# Quantum Field Theory WS 2025/26

Lecturers: Prof. Lorenzo Tancredi, Prof. Murad Alim
Assistants: Camilla Forgione, Felix Forner, Leonardo Sartori,
Dr. Chiara Savoini, Fabian Wagner, Dr. Denis Werth

**Sheet 08: Path integrals and Green Functions**

Please hand in your solutions on Moodle by **Friday, 12.12.25, 8am**

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## Exercise 1 - A non-standard path integral

Consider a one-dimensional quantum mechanical particle. In the path integral formalism, the transition amplitude from an initial state $|q_i\rangle$ at time $t_i$ to a final state $|q_f\rangle$ at time $t_f$ is usually given by the following path integral

$$\langle q_f, t_f | q_i, t_i \rangle = \mathcal{N} \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}[q(t)]\, e^{iS[q(t)]}\,, \tag{1}$$

over all possible trajectories $q(t)$ weighted by the exponential of the action for each path, given by $S[q(t)] \equiv \int_{t_i}^{t_f} \mathrm{d}t\, L[q(t), \dot{q}(t)]$. However, this expression is only true for a certain class of theories whose Hamiltonian is purely quadratic in the conjugate momentum. For this specific class, computing the (Gaussian) path integral over the momenta is equivalent to performing a Legendre transform, which trades the Hamiltonian $H$ for the Lagrangian $L$. This is not true in general: Hamiltonians are more *fundamental* than Lagrangians.

In this problem, we will explore a counterexample for which Eq. (1) does not hold.

Consider the Lagrangian

$$L[q, \dot{q}] = \frac{1}{2} f(q) \dot{q}^2\,. \tag{2}$$

1.1 Derive the Hamiltonian $H[q,p]$. Note that, even though it is quadratic in $p$, the corresponding coefficient is no longer constant in $q$.

1.2 By dividing the time interval $t_f - t_i$ in $N$ infinitesimal time steps $\Delta t$, express the transition amplitude

$$\langle q_f, t_f | q_i, t_i \rangle = \mathcal{N} \int \mathcal{D}[q(t)] \int \mathcal{D}[p(t)]\, \exp\left\{ i \int_{t_i}^{t_f} \mathrm{d}t\, (p\dot{q} - H[q,p]) \right\}\,, \tag{3}$$

as a path integral over the $q(t)$ trajectories only. You should find

$$\langle q_f, t_f | q_i, t_i \rangle = \mathcal{N}' \int \mathcal{D}[q(t)]\, \exp\left\{ i \int_{t_i}^{t_f} \mathrm{d}t\, (L[q,\dot{q}] + \delta(0)\Lambda(q)) \right\}\,, \tag{4}$$

where $\Lambda(q)$ is a function to be determined.

*<u>Hint</u>: keep in mind that in the continuum limit $N \to +\infty$ (for fixed $N\Delta t = t_f - t_i$) the following substitutions hold:*

$$\sum_{j=1}^{N} \Delta t \to \int_{t_i}^{t_f} dt\,, \quad \sum_{j=1}^{N} 1 \to \delta(0) \int_{t_i}^{t_f} dt \tag{5}$$

*where* $\delta(0) = \lim_{\Delta t \to 0} 1/\Delta t\,.$

## Exercise 2 - Exponential decay of the Feynman propagator outside the light cone

Consider the Feynman propagator of a free real scalar field of mass $m$

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}, \tag{6}$$

and define

$$\Delta t = x^0 - y^0, \quad \Delta\vec{r} = \vec{x} - \vec{y}, \quad \Delta r = |\Delta\vec{r}|. \tag{7}$$

In this exercise, you will show explicitly (by performing the Fourier integrals) that for space-like separations with $\Delta t = 0$, the propagator decades exponentially as $\Delta_F(\Delta r) \sim e^{-m\Delta r}$, for large $\Delta r$. This demonstrates that quantum correlations outside the light cone are exponentially suppressed, fully consistent with relativistic causality.

2.1 Perform the integral over $p^0$ in the definition of $\Delta_F$ using contour integration. Show that

$$\Delta_F(\Delta t, \Delta\vec{r}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[\Theta(\Delta t) e^{-iE_p\Delta t + i\vec{p}\cdot\Delta\vec{r}} + \Theta(-\Delta t) e^{+iE_p\Delta t + i\vec{p}\cdot\Delta\vec{r}}\right], \tag{8}$$

where $E_p = \sqrt{|\vec{p}| + m^2}$.

2.2 Specialize to space-like separation. Set $\Delta t = 0$ in the expression above and show that:

$$\Delta_F(\Delta\vec{r}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}\cdot\Delta\vec{r}}. \tag{9}$$

Choose the z–axis along $\Delta\vec{r}$, and rewrite the integral in spherical momentum coordinates. Show that:

$$\Delta_F(\Delta r) = -\frac{i}{2(2\pi)^2\Delta r} \int_{-\infty}^{+\infty} dp \frac{pe^{ip\Delta r}}{\sqrt{p^2 + m^2}}. \tag{10}$$

2.3 Consider the integrand

$$f(p) = \frac{pe^{ip\Delta r}}{\sqrt{p^2 + m^2}} \tag{11}$$

and

- identify its branch points in the complex $p$-plane;
- introduce branch cuts that run from the branch points upward/downward along the imaginary axis, and argue that for $\Delta r > 0$, the contour may be closed in the upper half-plane and deformed around the upper branch cut.

2.4 Parametrize the upper branch cut by $p = i\rho$, with $\rho \geq m$. Show that the propagator reduces to the integral:

$$\Delta_F(\Delta r) = \frac{1}{4\pi^2\Delta r} \int_m^{\infty} d\rho \frac{\rho e^{-\rho\Delta r}}{\sqrt{\rho^2 - m^2}}. \tag{12}$$

2.5 Show that the integral is dominated by the region near $\rho = m$ when $\Delta r \to \infty$. By writing $\rho = m + \eta$ with $\eta \ll m$, demonstrate that:

$$\Delta_F(\Delta r) \sim e^{-m\Delta r} \quad \text{as} \quad \Delta r \to \infty. \tag{13}$$

## Exercise 3 - Two-point function in $\phi^4$ theory

In the lectures, we calculated the two-point function in $\phi^3$ theory to second order in the coupling. In the following problem, we will consider the position-space two-point function in the so-called $\phi^4$ theory where

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4. \tag{14}$$

In the lecture we showed that

$$\begin{aligned}
&\int \mathcal{D}\phi\, e^{i\int d^4x(\mathcal{L}_0+\mathcal{L}_{\text{int}})}\phi(x_1)\ldots\phi(x_n) = \\
&= \int \mathcal{D}\phi\, e^{i\int d^4x\mathcal{L}_0}\left(\sum_{m=0}^{\infty}\frac{i^m}{m!}\int d^4z_1\ldots d^4z_m\mathcal{L}_{\text{int}}(z_1)\ldots\mathcal{L}_{\text{int}}(z_m)\,\phi(x_1)\ldots\phi(x_n)\right) = \\
&= \sum_{m=0}^{\infty}\frac{i^m}{m!}\int d^4z_1\cdots d^4z_m\ [\text{all pairwise contractions of the fields in } \mathcal{L}_{\text{int}}\ (z_1)\cdots\mathcal{L}_{\text{int}}\ (z_m)\,\phi(x_1)\cdots\phi(x_n)]
\end{aligned} \tag{15}$$

3.1 Use this result to derive $\int \mathcal{D}\phi\, e^{i\int d^4x(\mathcal{L}_0+\mathcal{L}_{\text{int}})}\phi(x_1)\,\phi(x_2)$ in $\phi^4$-theory to $\mathcal{O}(\lambda^2)$.
Draw the corresponding Feynman diagrams for the various contributions you will obtain.

3.2 Calculate the two-point function

$$\langle\Omega|T\{\phi(x_1)\,\phi(x_2)\}|\Omega\rangle = \frac{\int \mathcal{D}\phi\, e^{i\int d^4x(\mathcal{L}_0+\mathcal{L}_{\text{int}})}\phi(x_1)\,\phi(x_2)}{\int \mathcal{D}\phi\, e^{i\int d^4x(\mathcal{L}_0+\mathcal{L}_{\text{int}})}} \tag{16}$$

to order $\mathcal{O}(\lambda^2)$, and draw the corresponding Feynman diagrams.
What happens to the vacuum diagrams?