Quantum Field Theory WS 2025/26

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Sheet 09: Feynman Rules and Dimensional Regularization

Please hand in your solutions on Moodle by **Friday, 19.12.25, 8am**

Exercise 1 - Feynman diagrams and Feynman rules

a) Draw all diagrams contributing to

1. the two-point function at $\mathcal{O}(\lambda^2)$ in the theory of a *complex scalar field* with interaction

$$-\frac{\lambda}{4}(\phi^\dagger\phi)^2\,, \tag{1}$$

2. the three-point function at $\mathcal{O}(g^3)$ in the theory of a *real scalar field* with interaction

$$-\frac{g}{3!}\phi^3\,. \tag{2}$$

For the connected diagrams, determine the **symmetry factor**, i.e. the number of contractions resulting in identical diagrams, multiplied by $1/n!$ from the expansion of the exponential of the interaction Lagrangian to order n, and by the constant factors present in the vertices.
Note that there are two ways of computing the symmetry factor. One is to count all possible pairwise contractions giving rise to the same diagram. This is clearly very cumbersome and quickly becomes unfeasible as we go higher in perturbative order. Since drawing Feynman diagrams is easier, the second method consists of counting how many recombinations of the internal legs and internal vertices lead to the same diagram. Use the latter approach to find the symmetry factors in this problem.

b) Choose one of the $\mathcal{O}(\lambda^2)$ diagrams from case 1. above and write down its complete mathematical expression using the momentum-space Feynman rules.

c) Determine the momentum-space Feynman rule for the following terms in an interaction Lagrangian:

$$1)\quad \lambda\phi_1\phi_2\phi_1\phi_2 \tag{3}$$
$$2)\quad \lambda\phi^2(\partial_\mu\phi)(\partial^\mu\phi) \tag{4}$$
$$3)\quad gf^{ABC}(\partial_\mu A_\nu^A)A^{\mu B}A^{\nu C}\,, \tag{5}$$

where ϕ, ϕ_1, ϕ_2 denote (different) real scalar fields, and A_μ^A is a real vector field carrying an index A associated with an internal symmetry group. The coupling constants λ and g specify the strength of the interaction, and f^{ABC} is a totally antisymmetric structure constant defining the Lie algebra of the internal symmetry group.
The summation convention is implied for all index types.

Hint: Before computing the Feynman rule associated with $gf^{ABC}(\partial_\mu A_\nu^A)A^{\mu B}A^{\nu C}$, think about the possible form such a vertex can take and try to write down the result without explicit computation.

Exercise 2 - Mass term as an interaction

In the lectures, we have made a distinction between *kinetic terms*, which are bilinear in the fields, and *interactions*, which involve three or more fields. Time evolution with the kinetic terms is solved exactly as part of the free Hamiltonian H_0.
In this problem, we define the free Hamiltonian only as the bilinear term with two derivatives, while we treat the mass term as an interaction H_{int}. The corresponding Lagrangian densities read

$$\mathcal{L}_0 = -\frac{1}{2}\phi\Box\phi\,, \quad \mathcal{L}_{\text{int}} = -\frac{1}{2}m^2\phi^2\,. \tag{6}$$

In the following, you will explore some implications of this choice.

a) Draw the Feynman graphs that contribute to the two-point function $\langle 0|\,T\{\phi(x)\phi(y)\}\,|0\rangle$ up to $\mathcal{O}(m^6)$ in the theory described by Eq.(6). Write down the mathematical expression for each graph, paying attention to combinatorial factors, and use your result to guess the expression for the contribution at $\mathcal{O}(m^{2n})$.

b) Sum the series to all orders in m^2, and show that you recover the standard result obtained with the conventional choice
$$\mathcal{L}_0 = -\frac{1}{2}\phi(\Box + m^2)\phi\,, \quad \mathcal{L}_{\text{int}} = 0\,. \tag{7}$$

c) In this subproblem, you will reproduce the same result using classical equations of motion. We recall that the solution to the equation of motion for a massless scalar sourced by an external current J, $\mathcal{L} = -\frac{1}{2}\phi\Box\phi + \phi J$, is $\phi_0 = (1/\Box)J$. Upon adding the perturbation $\Delta\mathcal{L} = -\frac{1}{2}m^2\phi^2$, solve the equation of motion for the perturbed scalar $\phi = \phi_0 + \Delta\phi$ order by order in m^2, and show that you recover the standard result $\phi = \frac{J}{\Box + m^2}$.

Exercise 3 - Gamma and Beta Functions

In this exercise we consider two important functions that appear ubiquitously in loop calculations.

1. The **Gamma Function** is defined as

$$\Gamma(z) = \int_0^\infty \mathrm{d}t\, t^{z-1}e^{-t}\,, \qquad \mathrm{Re}(z) > 0\,. \tag{8}$$

Prove that

$$\Gamma(1+z) = z\Gamma(z)\,, \tag{9}$$

and use this formula to extend the definition of the Gamma function to $\mathrm{Re}(z) < 0$. For which values of z can the function *not be defined*? How does it behave close to those points?

2. Consider $0 < \epsilon \ll 1$ a small, positive, parameter. Prove that

$$\Gamma(1+\epsilon) = 1 - \epsilon\gamma_E + \epsilon^2\left(\frac{\gamma_E^2}{2} + \frac{\pi^2}{12}\right) + \mathcal{O}(\epsilon^3) \tag{10}$$

where

$$\gamma_E = -\int_0^\infty e^{-x}\ln x\, dx = +0.577216...$$

is the Euler-Mascheroni constant.

3. Show that for $0 < \epsilon \ll 1$

$$\Gamma(\epsilon - 1) = -\frac{1}{\epsilon} + (\gamma_E - 1) + \left(-1 + \gamma_E - \frac{\gamma_E^2}{2} - \frac{\pi^2}{12}\right)\epsilon + O\left(\epsilon^2\right) . \tag{11}$$

4. Prove that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\,, \qquad \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}\,. \tag{12}$$

5. The **Beta Function** is defined as

$$B(x,y) = \int_0^1 \mathrm{d}t\, t^{x-1}(1-t)^{y-1} = \int_0^\infty \mathrm{d}t\, \frac{t^{x-1}}{(t+1)^{x+y}}\,. \tag{13}$$

Show that the Beta function can be related to products of Gamma functions as follows

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,. \tag{14}$$

Exercise 4 - Towards dimensional regularisation of Feynman integrals

As a warm up, repeat the steps done in class, and compute in *dimensional regularization* the Euclidean massive tadpole with its propagator raised to a general power α, i.e.

$$\mathcal{T}_E(D;\alpha,m) = \int \frac{d^D k_E}{(2\pi)^D}\frac{1}{(k_E^2+m^2)^\alpha}\,. \tag{15}$$

Now, consider a so-called **massive scalar one-loop bubble** with equal masses and external momentum p in dimensional regularization:

$$\mathcal{B}(D;p^2,m) = \int \frac{d^D k}{(2\pi)^D}\frac{1}{(k^2-m^2+i0^+)((k+p)^2-m^2+i0^+)}\,. \tag{16}$$

a1) Estimate the ultra-violet degree of divergence of the integral introducing a cut-off Λ. How does it behave compared to the one-loop tadpole with $\alpha = 1$?

a2) Consider the Euclidean bubble $\mathcal{B}_E(D; p_E^2, m)$ defined as

$$\mathcal{B}_E(D; p_E^2, m) = \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{(k_E^2 + m^2)((k_E + p_E)^2 + m^2)}\,, \tag{17}$$

where $k_E^2 = -k^2\,, p_E^2 = -p^2$ are Euclidean momenta. How does $\mathcal{B}(D; p^2, m)$ relate to $\mathcal{B}_E(D; p_E^2, m)$?

a3) Introduce the *Feynman parametrization*

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}\,, \tag{18}$$

where A and B stand for any pair of denominators in the scalar integral.
Use it to show that $\mathcal{B}_E(D; p_E^2, m)$ can be written as

$$\mathcal{B}_E(D; p_E^2, m) = \int_0^1 dx \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{(k_E^2 + \Delta)^2}\,, \tag{19}$$

where $\Delta = m^2 + x(1-x)p_E^2$.
<u>Hint:</u> keep in mind that scalar Feynman integrals in dimensional regularization are invariant under translation of the loop momentum.

a4) In Eq. (19) you can now recognize the expression of a massive tadpole with exponent $\alpha = 2$. Exploit the result found in Eq. (15) to solve the D-dimensional integral over k_E.

a5) Finally, expand your result for $D = 4 - 2\epsilon$ and extract the coefficient of the $1/\epsilon$ pole.