Quantum Field Theory WS 2025/26

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Sheet 06: Quantization of free fields

Please hand in your solutions on Moodle by **Friday, 28.11.25, 8am**

Exercise 1 - Microcausality and the Schrödinger Lagrangian

For a scalar field $\Phi(x)$, the free equation of motion is given by the usual Klein-Gordon Lagrangian

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi. \tag{1}$$

If $\Phi(x)$ is non-hermitian, a general solution to the equation above for $p_0 = E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}$ is given by

$$\Phi(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} e^{-iE_\mathbf{p}t + i\mathbf{p}\cdot\mathbf{x}} a(p), \tag{2}$$

with a generally complex coefficient $a(p)$. Despite solving correctly the equation of motion associated to Eq. (1), this solution is not consistent with *relativistic causality*.

1. Promote $\Phi(x)$ to an operator imposing

$$[a(p), a^\dagger(k)] = (2\pi)^3 2E_\mathbf{p} \delta^{(3)}(\mathbf{p} - \mathbf{k})\,.$$

   What is the commutator between $\Phi(x)$ and its conjugate momentum? Does it vanish for space-like separated x and y? Does this suffice to claim that relativistic causality is broken?

Since the ansatz of Eq. (3) is not appropriate to describe a relativistic field, let us reduce ourselves to a non-relativistic setting.

2. Using the approximation $\Phi(x) = e^{-imt}\phi(x)/\sqrt{2m}$ show that the Klein-Gordon Lagrangian reduces to

$$\mathcal{L} = \phi^\dagger \left[i\partial^0 + \frac{\vec{\nabla}^2}{2m} \right] \phi. \tag{3}$$

   Using the ansatz of Eq. (2), show that the field respects the dispersion relation $p_0 = \mathbf{p}^2/2m$.

3. Show that the wave function $\psi(x) = \langle \Omega | \phi(x) | \Psi \rangle$ of *any* single-particle state $|\Psi\rangle$ in the Fock space satisfies the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t, \mathbf{x}) = H_0 \psi(t, \mathbf{x}) \qquad \text{where} \qquad H_0 = -\frac{\nabla^2}{2m}. \tag{4}$$

4. What would be the corresponding equation for a two-particle state with wave function $\psi(t, \mathbf{x}, \mathbf{y}) = \langle \Omega | \phi(x)\phi(y) | \Psi \rangle$? Sketch the derivation for a general n-particle state. What kind of system does this field theory describe?

As will be discussed in class, if the Lagrangian of Eq. (3) were to describe a fermion field ψ, this would now satisfy canonical anti-commutation relations (CAR):

$$\{a(k), a^\dagger(p)\} = (2\pi)^3 2E_\mathbf{p} \delta^{(3)}(\mathbf{k} - \mathbf{p})$$
$$\{a(k), a(p)\} = 0\,, \qquad \{a^\dagger(k), a^\dagger(p)\} = 0.$$

5. Repeat the analysis of points 1. to 4. and discuss which results change for the fermionic case.

Problem 2 - Canonical Quantization of a Rescaled Scalar Field

Consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2} Z^2 \, (\partial_\mu \phi)^2 - \frac{1}{2} \mu^2 \phi^2 \,, \tag{5}$$

where Z is a real positive constant and ϕ is a real scalar field.

a) In order to easily quantize this theory one can rescale the field $\phi \to \tilde{\phi} = \phi/Z$ and follow the quantization procedure outlined in the lecture. What is the mass of the excitations of the field $\tilde{\phi}$? Write down explicitly the eigenvalue of the Hamiltonian for a state with one excitation of momentum p, $E^Z_{\mathbf{p}}$:

$$H|\,p\rangle = H\, a^\dagger(p)|\Omega\rangle = E^Z_{\mathbf{p}}\,|p\rangle \,, \tag{6}$$

where $|0\rangle$ is the vacuum state and the eigenvalue $E^Z_{\mathbf{p}}$ depend in general also on Z.

b) Suppose now that you want to quantize this theory <u>without</u> rescaling the field as in point a).

b_1) Derive the expression for the conjugate momentum $\pi(x)$ and write down the explicit form of the Hamiltonian in terms of ϕ and π.

b_2) Prove that, in order to get the right equation of motion for the field $\phi(x)$

$$i\,\frac{d}{dt}\phi(t,\mathbf{x}) = [\phi, H]$$

one needs to impose the canonical equal-time commutation relations on $\phi(t,\mathbf{x})$ and $\pi(t,\mathbf{y})$

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] = i\,\delta^{(3)}(\mathbf{x}-\mathbf{y})\,. \tag{7}$$

b_3) Decompose the field $\phi(x)$ in Fourier modes by introducing creation and annihilation operators $b^\dagger(p)$, $b(p)$, which may differ from the $a^\dagger(p)$ and $a(p)$ introduced in point a). Which commutation relations do you have to impose on $b^\dagger(p)\, b(p)$ to make $\phi(x)$ and $\pi(x)$ satisfy the canonical equal-time commutation relations, Eq. (7)?

b_4) Use the previous results to diagonalize the Hamiltonian H, (i.e. write the Hamiltonian in terms of creation and annihilation operators $b(p)\,, b^\dagger(p)$). Show that the energy of a single particle state defined as

$$|\tilde{p}\rangle = b^\dagger(p)|0\rangle$$

is $E^Z_{\mathbf{p}}$, the same as in Eq. (6).

Exercise 3 - Intrinsic Parity

To the extent that the symmetry under the transformation $\mathbf{x} \to -\mathbf{x}$ is really valid, there must exist a unitary operator $\mathbf{P}$ under which free multiparticle states transform as a direct product of single-particle states:

$$\mathbf{P}\Psi_{p_1\sigma_1 n_1;\, p_2\sigma_2 n_2;\,\ldots} = \eta_{n_1}\eta_{n_2}\ldots\Psi_{\mathscr{P}p_1\sigma_1 n_1;\,\mathscr{P}p_2\sigma_2 n_2;\,\ldots} \tag{8}$$

where η_n is the intrinsic parity of particles of species n, and $\mathscr{P}$ reverses the space components of p^μ. Being $\mathbf{P}$ unitary, the η_n just amount to phases, which can either be inferred from dynamical models or from experiment. However, neither of the two can provide a unique determination of the η_n. This is because we are always free to redefine $\mathbf{P}$ by combining it with any conserved internal symmetry operator. For instance, if $\mathbf{P}$ is conserved, then so is

$$\mathbf{P}' = \mathbf{P}\exp\left(i\alpha B + i\beta L + i\gamma Q\right), \tag{9}$$

where B, L and Q are respectively baryon number, lepton number, and electric charge, and α, β and γ are arbitrary real phases.

1. Find a judicious choice of the parameters α, β and γ so that we can define the intrinsic parities of an electron, a proton, and a neutron to all be $+1$.

   Hint: In Eq. (9) the quantities B, L, and Q are conserved numbers assigned to each particle due to global symmetries. Protons and neutrons each have baryon number $B = 1$, while their antiparticles have $B = -1$, and all other particles in this problem have $B = 0$. Electrons, muons, and neutrinos have lepton number $L = 1$, their antiparticles have $L = -1$, and protons and neutrons have $L = 0$. The electric charge Q is the usual one, namely ($Q(p) = +1$, $Q(e^-) = -1$, etc.).

2. The redefinition in Eq. (9) effectively modifies the parity eigenvalues η_n. Show that, provided that we have have enough conserved internal symmetries which form continuous groups of phase transformations, it is always possible to introduce an operator $I_{\mathbf{P}}$ and redefine $(\mathbf{P}')^2 = (I_{\mathbf{P}}\mathbf{P})^2 = +1$ so that every particle species has intrinsic parity equal to either ± 1.

3. Consider now the case of a purely fermionic system. It is a consequence of angular-momentum conservation that the total number F of all particles of half-integer spin can only change by even numbers, so the internal symmetry operator $(-1)^F$ is conserved. Suppose we start from a choice of $\mathbf{P}$ such that $\mathbf{P}^2 = (-1)^F$, which constraints should now B, L and Q satisfy to ensure that a redefinition of $\mathbf{P}$ as per the previous point is always possible?

4. Is there a particle not fulfilling this requirement? What is it and why? How can we redefine its intrinsic parity?

Exercise 4 - Lorentz Transformations of Spinors

In the lectures, we will soon derive transformation laws for the creation/annihilation operators $a_s(p), a_s^\dagger(p)$ of a spinor field ψ from the transformation law of a general state $|p, s\rangle$. Indeed, up to a normalisation

$$U(\Lambda, 0)|p, s\rangle = \sum_{s'} D_{ss'}\left(W_{\Lambda^{-1},p}\right)|\Lambda^{-1}p, s'\rangle, \tag{10}$$

where the coefficients $D_{ss'}\left(W_{\Lambda^{-1},p}\right)$ provide little group representations as given in Lecture 4, we can find that

$$U(\Lambda, 0)a_s^\dagger(p)U(\Lambda^{-1}, 0) = \sum_{s'} D_{ss'}\left(W_{\Lambda^{-1},p}\right)a_s^\dagger(\Lambda^{-1}p) \tag{11}$$

$$U(\Lambda, 0)a_s(p)U(\Lambda^{-1}, 0) = \sum_{s'} D^*_{ss'}\left(W_{\Lambda^{-1},p}\right)a_s(\Lambda^{-1}p), \tag{12}$$

with the very same laws applying to $b_s(p), b_s^\dagger(p)$.

1. Starting from considering $\langle\Omega|\psi_s(x)|\Psi\rangle$ and $\langle\Psi|\psi_s(x)|\Omega\rangle$ (where $|\Psi\rangle$ is a suitable one-particle state), derive now the transformation laws for $u_s(p)$ and $v_s(p)$ under a generic Lorentz transformation Λ.

2. Specialize to the case of $\mathbf{P}, \mathbf{T}$ and $\mathbf{C}$. How do $u_s(p)$ and $v_s(p)$ transform in this case?

3. Consider now the fields $\psi, \bar{\psi}$ and the bilinears $\bar{\psi}\psi_s, \bar{\psi}\gamma^\mu\psi, i\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi, \bar{\psi}\gamma^\mu\gamma^5\psi$ and $i\bar{\psi}\gamma^5\psi$. How do they transform under $\mathbf{P}, \mathbf{T}$ and $\mathbf{C}$ and combinations thereof?

4. Let $\phi(x)$ be a complex Klein-Gordon field. Find unitary operators $\mathbf{P}, \mathbf{C}$ and an anti-unitary operator $\mathbf{T}$ (all defined in terms of their action on the annihilation operators $a(p)$ and $b(p)$ for the Klein- Gordon particles and antiparticles) that give the following transformations of the Klein-Gordon field:

$$\mathbf{P}\phi(t, \mathbf{x})\mathbf{P} = \phi(t, -\mathbf{x}), \tag{13}$$
$$\mathbf{T}\phi(t, \mathbf{x})\mathbf{T} = \phi(-t, \mathbf{x}), \tag{14}$$
$$\mathbf{C}\phi(t, \mathbf{x})\mathbf{C} = \phi^*(t, \mathbf{x}). \tag{15}$$

   Find the transformation properties of the components of the current

$$J^\mu = i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) \tag{16}$$

   under $\mathbf{P}, \mathbf{C}$ and $\mathbf{T}$.

5. Show that any Hermitian Lorentz-scalar local operator built from ψ, ϕ and their conjugates has $\mathbf{CPT} = +1$. How would you write an operator violating $\mathbf{CPT}$-invariance?