The Real & Complex scalar fields; a first look at gauge invariance

QFT 1

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Until now we have been very general and abstract. We will now start applying what we learnt to the case of the SCALAR FIELD $\Rightarrow$ as we have seen generalizing this to HIGHER SPINS will require to use fields that transform appropriately under Lorentz Transformations

Start considering a single, REAL SCALAR FIELD $\phi(x)$

We want a Lorentz invariant action with non-trivial dynamics $\Rightarrow$ to have dynamics, $\mathcal{L}$ must depend on $\partial_\mu \phi$! Simplest example is then

$$\mathcal{L} = \frac{1}{2} \underbrace{\partial_\mu \phi \, \partial^\mu \phi}_{\text{Lorentz index saturated}} + \underbrace{P(\phi)}_{\text{any function of } \phi \text{ automatically invariant}}$$

there are various arguments to "restrict" $P(\phi)$

$\Rightarrow$ imagine we can Taylor expand it close to $\phi = 0$

$$P(\phi) = p_0 + p_1\phi + p_2\phi^2 + \cdots$$

constant shift — linear — quadratic term

Let's stop at these first three terms

1) p_0 is a constant shift in $\mathcal{L}$ [& in $\mathcal{H}$]

$\Rightarrow$ it definitely does not change the equations of motion

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0}$$

we can forget about it

2) linear term does change EoM but in a trivial way, in fact imagine to redefine the field

$$\phi \to \varphi + \phi_0 \quad \text{then}$$

$$\mathcal{L} \to \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + p_1 \varphi + \cancel{p_1 \phi_0} \quad \text{constant}$$

$$+ p_2 \varphi^2 + \underbrace{\cancel{p_2 \phi_0^2}}_{\text{constant}} + 2 p_2 \varphi \phi_0$$

$$= \underbrace{\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + p_2 \varphi^2}_{\text{quadratic}} + \cancel{(p_1 + 2 p_2 \phi_0)\varphi}$$

choose $\phi_0 = -\frac{p_1}{2 p_2}$

removes linear term!

$\Rightarrow$ we will see linear terms can also be interpreted as interactions with external sources!

So it make sense to start considering just a quadratic term (on higher orders more later)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi \underbrace{-\frac{1}{2}m^2}\phi^2$$

"m" interpreted as mass of ϕ!

Computing EoM we get

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \; ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\Rightarrow \partial_\mu \partial^\mu \phi + m^2 \phi = \boxed{(\Box + m^2)\phi = 0}$$

(Free) Klein-Gordon Equation

It's easy to see that plane waves solve this eq.

$$\varphi(p) = e^{\pm ip\cdot x} \Rightarrow \begin{array}{l} \partial_\mu \varphi = \pm i p_\mu e^{\pm ip\cdot x} \\ \Box \varphi = -p^2 e^{\pm ip\cdot x} \end{array}$$

so we get $(\Box + m^2)\varphi(p) = (-p^2 + m^2)\varphi(p) = 0$

$\Rightarrow$ imposes the dispersion relation $\begin{cases} p^2 = m^2 \\ p_0^2 = \vec{p}^2 + m^2 = E_p^2 \end{cases}$

this demonstrates why we identified m with the "mass"

$\Rightarrow$ we impose then $m > 0$; $m \in \mathbb{R}^+$

Since $\phi(x)$ must be REAL, the most general solution can then be written as a superposition of:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3(2E_p)} \left[a(p)\, e^{-ip\cdot x} + a(p)^* \, e^{ip\cdot x} \right]_{p_0 = E_p}$$

Lorentz inv. int measure

$E_p = \sqrt{\vec{p}^2 + m^2}$

sometimes called

$\omega_p = E_p$!

Fourier coefficients

guarantee reality condition

$\phi^*(x) = \phi(x)$

We will use a DIFF ONE when QUANTIZING !

$\Rightarrow$ solutions oscillate with $e^{\pm i E_p t}$

$\uparrow$ positive & negative frequency modes

Let's compute the Hamiltonian density $\Rightarrow$ Energy

In general we had:

$$\mathcal{H} = \pi_i \dot{\phi}_i - \mathcal{L} \quad ; \quad \pi_i = \frac{\partial \mathcal{L}}{\partial [\partial_0 \phi_i]}$$

in this case only one field $\Rightarrow$ $\left[\pi = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} = \partial_0 \phi \right]$

$$\Rightarrow \mathcal{H} = [\partial_0 \phi]^2 - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2$$

$\uparrow$ here overall sign of $\mathcal{L}$ is important

$$= \frac{1}{2} \left[(\partial_0 \phi)^2 + (\vec{\nabla} \phi)^2 + m^2 \phi^2 \right] \geq 0$$

it would NOT be ≥ 0 if $\mathcal{L} \to -\mathcal{L}$!

Similarly, for the full Energy-momentum tensor

$$T^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \delta^\mu{}_\nu \mathcal{L}$$

$$\Rightarrow \quad \underbrace{T^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi - g^{\mu\nu} \mathcal{L}}$$

In the case of scalar field, $T^{\mu\nu}$ is manifestly symmetric!

$\Rightarrow$ no need to add anything to it!

Let's compute now explicitly the conserved current associated to Lorentz invariance $\Rightarrow$

$$j^\mu_{\rho\sigma} = -T^\mu{}_\nu \frac{1}{2}\left[\delta^\nu_\rho x_\sigma - \delta^\nu_\sigma x_\rho\right]$$

which was derived for <u>scalar</u> field ($\Delta\phi = 0$) removing a numerical overall $(+\frac{1}{2})$ factor

$$j^{\mu}_{\rho\sigma} = T^{\mu}_{\sigma} x_{\rho} - T^{\mu}_{\rho} x_{\sigma}$$ and the conserved charge is as we saw

$$j^{0}_{\rho\sigma} = T^{0}_{\sigma} x_{\rho} - T^{0}_{\rho} x_{\sigma}$$

$$= (\partial^0\phi)(\partial_\sigma\phi) x_\rho - (\partial^0\phi)(\partial_\rho\phi) x_\sigma$$

$$= (\partial^0\phi)\left[x_\rho \partial_\sigma - x_\sigma \partial_\rho \right] \phi$$

such that, we find $L_{\mu\nu} = i\,(x_\mu \partial_\nu - x_\nu \partial_\mu)$

$\Rightarrow$ Lorentz generator on scalar field

$$j^{0}_{\rho\sigma} = -\,i\,(\partial^0\phi)\, L_{\rho\sigma}\, \phi$$ focus now on spatial components

$$M^{ij} = -\,i \int d^3x\, (\partial^0\phi) L^{ij} \phi$$ and do int by parts on spatial derivatives

$$+\int d^3x\, (\partial_0\phi)\,[x^i\partial^j - x^j\partial^i]\,\phi \;=$$

$$= \cancel{\text{Boundary}} - \int d^3x\, \phi\left(\partial^j[x^i(\partial_0\phi)] - \partial^i[x^j(\partial_0\phi)]\right)$$

$$= -\int d^3x\, \phi\,(x^i\partial^j - x^j\partial^i)(\partial_0\phi)$$

$$= +i\int d^3x\, \phi\, L^{ij}(\partial_0\phi)$$

so we can also use this to write

$$M^{ij} = \frac{i}{2}\int d^3x\,[\phi\, L^{ij}(\partial_0\phi) - (\partial_0\phi)\, L^{ij}\phi]$$

$$= \frac{i}{2}\int d^3x\, \phi\, \overleftrightarrow{\partial_0}\, L^{ij}\,\phi$$

$$= \langle\phi|L^{ij}|\phi\rangle$$

with a newly introduced scalar product $\Rightarrow$

$$\boxed{\langle \phi_1 | \phi_2 \rangle = \frac{i}{2} \int d^3x \; \phi_1 \overleftrightarrow{\partial_0} \phi_2}$$ Klein-Gordon inner product

where

$$f \overleftrightarrow{\partial_\mu} g = f \partial_\mu g - (\partial_\mu f) g$$

this scalar product has the following important properties

$\boxed{1.}$ it's time-independent $\Rightarrow$ important to define a Hilbert-space structure

$$\partial_0 \langle \phi_1 | \phi_2 \rangle = \frac{i}{2} \int d^3\vec{x} \; \partial_0 \left[\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1 \right]$$

$$= \frac{i}{2} \int d^3\vec{x} \left[\phi_1 \partial_0^2 \phi_2 - \phi_2 \partial_0^2 \phi_1 \right]$$

$= \text{use } (\partial_0^2 - \nabla^2 + m^2)\phi = 0$

$$= \frac{i}{2} \int d^3\vec{x} \left[\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1 \right]$$

now do integration by parts to write

$$= -\int d^3\vec{x} \, \cancel{\left[\nabla\phi_1 \nabla\phi_2 - \nabla\phi_2 \nabla\phi_1 \right]} + \cancel{\text{boundary}} = 0 \, !$$

$\boxed{2.}$ this scalar product is NOT POSITIVE DEFINITE

$\Rightarrow$ it does not allow to interpret ϕ as a single-particle wave function, as we already know !

IMPORTANT because, as in $M^{ij} = \langle \phi | L^{ij} | \phi \rangle$

provides connection between

$M^{ij} \rightarrow$ CHARGES

$L^{ij} \rightarrow$ GENERATORS of symmetry as operators acting on fields [INF. DIM. REPRESENTATION]

similarly you can prove

$$P^\mu = \langle \phi | i\partial^\mu | \phi \rangle$$

$P^\mu \rightarrow$ charge, conserved momentum

$i\partial^\mu \rightarrow$ generator acting on fields

WHAT ABOUT HIGHER ORDERS IN ϕ^n ?

From previous discussion you can imagine that they represent INTERACTIONS $\Rightarrow$ they modify K.G. equation.

One thing to realize is that they have a "higher mass dimension"

$$S = \int d^4x \, \mathcal{L} \quad \text{DIMENSIONLESS}; \quad [\mathcal{L}] = 4$$

$$[d^4x] = -4$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \qquad [\phi] = 1 \;\; !$$

$$+ \underbrace{\lambda_4}_{\text{dimensionless}}\phi^4 + \frac{\lambda_5}{\Lambda}\phi^5 + \frac{\lambda_6}{\Lambda^2}\phi^6 + \dots$$

Some dimensionFULL scale Λ

if λ_i all dimensionless

$\Rightarrow$ we call ϕ^6 dim $-$ 6 interaction $\Rightarrow$ it MUST come "suppressed" by some energy scale Λ

$\Rightarrow$ LATER we will see how this connects to RENORMALIZATION

COMPLEX SCALAR FIELD

We would like to build a field theory in which we can provide a definition of the ELECTRIC CHARGE or a CONSERVED CHARGE, so associated to symmetry.

In Minkowski space we already considered all symmetries $\Rightarrow$ we need something more!

Something that acts on space of ϕ_i $\Leftarrow$ need more than one field [note, spinor or vector fields have more components, but their transformation is FIXED by Lorentz]

Simplest case $\Rightarrow$ 2 REAL FIELDS ϕ_1, ϕ_2 $\Rightarrow$

1 complex field $\begin{cases} \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \end{cases}$

Obvious generalization of our $\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi^* \phi$

which gives two independent K.G. equations

$$(\Box + m^2)\phi = 0 \; ; \quad (\Box + m^2)\phi^* = 0$$

Following our discussion for REAL FIELD we can write for a general solution the Fourier Expansion

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 (2E_p)} \left(a(p)\, e^{-ip\cdot x} + b^*(p)\, e^{ip\cdot x} \right)\Big|_{p^0 = E_p}$$

where a_p & b_p are independent since $\phi \neq \phi^*$

this $\mathcal{L}$ has a new global symmetry

$$\phi \to e^{+i\alpha}\phi \; ; \quad \phi^* \to e^{-i\alpha}\phi^*$$

U(1) symmetry !

IF α is a constant then

$$(\phi^*\phi)' = \phi^*\phi \; ; \quad (\partial_\mu \phi^* \partial^\mu \phi)' \to (\partial_\mu \phi^* \partial^\mu \phi)$$

$\Rightarrow$ $\mathcal{L}$ is invariant under GLOBAL U(1)

Let's compute Noether current

INFINITESIMALLY $\phi = \phi + i\alpha\phi$; $\phi^* = \phi^* - i\alpha\phi^*$

$$\delta X^\mu = \varepsilon^a Y_a^\mu(x) = 0 \quad (\text{global, no } x^\mu\text{ !})$$

$$\Delta\phi_i = \varepsilon^a G_{i,a}(\phi, \partial\phi) = \begin{cases} i\phi\alpha \\ -i\phi^*\alpha \end{cases} \quad \varepsilon^a = \alpha\,!$$

Conserved current becomes

$$J_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} G_{i,a} \cancel{- \Theta^\mu_\nu Y_a^\nu}$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}[i\phi] + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}[-i\phi^*]$$

$$= i\phi\,\partial^\mu\phi^* - i\phi^*\partial^\mu\phi$$

$$= i\left[\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right] \Rightarrow \boxed{\partial_\mu J^\mu = 0}$$

or it can be verified using K.G. equations !

CONVENTIONALLY $J^\mu = -J^\mu$ change sign !

Conserved charge is then

$$Q = \int j^0 \, d^3\vec{x} = -i \int d^3\vec{x} \left[\phi \, \partial^0 \phi^* - \phi^* \partial^0 \phi \right]$$

$$= i \int d^3\vec{x} \left[\phi^* \overleftrightarrow{\partial^0} \phi \right]$$

with $\frac{dQ}{dt} = 0$ $\Rightarrow$ Does it have a physical interpretation ?

Defining $\langle \phi_1 | \phi_2 \rangle = i \int d^3x \, \phi_1^* \overleftrightarrow{\partial_0} \phi_2$ which is also time indep. (EXERCISE)

then we see that $Q = \langle \phi | \phi \rangle$ is expectation value of identity operator $\Rightarrow$ $\mathbb{1}$ is GENERATOR of $U(1) = \underline{\underline{e^{i\theta \mathbb{1}}}}$

$\Rightarrow$ it might be disturbing that such a GLOBAL transformation implies that we should perform a transformation on $\phi(x)$ everywhere at once

it might seem to contradict relativity

why should the field at $x = x_1$ be influenced by the field at $x = x_2$ if x_2 is very far away?

It seems then natural to try to promote this symmetry to a <u>LOCAL SYMMETRY</u>

$$\begin{cases} \phi(x) \to e^{i\alpha(x)}\phi(x) \\ \phi^*(x) \to e^{-i\alpha(x)}\phi^*(x) \end{cases} \Rightarrow$$

the phase of ϕ can be fixed differently at each space-time point!

now it's still true that $(\phi^*\phi)' \longrightarrow (\phi^*\phi)$

but the kinetic term is <u>NOT INVARIANT</u>

$$(\partial_\mu\phi)' = \partial_\mu e^{i\alpha(x)}\phi = i[\partial_\mu\alpha(x)]e^{i\alpha(x)}\phi(x) + e^{i\alpha}\partial_\mu\phi$$

$$(\partial_\mu\phi^*)' = -i[\partial_\mu\alpha(x)]e^{-i\alpha(x)}\phi^*(x) + e^{-i\alpha}\partial_\mu\phi^*$$

to first order in $\alpha(x)$

$$(\partial_\mu \phi^* \partial^\mu \phi)' = (\partial_\mu \phi^*)\partial^\mu \phi + i(\partial_\mu \alpha)\left[\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi\right]$$

$$= (\partial_\mu \phi^*)\partial^\mu \phi + [\partial_\mu \alpha]\, j^\mu$$

conserved current !

so $\mathcal{L}' = \mathcal{L} + [\partial_\mu \alpha]\, j^\mu$ not invariant !

We can easily try to fix this "by hand" changing $\mathcal{L}$, but we want to do this differently $\Rightarrow$ the problem is indeed a purely geometrical problem

$\Rightarrow$ how do we properly "take derivatives" ?

if $\phi(x)$ can be redefined by LOCAL PHASE $e^{i\alpha(x)}\phi(x)$

then ∂_μ is not a well defined derivative operator!

$\Rightarrow$ the QUESTION is : how do we compare two fields at two different SPACE-TIME points in a UNAMBIGUOUS WAY ?

$$|\phi(y) - \phi(x)| \neq |e^{i\alpha(y)}\phi(y) - e^{i\alpha(x)}\phi(x)|$$

unless $\alpha(x)$ = constant !

if you studied (or are studying) General Relativity you will recognize the same problem there : how do I compare two vectors in a curved space-time ? $\Rightarrow$ we need a UNIQUE way to "transport" vectors (or fields) at the same phase-space point and compare them

In GR this operation is called PARALLEL TRANSPORT

In QFT the problem is very similar $\Rightarrow$ the local symmetry introduces something that can be interpreted as a CURVATURE in some space.

In this case, the solution comes from introducing a BI-LOCAL FIELD called a WILSON LINE $W(x,y)$ with the following transformation properties

$$\begin{cases} W(x,y) \xrightarrow[\text{local}]{U(1)} e^{i\alpha(x)}\, W(x,y)\, e^{-i\alpha(y)} \\ W(x,x) = 1 \end{cases}$$

With this, consider now the quantity

$$W(x,y)\,\phi(y) - \phi(x) \quad \text{under } U(1) \text{ local}$$

$$\rightarrow e^{i\alpha(x)} \left[W(x,y)\,\phi(y) - \phi(x) \right] \quad \text{so it}$$

can be used to define a "DIFFERENCE"

In geometric language, the Wilson line parallel transports the field $\phi(y) \to$ to x and allows us to COMPARE it to $\phi(x)$ UNAMBIGUOUSLY

$| W(x,y)\phi(y) - \phi(x) |$ is $O(1)$ - local invariant!

Now that we have a way to take DIFFERENCES, we can also define a DERIVATIVE

$$D_\mu \phi(x) = \lim_{\delta x^\mu \to 0} \frac{W(x, x+\delta x)\,\phi(x+\delta x) - \phi(x)}{\delta x^\mu}$$

since $W(x,x) = 1$, we can Taylor expand W as

$$W(x, x+\delta x) = 1 + \delta x^\mu B_\mu(x) + \dots$$

new 4-vector field!

but we need to make sure that W transforms properly $\Rightarrow$ this fixes transformation of B_μ

$$W(x, x+\delta x) \xrightarrow{U(1)} e^{i\alpha(x)} W(x, x+\delta x)\, e^{-i\alpha(x+\delta x)}$$

and expanding in δx we write

$$\alpha(x+\delta x) = \alpha(x) + i(\partial_\mu \alpha)\, \delta x^\mu + \dots$$

$$W(x, x+\delta x) \xrightarrow{U(1)} \cancel{e^{i\alpha(x)}} W(x, x+\delta x)\, \cancel{e^{-i\alpha(x)}} \left[1 - i\partial_\mu \alpha\, \delta x^\mu\right]$$

now using

$$W(x, x+\delta x) = 1 + \delta x^\mu B_\mu(x) + \dots$$

$$= 1 + \delta x^\mu \left[B_\mu(x) - i\partial_\mu \alpha \right] + \text{higher orders}$$

$$\overset{!}{=} 1 + \delta x^\mu B'_\mu(x) + \dots$$ if we had transformed W in terms of B_μ

$\Rightarrow$ this implies $\boxed{B'_\mu(x) = B_\mu(x) - i\partial_\mu \alpha}$

with this the new derivative becomes

$$D_\mu \phi = \lim_{\delta x^\mu \to 0} \frac{\phi(x+\delta x) - \phi(x) + \delta x_\nu B^\nu(x)\phi(x)}{\delta x_\mu}$$

$$= \partial_\mu \phi + B_\mu(x)\phi(x)$$

and we know that $D_\mu \phi \to e^{i\alpha(x)} D_\mu \phi$

this is called a COVARIANT DERIVATIVE as it transforms covariantly under U(1) local !

$B_\mu(x)$ is a new vector field that takes the role of a CONNECTION. The point is that, with this D_μ, I can now construct a U(1) local - invariant Lagrangian by $\partial_\mu \to D_\mu$

$$\mathcal{L} = [D_\mu \phi]^* D^\mu \phi - m^2 \phi^* \phi$$

At this point, you might have recognized that this is nothing but the Gauge Invariance we learned when talking about Electrodynamics

$\Rightarrow$ redefining $B_\mu(x) = +i\, q\, A_\mu(x)$

we have $\boxed{D_\mu = \partial_\mu + i q A_\mu}$ when acting on $\underline{\underline{\phi(x)}}$!

minimal coupling in Electro Dynamics

A_μ seems to naturally be the Electromagnetic Field

q is charge of field ϕ in e-units

$$B_\mu \to B_\mu - i \partial_\mu \alpha \Rightarrow A_\mu \to A_\mu - \frac{1}{q} \partial_\mu \alpha$$

GAUGE TRANSFORMATION

So just by requesting invariance of original $\mathcal{L}$ under a GAUGE TRANSFORMATION we got for free A^μ !

Problem : In our $\mathcal{L}$ there is no kinetic term for A^μ, but this we know well how to build

$$\mathcal{L}_{KIN} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \, ; \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$F^{\mu\nu}$ is gauge invariant

$\mathcal{L}_{KIN}$ gives MAXWELL EQUATIONS (see next lecture!)

So we have now

$$\mathcal{L} = [D_\mu \phi]^* D^\mu \phi - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

Lagrangian of a complex scalar field ϕ coupled to electromagnetic Field, which is GAUGE INVARIANT!

Note $[D_\mu \phi]^* \overset{?}{=} D_\mu \phi^*$ if $\boxed{D_\mu = \partial_\mu - iqA_\mu}$

when acting on ϕ^* !!

$\Rightarrow$ the form of D_μ changes depending on the field it acts onto !

this is the right way to think about it ! $\Rightarrow$

the covariant derivative transforms as the field it acts on

$$\phi \to e^{i\alpha} \phi \qquad D_\mu \phi \to e^{i\alpha} D_\mu \phi$$

$$\phi^* \to e^{-i\alpha} \phi \qquad D_\mu \phi^* \to e^{-i\alpha} D_\mu \phi^*$$

which implies $\underline{D_\mu \phi^* = (\partial_\mu - iqA_\mu)\phi^*}$!

right way to think about it

(exercise !)