Quantum Field Theory WS 2025/26

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Sheet 07: Spin 1/2 and introduction to path integral

Please hand in your solutions on Moodle by **Friday, 05.12.25, 8am**

Exercise 1 - Canonical Quantization of the free spinor field

In the lecture we have seen that in order to quantize the free spinor field we proceed as follows. We decompose fields and their adjoints into creation and annihilation operators

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} \sum_{s=1,2} \left[a_s(p) u_s(p) e^{-ip\cdot x} + b_s^\dagger(p) v_s(p) e^{ip\cdot x} \right] , \tag{1}$$

with

$$\{a_r(p), a_s^\dagger(q)\} = \{b_r(p), b_s^\dagger(q)\} = 2E_p (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \delta_{rs} \tag{2}$$

and all other anticommutators equal to zero.
Assuming these anticommutation relations for the creation and annihilation operators:

1. Show that the fields satisfy the expected anticommutation relations at **equal times**

$$\{\psi_a(t, \vec{x}), \psi_b^\dagger(t, \vec{y})\} = \delta^3(\vec{x} - \vec{y}) \delta_{ab} , \tag{3}$$

   where a, b are the spinor indices.

2. Prove that, at **generic times**, instead

$$\{\psi_a(x), \overline{\psi}_b(y)\} = (i\partial\!\!\!/ + m)_{ab} \left[\Delta_+(x - y) - \Delta_-(x - y) \right] , \tag{4}$$

   where

$$\Delta_+(x - y) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} e^{-ip\cdot(x-y)} , \qquad \Delta_-(x - y) = \Delta_+(y - x) . \tag{5}$$

   Argue that this anticommutator is zero for spacelike separations.

   What does this imply for the *commutation relations* of operators built of an even number of fermionic operators[1]? To see how this works, consider $\mathcal{O}(x) = \overline{\psi}(x)\psi(x)$ and compute

$$[\mathcal{O}(x), \mathcal{O}(y)] = \dots$$

3. Given the form of the Hamiltonian $H = \int \mathrm{d}^3\vec{x} : \mathcal{H} :$ with $\mathcal{H} = \overline{\psi}(-i\gamma^j \partial_j + m)\psi$ prove that

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} \sum_{s=1,2} E_p \left[a_s^\dagger(p) a_s(p) + b_s^\dagger(p) b_s(p) \right] . \tag{6}$$

[1] This goes under the name of "microcausality".

4. Given the form of the momentum $\vec{P} = \int \mathrm{d}^3\vec{x} : \psi^\dagger(-i\vec{\nabla})\psi :$ prove that

$$\vec{P} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} \sum_{s=1,2} \vec{p} \left[a_s^\dagger(p) a_s(p) + b_s^\dagger(p) b_s(p)\right] \ . \tag{7}$$

5. Given the form of the U(1) charge operator $Q = \int \mathrm{d}^3\vec{x} : \psi^\dagger\psi :$ show that

$$Q = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} \sum_{s=1,2} \left[a_s^\dagger(p) a_s(p) - b_s^\dagger(p) b_s(p)\right] \ . \tag{8}$$

Exercise 2 - Multidimensional Gaussian integral with sources

In preparation of the path-integral quantization, consider an n-dimensional real vector $x = (x_1, \dots, x_n)^T$ and an $n \times n$ real, symmetric, positive-definite matrix A. Let J be an n-dimensional vector, which we will refer to as "source".

1. Compute the integral

$$Z[J] \equiv \int_{\mathbb{R}^n} \mathrm{d}^n x \ \exp\left(-\tfrac{1}{2} x^T A x + J^T x\right) \ , \tag{9}$$

and show that

$$Z[J] = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left(\frac{1}{2} J^T A^{-1} J\right) \ . \tag{10}$$

2. Consider now an "operator" $\mathcal{O}(x)$[2] which can be written as a polynomial in the various x_i, for example

$$\mathcal{O}(x) = a^{(0)} + a_i^{(1)} x_i + a_{ij}^{(2)} x_i x_j + \dots \ .$$

We *define* its expectation value with the source J to be

$$\langle \mathcal{O}(x) \rangle_J = \frac{1}{Z[J]} \int d^n x \, \mathcal{O}(x) \ \exp\left(-\tfrac{1}{2} x^T A x + J^T x\right) \ . \tag{11}$$

By taking derivatives of $Z[J]$ with respect to J_i, show that: (i) x_i has zero expectation value when the source is put to zero $\langle x_i \rangle_{J=0} = 0$, and ($ii$) the expectation value (again for $J = 0$) of $x_i x_j$ is given by $\langle x_i x_j \rangle_{J=0} = (A^{-1})_{ij}$.

3. Let B be an antisymmetric $n \times n$ matrix, and consider the following integral

$$I \equiv \int_{\mathbb{R}^n} \mathrm{d}^n x \ \exp\left(-\tfrac{1}{2} x^T A x + \tfrac{1}{2} x^T B x\right) \ . \tag{12}$$

Show that $I = Z[0]$, and comment on the implication for path integrals with first-order derivative terms, e.g. of the form $\int \mathrm{d}t \phi \dot{\phi}$.

[2] We call this "an operator" thinking about the generalization of this formalism to quantum field theory, but for now this is just a name.

Exercise 3 - Path integral for statistical mechanics

In this problem, we study the path integral formulation in statistical mechanics. The theory can be described by the partition function

$$Z \equiv \mathrm{Tr}\, e^{-\beta H[q,p]}\,, \tag{13}$$

where $\beta = 1/T$ is the inverse temperature (we set the Boltzmann constant to unity $k_{\mathrm{B}} = 1$), and $H[q,p]$ is the Hamiltonian of the system, function of the d-dimensional generalized coordinates $q \equiv (q_1, \dots, q_d)^T$ and $p \equiv (p_1, \dots, p_d)^T$. Let us assume that the Hamiltonian has the following form

$$H[q,p] = \frac{p^2}{2m} + V(q)\,, \tag{14}$$

where $V(q)$ is a general potential, and m is a mass parameter. We assume that the eigenstates of both q and p form a complete orthonormal basis of the Hilbert space:

$$\mathbb{1} = \int \mathrm{d}^d q\, |q\rangle\,\langle q|\;, \quad \mathbb{1} = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, |p\rangle\,\langle p|\;. \tag{15}$$

1. By discretizing the quantity $e^{-\beta H[q,p]}$ into N factors

   $$e^{-\beta H} = \underbrace{e^{-\epsilon H} \dots e^{-\epsilon H}}_{\text{N factors}} \quad \text{with} \quad \epsilon = \beta/N\,, \tag{16}$$

   show that, in the limit $N \to \infty$, the partition function can be written as the following path integral

   $$Z = \int \mathcal{D}q\, \exp\left(-\oint \mathrm{d}\tau L_E(\tau)\right)\,, \tag{17}$$

   where τ is the Euclidean time and $\oint \mathrm{d}\tau$ stands for the integral over the Eucilead circle of radius β (with periodic boundary conditions $q(0) = q(\beta)$). The integration measure is defined by

   $$\mathcal{D}q = \lim_{N\to\infty} \left[\frac{m}{2\pi\epsilon}\right]^{Nd/2} \prod_{i=0}^{N} \mathrm{d}^d q_i\,, \tag{18}$$

   and $L_E(\tau)$ is a Lagrangian in Euclidean form

   $$L_E(\tau) = \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + V(q(\tau))\,. \tag{19}$$

2. Evaluate (up to a possibly divergent and β-dependent constant) the path integral (17) for a uni-dimensional harmonic oscillator

   $$L_E(\tau) = \frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\omega^2 q^2\,, \tag{20}$$

by introducing the following Fourier decomposition

$$q(\tau) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{+\infty} q_n \, e^{2i\pi n\tau/\beta} \, . \tag{21}$$

<u>*Hint*</u>*: use the (not so well-known) formula*

$$\sinh(z) = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(n\pi)^2} \right) \, . \tag{22}$$

3. Comment on the zero-temperature limit $\beta\omega \gg 1$.