

Classical Field Theory:

spin- $\frac{1}{2}$ fields

QFT

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As far as we know, matter is made by spin- $\frac{1}{2}$ elementary components $\Rightarrow$ electrons, quarks, protons...

Let us therefore see how we can build a field theory for spin $\frac{1}{2}$ (classical) fields

We have seen that fundamental building blocks from the point of view of Poincaré are Weyl spinors

start with $(\frac{1}{2}, 0)$ LEFT-HANDED ψ_L

First, we need a KINETIC TERM which is Lorentz invariant $\Rightarrow$ must involve ∂_μ and so we need a 4-vector to get a SCALAR! ∇

We have seen that $\psi_L^\dagger \bar{\sigma}^\mu \psi_L$ behaves indeed,

as 4-vector under Lorentz (see Lecture 3)

$$\left[\begin{array}{l} \text{Similarly } \psi_R^\dagger \sigma^\mu \psi_R \text{ is also 4-vector} \\ \text{and remember } \sigma^\mu = (1, \vec{\sigma}) \\ \bar{\sigma}^\mu = (1, -\vec{\sigma}) \end{array} \right]$$

$\Rightarrow$ can we build something else?

$\psi_L^\dagger \psi_L$ IS NOT LORENTZ INVARIANT

we would need $\psi_L^\dagger \psi_R$ or $\psi_R^\dagger \psi_L$

We can therefore build simplest Lorentz scalar or

$$\mathcal{L}_K = i \psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L \quad [\psi_L]_E = \frac{3}{2} !$$

$\downarrow$

non-dimension

where the i is needed to get $\int d^4x \mathcal{L}_K \in \mathbb{R}$

$$(i \psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L)^* = -i [\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L]^*$$

$$= -i [\partial_\mu \psi_L]^\dagger (\bar{\sigma}^\mu)^\dagger \psi_L$$

$$= -i \cancel{\partial_\mu [\psi_L^\dagger \bar{\sigma}^\mu \psi_L]} + i \psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L$$

Total derivative

where I used $(\bar{\sigma}^\mu)^\dagger = \sigma^\mu$! hermiticity

if we consider this as the complete $\mathcal{L}$ then
Equations of motion become

$$\psi_L: \quad \frac{\partial \mathcal{L}}{\partial \psi_L} - \cancel{\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_L}} = 0$$

$$\Rightarrow \bar{\sigma}^\mu \partial_\mu \psi_L = 0 \quad \text{Weyl Equation}$$

FIRST ORDER!

Let's now multiply this by $\sigma^\nu \partial_\nu$ then

$$\text{using } \sigma^\nu \bar{\sigma}^\mu \partial_\mu \partial_\nu = \frac{1}{2} [\sigma^\nu \bar{\sigma}^\mu + \bar{\sigma}^\mu \sigma^\nu] \partial_\mu \partial_\nu$$

$$= \frac{1}{2} \{ \sigma^\nu, \bar{\sigma}^\mu \} \partial_\mu \partial_\nu \quad \text{and}$$

$$\{ \sigma^\nu, \bar{\sigma}^\mu \} = \begin{cases} 2\mathbb{1} & \text{for } \mu = \nu = 0 \\ \{1, -\sigma^i\} = -2\sigma^i & \text{for } \mu=i, \nu=0 \\ \{ \sigma^i, \mathbb{1} \} = +2\sigma^i & \text{for } \mu=0, \nu=i \\ -\{ \sigma^i, \sigma^j \} = -2\delta^{ij} \mathbb{1} & \text{for } \mu=i, \nu=j \end{cases}$$

then $\frac{1}{2} \{ \sigma^\nu, \bar{\sigma}^\mu \} \partial_\mu \partial_\nu = \partial_0^2 - \partial_i^2 = \square$

which then gives

$$\sigma^\nu \partial_\nu [\bar{\sigma}^\mu \partial_\mu \psi_L] = \square \psi_L = 0$$

AGAIN we're back to Klein-Gordon!

Consider then a plane-wave solution, say

$$\tilde{\psi}_L(p) = u_L(p) e^{-ip \cdot x} \Rightarrow p^2 = 0 \\ E_p = |\vec{p}|$$

but we also have 1st order equation, which is

$$\bar{\sigma}^\mu \partial_\mu u_L e^{-ip \cdot x} = -i \bar{\sigma}^\mu p_\mu u_L e^{-ip \cdot x} = 0$$

$$\Rightarrow \bar{\sigma}^\mu p_\mu u_L = 0 \quad (E_p + \vec{\sigma} \cdot \vec{p}) u_L = 0$$

↑
because $\bar{\sigma}^\mu = (1, -\vec{\sigma})!$

$$\Rightarrow \boxed{\frac{\vec{\sigma} \cdot \vec{p}}{E_p} u_L = -u_L} \quad \text{with } E_p = |\vec{p}|$$

remember now that for spin $\frac{1}{2}$ field the angular momentum is given by $\vec{J} = \frac{\vec{\sigma}}{2}$

$$\Rightarrow \frac{\hat{p} \cdot \vec{J}}{|\vec{p}|} u_L = -\frac{1}{2} u_L \quad \text{with } \frac{\hat{p}}{|\vec{p}|} = \frac{\vec{p}}{|\vec{p}|}$$

which means that ψ_L has projection of spin along direction of momentum $= -\frac{1}{2}$

ψ_L has **HELICITY** $= -\frac{1}{2}$, which is

what we expect from little group massless spin $\frac{1}{2}$!

For now this is a statement about the Weyl field which belongs to $(\frac{1}{2}, 0)$ (REP of LORENTZ)

$\Rightarrow$ After Quantization it'll become about PARTICLES!

We can compute the energy-momentum tensor :

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_L)} \partial^\nu \psi_L + \cancel{\partial^\nu \psi_L^\dagger} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_L^\dagger)} - \cancel{g^{\mu\nu} \mathcal{L}}$$

$\Downarrow$
 because on
 E.O.M $\mathcal{L} = 0$
 $\bar{\sigma}^\mu \partial_\mu \psi_L = 0$

$$= i \psi_L^\dagger \bar{\sigma}^\mu \partial^\nu \psi_L$$

such that $\Theta^{00} = i \psi_L^\dagger \partial^0 \psi_L \leadsto$ HAMILTONIAN

Also, since this is a COMPLEX FIELD, we can also

consider U(1) transformation $\psi_L \rightarrow e^{i\theta} \psi_L$

$\mathcal{L} = i \psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L$ is INVARIANT ∇

We find conserved current $j^\mu = \psi_L^\dagger \bar{\sigma}^\mu \psi_L$

(From Noether theorem)

such that the conserved charge is

$$Q = \int d^3x \psi_L^\dagger \psi_L$$

this can also be
interpreted as electric charge!
(see later!)

Notice also that we can start from EQUIVALENT $\mathcal{L}$

$$\mathcal{L} = i \psi_L^\dagger \bar{\sigma}_\mu \partial^\mu \psi_L - \underbrace{\frac{i}{2} \partial^\mu [\psi_L^\dagger \bar{\sigma}_\mu \psi_L]}_{\text{total derivative}}$$

$$= \frac{i}{2} [\psi_L^\dagger \bar{\sigma}_\mu \partial^\mu \psi_L - \partial^\mu \psi_L^\dagger \bar{\sigma}_\mu \psi_L]$$

$$= \frac{i}{2} \psi_L^\dagger \bar{\sigma}_\mu \overset{\leftrightarrow}{\partial}^\mu \psi_L \quad \underline{\text{SAME Equations of motion}}$$

Conserved current looks different, but charge are
the same ! **EXERCISE**

Now of course this theory is NOT parity invariant
since P sends $\psi_L \rightarrow \psi_R$

so let's first construct the equivalent theory
for a RIGHT-HANDED WEYL SPINOR

$$\mathcal{L}_R = i \psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$$

where similarly EoM $\sigma^\mu \partial_\mu \psi_R = 0$

$$\Rightarrow \square \psi_R = 0 \text{ multiplying by } \bar{\sigma}^\nu \partial_\nu$$

but now writing out components we get

$$\psi_R = u_R e^{-ip \cdot x} \quad \text{then} \quad \underline{\phi^2 = 0} ; \quad \underline{E_p = |\vec{p}|}$$

$$-i \sigma^\mu p_\mu u_R = 0 \Rightarrow (E_p - \vec{\sigma} \cdot \vec{p}) u_R = 0$$

$$\vec{J} \cdot \hat{\vec{p}} u_R = +\frac{1}{2} u_R \quad \underline{\text{helicity } +\frac{1}{2} !}$$

DIRAC FIELDS

We would like to construct now a theory with both ψ_L & $\psi_R \Rightarrow$ PARITY INVARIANCE !

now I can consider also $\psi_L^\dagger \psi_R$ & $\psi_R^\dagger \psi_L$!

they have dim $[\psi^\dagger \psi] = 3$, so they are natural mon terms

$$\mathcal{L}_0 = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$$

$$+ a \psi_L^\dagger \psi_R + b \psi_R^\dagger \psi_L$$

$a, b \propto m$
mon of ψ !

rearrange them into

parity even & odd combinations

$$+ \mathcal{Q}_E (\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) + \mathcal{Q}_O (\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L)$$

pseudo-scalar

exclude for P-EVEN 9

$\Rightarrow$ we focus on parity EVEN Theory (Dirac theory)

Kinetic part is PARITY EVEN because

$$P: \psi_L \rightarrow \psi_R \quad P: \psi_R \rightarrow \psi_L$$

$$P: \partial_\mu \rightarrow (\partial_0, -\partial_i) \Rightarrow P: \partial_\mu \bar{\psi} = \partial_\mu \psi \quad \nabla$$

Fix $a_0 = 0$; $a_E = -m$, as we see now, mass term

As usual, we can work out Euler-Lagrange Eqs

$$\left. \begin{aligned} \bar{\psi} i \partial_\mu \psi_L &= m \psi_R \\ \psi i \partial_\mu \psi_R &= m \psi_L \end{aligned} \right\} \begin{array}{l} \text{vary } \psi_L, \psi_L^* \\ \psi_R, \psi_R^* \\ \text{independently!} \end{array}$$

As before, take first Eq and multiply $(i \sigma^\nu \partial_\nu)$

$$\text{LHS} = -\sigma^\nu \bar{\psi} i \partial_\nu \partial_\mu \psi_L = -\frac{1}{2} \{\sigma^\nu, \bar{\psi} i\} \partial_\nu \partial_\mu \psi_L$$

$$= -\square \psi_L \quad ; \quad \text{RHS} = m i \sigma^\nu \partial_\nu \psi_R = m^2 \psi_L$$

so we get $-\square \psi_L = m^2 \psi_L$

$\Rightarrow (\square + m^2) \psi_L = 0$ massive Klein-Gordon Eq!

and also $(\square + m^2) \psi_R = 0$

↓
"m" is indeed right
mass parameter for ψ_L, ψ_R

We can now recast everything in terms of one
single DIRAC SPINOR

$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \equiv (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ as discussed

this is a special representation for Dirac's spinors
called CHIRAL REPRESENTATION, we will see
more representations later on.

In this representation we can DEFINE Dirac's γ matrices

$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \Rightarrow \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ 4×4
MATRIX 11

such that $\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1}_{4 \times 4}$ CLIFFORD ALGEBRA

↑
Dirac Spinor space

$\Rightarrow \gamma^\mu$ 4×4 matrices, smallest possible DIMENSION!
(more LATER)

With this notation the two "Weyl" massive eps become

$$(i \partial_\mu \gamma^\mu - m) \psi_D = 0$$

or $(i \not{\partial} - m) \psi_D = 0$ where $\not{\partial} = \gamma^\mu \partial_\mu$

⏟
DIRAC EQUATION !

We can also derive this equation directly from a $\mathcal{L}$

$$\mathcal{L}_D = \bar{\psi}_D (i \not{\partial} - m) \psi_D = \bar{\psi} (i \not{\partial} - m) \psi$$

↑
Drop "D" for simplicity

And we defined DIRAC ADJUGATE $\bar{\psi} = \psi^\dagger \gamma^0$

$\Rightarrow \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is necessary to have right kinetic & mass terms in terms of ψ_L & ψ_R !

γ^μ matrices are a representation of $Cl(1,3)$

Interestingly, consider $\Sigma_{\mu\nu} = \frac{i}{4} [\gamma_\mu, \gamma_\nu]$

you can prove that

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = i(g_{\nu\rho}\Sigma_{\mu\sigma} - g_{\mu\rho}\Sigma_{\nu\sigma} - g_{\nu\sigma}\Sigma_{\mu\rho} + g_{\mu\sigma}\Sigma_{\nu\rho})$$

(EXERCISE)

$\leadsto$ Clifford algebra "contains" a representation of Lorentz algebra ! $\hookleftarrow$

remember we wrote for a Lorentz transformation (see Lecture 3)

$$\Lambda = e^{-\frac{i}{2} \omega_{\mu\nu} J^{\mu\nu}}$$

then we can generate a REPRESENTATION of Lorentz on
DIRAC-SPINOR SPACE !

$$\Lambda_S = S = e^{-\frac{i}{2} \omega_{\mu\nu} \Sigma^{\mu\nu}}$$

↑ generators on SPINOR
SPACE

and one can see explicitly that

$$S \psi = e^{-\frac{i}{2} \omega_{\mu\nu} \Sigma^{\mu\nu}} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{bmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{bmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

where $\Lambda_L = e^{(-i\vec{\sigma} - \vec{\eta}) \frac{\vec{\sigma}}{2}}$; $\Lambda_R = e^{(-i\vec{\sigma} + \vec{\eta}) \frac{\vec{\sigma}}{2}}$

$$\theta^i = \frac{1}{2} \epsilon^{ijk} \omega_{jk} ; \quad \eta^i = \omega^{i0}$$

⇒ of course, this is a REDUCIBLE representation

$$\text{since Dirac} = \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right) !$$

Now back to Dirac Equation & Dirac Spinors

We define a 5th γ matrix γ_5

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix} \quad \text{in chiral representation}$$

with $\boxed{\{\gamma^\mu, \gamma^5\} = 0} \Leftarrow$ important EXERCISE!

With this we see $P_L = \frac{1-\gamma_5}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$

$$P_R = \frac{1+\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

Project out Left & Right components of spinors

$$\Rightarrow P_L \cdot P_L = P_L; \quad P_R P_R = P_R; \quad P_L P_R = P_R P_L = 0$$

projector operators ▼

$$P_L \psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \quad P_R \psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$$

Now, could we have guessed the form of $\mathcal{L}_D$ using usual requirements? $\Rightarrow$ Lorentz invariance
Parity invariance etc

Start noticing that γ^μ is 4×4 matrix

- a general 4×4 matrix has 16 (complex) entries
- we can build 16 independent 4×4 matrices

$$\{ 1, \gamma^\mu, \gamma^5, \gamma^\mu \gamma^5, \Sigma^{\mu\nu} \}$$

10 matrices

$$\Sigma^{\mu\nu} = -\Sigma^{\nu\mu}$$

6 independent ones

$\Rightarrow$ Any 4×4 $M =$ linear combination of these 16 !

this allows us to classify all possible terms in

a Lagrangian involving DIRAC SPINORS

$$\mathcal{L}_D = a \bar{\psi} \psi + b \bar{\psi} \gamma^5 \psi + c_\mu \bar{\psi} \gamma^\mu \psi + d_\mu \bar{\psi} \gamma^5 \gamma^\mu \psi + e_{\mu\nu} \bar{\psi} \Sigma^{\mu\nu} \psi$$

• We have seen $\bar{\psi} \psi = \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$ PARITY EVEN SCALAR

• now check: $\bar{\psi} \gamma^5 \psi = \psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L$ PARITY ODD

exclude for P-invariant theory! { pseudo scalar under full Lorentz group (including P) }

• then we find: $\bar{\psi} \gamma^\mu \psi = \psi_L^\dagger \underbrace{\sigma^\mu}_{\text{both 4-vectors}} \psi_L + \psi_R^\dagger \sigma^\mu \psi_R$

this must transform as a 4 vector = γ^μ_ν "VECTOR CURRENT"

notice that under P $\psi_L \leftrightarrow \psi_R$

which implies $j^0 \rightarrow j^0$; $j^i \rightarrow -j^i$ TRUE 4-vector!

- on the other hand: $\bar{\psi} \gamma^5 \gamma^\mu \psi = \psi_L^\dagger \bar{\sigma}^\mu \psi_L - \psi_R^\dagger \bar{\sigma}^\mu \psi_R$
 $= \int_A^\mu$ AXIAL-VECTOR CURRENT

is a pseudo vector because
 $\Rightarrow$ exclude it for PARITY
 INVARIANT THEORY

$$\left. \begin{array}{l} j_A^0 \rightarrow j_A^0 \\ j_A^i \rightarrow +j_A^i \end{array} \right\} \text{ under PARITY!}$$

- fully $\bar{\psi} \Sigma^{\mu\nu} \psi$ is a G-tensor

$\Rightarrow$ From this we see that we end up with usual
FREE DIRAC EQUATION $b = d_\mu = 0$; $\begin{cases} c_\mu = i\partial_\mu \\ a = -m \end{cases}$

EXTRA PIECE

$\bar{\psi} \Sigma^{\mu\nu} \psi$ requires something to contract two
 indices: " $e_{\mu\nu}$ " which is ANTI-SYMMETRIC

$\rightarrow$ We can use $F^{\mu\nu} \Rightarrow e_{\mu\nu} = e F_{\mu\nu}$

$\Rightarrow e \bar{\psi} \Sigma^{\mu\nu} \psi F_{\mu\nu}$ non-minimal coupling! MAGNETIC DIPOLE INTERACTION

$[e] = \frac{1}{E}$ again "suppressed" $\rightarrow$ LATER!

So let us focus on our FREE DIRAC LAGRANGIAN

$$\mathcal{L}_0 = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi$$

We have seen that γ^μ generate CLIFFORD ALGEBRA
and we found one specific representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{CHIRAL REPR.}$$

in which $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

this is not the only one: consider $\psi' = U \psi$

U unitary matrix
 x^μ INDEPENDENT !

then $\psi = U^{-1} \psi' = U^\dagger \psi'$; $\bar{\psi} = (U^\dagger \psi')^\dagger \gamma^0 = \psi'^\dagger U \gamma^0$
 $\Downarrow$

$$\mathcal{L}_0 = \psi'^\dagger U \gamma^0 [i \gamma^\mu \partial_\mu - m] U^\dagger \psi'$$

$$\mathcal{L} = \bar{\psi}^t \underbrace{U \gamma_0 U^\dagger}_{\gamma_0'} [i \gamma^\mu \partial_\mu - m] U^\dagger \psi'$$

$$= [\bar{\psi}^t \gamma_0'] [i \partial_\mu U \gamma^\mu U^\dagger - m] \psi'$$

$$= \bar{\psi}' [i \partial_\mu \gamma^\mu' - m] \psi'$$

with γ^μ' in new representation $\gamma^\mu' = U \gamma^\mu U^\dagger$

Indeed this transformation does not change CLIFFORD ALG.

$$\{\gamma^\mu', \gamma^\nu'\} = \gamma^\mu' \gamma^\nu' + \gamma^\nu' \gamma^\mu' = U [\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu] U^\dagger$$

$$= 2g^{\mu\nu} \underbrace{UU^\dagger}_{\text{IF } U \text{ UNITARY!}} = 2g^{\mu\nu} \mathbb{1} \quad \checkmark$$

so indeed γ^μ' still represent same algebra. Dirac

Equation is invariant ; $\mathcal{L}$ also invariant !

Let us consider a specific choice for $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$\psi' = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_L + \psi_R \\ \psi_R - \psi_L \end{pmatrix} = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \quad \text{"STANDARD" representation}$$

in this case $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In this representation, ψ_L, ψ_R are more HIDDEN,
so it's less convenient in HIGH ENERGY LIMIT
 $\Rightarrow$ they are EIGENSTATES of helicity for $m=0$!

For LOW ENERGY, $m \neq 0$ and helicity is NOT
an eigenstate, in this case STANDARD REPR.
is more useful

What about solutions of the free Dirac Equation?

We have seen that L & R satisfy Klein-Gordon Eq

$$\Rightarrow (i\not{\partial} - m)\psi = 0 \quad \text{multiply by } -i\not{\partial}$$

$$+ \partial_\mu \partial_\nu \gamma^\mu \gamma^\nu \psi + m \underbrace{i\not{\partial} \psi}_{= m\psi} = 0$$

$$\square \psi + m^2 \psi = 0$$

$$(\square + m^2) \psi = 0$$

so again general solution can be decomposed in Fourier modes

$$\psi(x) = \sum_{s=1}^2 \int \frac{d^3 \vec{p}}{(2\pi)^3 (2E_p)} \left[u_s(p) a_s(p) e^{-ip \cdot x} + v_s(p) b_s^*(p) e^{ip \cdot x} \right]$$

$u_s(p), v_s(p)$ (4-DIM) SPINORS $s=1,2$ SPINS!

$u_s(p), v_s(p)$ & $a_{p,s}, b_{p,s}^*$ are different because

ψ COMPLEX FIELD! $\Rightarrow$ I could describe $a, b \rightarrow u, v$ but convenient for QUANTIZATION!

K.G. equation gives DISPERSION RELATION $p^2 = m^2$

DIRAC EQUATIONS gives more information on u, v !

$$(i\not{D} - m)\psi = 0 \quad \begin{cases} (\not{D} - m) u_s(p) \\ (\not{D} + m) v_s(p) \end{cases}$$

Let's look for an EXPLICIT representation for u, v
in CHIRAL BASIS

$$u(p) = \begin{pmatrix} u_L \\ u_R \end{pmatrix} \quad \text{and assume } m \neq 0$$

then we go to rest frame $p^\mu = (m, \vec{0})$

$$1] \quad m[\gamma^0 - 1] u_s(p) = 0$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$2] \quad m[\gamma^0 + 1] v_s(p) = 0$$

$$\Rightarrow 1] \quad u_s^R = u_s^L$$

$$2] \quad v_s^R = -v_s^L \quad \nabla$$

so Dirac Equation says $\begin{Bmatrix} u^L, u^R \\ v^L, v^R \end{Bmatrix}$ are not independent $\Rightarrow$ $\text{SPIN } \frac{1}{2}$ has 2 degrees of freedom!

Let's focus on $u_s^{L,R}$

Define BASIS $\begin{cases} \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$
 $\xi_s^\dagger \xi_r = \delta_{sr}$

and normalize

$$u_L = u_R = \sqrt{m} \xi$$

in REST FRAME!

now we can get relation for general p^μ by BOOST

$$\Rightarrow (m, \vec{0}) \rightarrow (E, 0, 0, p_z) \text{ boost along } z$$

EXERCISE

$$u_s(p) = \begin{pmatrix} \left[\sqrt{E+p_z} \left(\frac{1-\sigma_3}{2} \right) + \sqrt{E-p_z} \left(\frac{1+\sigma_3}{2} \right) \right] \xi_s \\ \left[\sqrt{E+p_z} \left(\frac{1+\sigma_3}{2} \right) + \sqrt{E-p_z} \left(\frac{1-\sigma_3}{2} \right) \right] \xi_s \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\frac{1-\sigma_3}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1+\sigma_3}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Consider the high-energy limit $p_\mu \rightarrow (E, 0, 0, E)$:

$$u_s(p) \rightarrow \begin{pmatrix} \sqrt{2E} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \xi_s \\ \sqrt{2E} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \xi_s \end{pmatrix} \quad \begin{aligned} \xi_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \xi_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$$u_1(p) \rightarrow \sqrt{2E} \begin{pmatrix} 0 \\ \xi_1 \end{pmatrix}$$

$$u_2(p) \rightarrow \sqrt{2E} \begin{pmatrix} \xi_2 \\ 0 \end{pmatrix}$$

only RIGHT HANDED

only LEFT HANDED

note that $\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ SPIN UP ; $\xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ SPIN DOWN

Similarly for $v_s(p)$

$$v_s(p) = \begin{pmatrix} \left[\sqrt{E+p_2} \left(\frac{1-\sigma_3}{2} \right) + \sqrt{E-p_2} \left(\frac{1+\sigma_3}{2} \right) \right] \eta_s \\ - \left[\sqrt{E+p_2} \left(\frac{1+\sigma_3}{2} \right) + \sqrt{E-p_2} \left(\frac{1-\sigma_3}{2} \right) \right] \eta_s \end{pmatrix} \quad \begin{aligned} \eta_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \eta_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\Leftrightarrow \xi_s \end{aligned}$$

↑
different sign as $u_s(p)$!

and we can define $\bar{u}_s = u_s^\dagger \gamma^0$; $\bar{v}_s = v_s^\dagger \gamma^0$

$\Rightarrow$ Dirac Adjoint Spinors

the normalization we chose implies

$$\bar{u}_r(p) u_s(p) = 2m \delta_{rs} ; \quad \bar{v}_r(p) v_s(p) = -2m \delta_{rs}$$

$$u_r^\dagger(p) u_s(p) = 2E_p \delta_{rs} ; \quad v_r^\dagger(p) v_s(p) = 2E_p \delta_{rs}$$

+ Dirac ORTHOGONALITY $\begin{cases} \bar{u}_r(p) v_s(p) = 0 \\ \bar{v}_r(p) u_s(p) = 0 \end{cases}$

note that we can also rewrite spinors as

$$u_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi_s \\ \sqrt{p \cdot \bar{\sigma}} \xi_s \end{pmatrix} \quad v_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta_s \\ -\sqrt{p \cdot \bar{\sigma}} \eta_s \end{pmatrix}$$

$\Rightarrow \sqrt{\text{MATRIX}}$ implies POSITIVE ROOT OF EACH EIGENVALUE

this form is valid also for p^μ not along z-direction

$$\Rightarrow \text{remember } (\not{p} \cdot \sigma)(\not{p} \cdot \bar{\sigma}) = p^\mu p^\nu \sigma_\mu \bar{\sigma}_\nu = p^2 = m^2$$

SPIN SUMS

For later, it will be useful to be able to sum over all polarizations of particles involved

notice:

$$\sum_{s=1,2} u_s(p) \bar{u}_s(p) = \sum_s \begin{pmatrix} \sqrt{p \cdot \sigma} \xi_s \\ \sqrt{p \cdot \bar{\sigma}} \xi_s \end{pmatrix} (\xi_s^\dagger \sqrt{p \cdot \bar{\sigma}}, \xi_s^\dagger \sqrt{p \cdot \sigma})$$

$$= \sum_s \begin{bmatrix} m \xi_s \xi_s^\dagger & p \cdot \sigma \xi_s \xi_s^\dagger \\ p \cdot \bar{\sigma} \xi_s \xi_s^\dagger & m \xi_s \xi_s^\dagger \end{bmatrix} = \begin{bmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{bmatrix}$$

because $\sum_s \xi_s \xi_s^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ identity !

$$= \not{\gamma} \cdot p + m = \not{\not{p}} + m \quad \text{true in every} \\ \text{repres. for } \gamma^\mu$$

Similarly we get

$$\sum_s v_s(p) \bar{v}_s(p) = \not{p} - m$$

$\left[\begin{array}{l} \bar{u}u, \bar{v}v \text{ numbers} \\ u\bar{u}, v\bar{v} \text{ } 4 \times 4 \text{ MATRICES!} \end{array} \right.$

As a last comment, the DIRAC LAGRANGIAN has an extra set of Global Symmetries

let us go back to CHIRAL BASIS

$$\mathcal{L}_D = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$

imagine now to generalize $U(1)$ transform. of complex scalar field as follows

$$\psi_L \rightarrow e^{i\partial_L} \psi_L \quad \psi_R \rightarrow e^{i\partial_R} \psi_R \quad \partial_L \neq \partial_R$$

$U(1) \times U(1)$ transformation $\longleftarrow$ in general

this is clearly a symmetry of $\mathcal{L}_D$ if $m=0$

note now two special cases

• $\partial_L = \partial_R = \alpha$ $\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \rightarrow e^{i\alpha} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ $\leftrightarrow$ DIRAC SPINOR

$$\psi \rightarrow e^{i\alpha} \psi$$

the $\mathcal{L}_D$ is invariant under this transformation even if $m \neq 0$!

• $\partial_R = -\partial_L = \beta$ $\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \rightarrow \begin{pmatrix} e^{-i\beta} \psi_L \\ e^{+i\beta} \psi_R \end{pmatrix} = e^{i\beta \gamma_5} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

because $\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Downarrow$
symmetry only if $m=0$

EXERCISE :

prove that they are symmetries using $\mathcal{L}_D$ in terms of ψ

in fact $m\bar{\psi}\psi \rightarrow m(\psi^\dagger \gamma^0 \psi)' = m[e^{i\beta\gamma^5} \psi]^\dagger \gamma^0 e^{i\beta\gamma^5} \psi$

$$= m\psi^\dagger \underbrace{e^{-i\beta\gamma^5} \gamma^0 e^{i\beta\gamma^5}}_{\gamma^0} \psi$$

$$= m\bar{\psi} e^{+2i\beta\gamma^5} \psi \quad \text{not invariant!}$$

the Noether currents associated to the two symmetries are

$$J_V^\mu = \bar{\psi} \gamma^\mu \psi \quad \text{for } U(1) \text{ vector } e^{i\alpha} \psi$$

$$J_A^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi \quad \text{for } U(1) \text{ axial } e^{i\beta\gamma^5} \psi$$

↑

this is called a CHIRAL TRANSFORMATION & we

say that massless Dirac theory is CHIRAL SYMMETRIC

Note also $\partial_\mu J_A^\mu = 2im\bar{\psi}\gamma^5\psi$

m breaks
chiral symmetry

EXERCISE !

Making $U(1)_V$ a LOCAL SYMMETRY

When discussing the scalar (complex) field, we have seen that to promote GLOBAL $U(1)$ to a LOCAL symmetry we need to introduce a new spin 1 vector field A_μ [connection] which defines a COVARIANT DERIVATIVE

$$\partial_\mu \phi \rightarrow D_\mu \phi = (\partial_\mu + iq A_\mu) \phi$$

↑
charge

↑
electromagnetic field !

If you go back to our derivations, you will see that nothing depends on LORENTZ PROPERTIES of the field $\phi \Rightarrow$ we can repeat the very same argument for ψ

if we want invariance under LOCAL $U(1)$

$$\psi \rightarrow e^{i\alpha(x)} \psi$$

We need to CHANGE $\partial_\mu \rightarrow D_\mu = \partial_\mu + iqA_\mu$

$$\mathcal{L}_0 \rightarrow \bar{\psi} [i\gamma^\mu D_\mu - m] \psi$$

$$= \bar{\psi} [i\gamma^\mu \partial_\mu - m] \psi - \underbrace{q A_\mu \bar{\psi} \gamma^\mu \psi}$$

minimal coupling of

A_μ to ψ through
vector current j_V^μ !

CURRENT REMAINS CONSERVED

so theory still has

GLOBAL $U(1)$ ✓



$$Q = \int d^3\vec{x} \bar{\psi} \gamma^0 \psi$$

$$= \underbrace{\int d^3x \psi^\dagger \psi}$$

will become electric charge after QUANTIZATION !

if we add to this $\mathcal{L}$ the kinetic term for A_μ we get $\mathcal{L}_{\text{QED}}$ (QUANTUM) ELECTRODYNAMICS
↑
we have not quantized it yet!

$$\mathcal{L}_{\text{QED}} = \bar{\psi} [i \not{D} - m] \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

So we see again that by making global symmetries LOCAL for SCALAR & SPINOR FIELDS (MATTER!) we naturally generate SPIN 1 fields

In the next lecture we will see what we can say classically about these theories ...