Quantum Field Theory WS 2025/26

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Sheet 05: Classical Spinor Fields

Please hand in your solutions on Moodle by **Friday, 21.11.2025, 8am**

Exercise 1 - Gamma Matrices

For this exercise, let us introduce the short-hand notation

$$\Gamma^{a_1 \dots a_n} \equiv \gamma^{a_1} \cdots \gamma^{a_n}$$

for chains of gamma matrices, where every a_j may either be a Lorentz-index, $a_j = \mu_j$, or $a_j = 5$.

1. Using only

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}, \\
\sigma^{\mu\nu} &= \frac{i}{2}[\gamma^\mu, \gamma^\nu] = 2\Sigma^{\mu\nu}, \\
\gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma,
\end{aligned}$$

prove the following identities (without using any explicit representations of gamma matrices):

$$\Gamma^{\alpha}{}_{\alpha} = 4 \tag{1}$$
$$\Gamma^{\alpha\mu}{}_{\alpha} = -2\gamma^\mu \tag{2}$$
$$\Gamma^{\alpha\mu\nu}{}_{\alpha} = 4g^{\mu\nu} \tag{3}$$
$$\gamma^\alpha \sigma^{\mu\nu}\gamma_\alpha = 0 \tag{4}$$
$$\Gamma^{\alpha\mu\nu\rho}{}_{\alpha} = -2\Gamma^{\rho\nu\mu} \tag{5}$$
$$(\gamma^0)^2 = \mathbb{1}, \quad (\gamma^i)^2 = -\mathbb{1} \quad \text{and} \quad (\gamma_5)^2 = \mathbb{1} \tag{6}$$
$$\{\gamma_5, \gamma^\mu\} = 0 \tag{7}$$
$$\mathrm{tr}(\Gamma^{\mu_1 \cdots \mu_{2n+1}}) = 0 \quad \forall n \in \mathbb{N}_0 \tag{8}$$
$$\mathrm{tr}(\Gamma^{\mu_1 \cdots \mu_{2n+1}5}) = 0 \quad \forall n \in \mathbb{N}_0 \tag{9}$$
$$\mathrm{tr}(\gamma^5) = 0 \tag{10}$$
$$\mathrm{tr}(\Gamma^{\mu\nu}) = 4g^{\mu\nu} \tag{11}$$
$$\mathrm{tr}(\Gamma^{\mu\nu}{}_5) = 0 \tag{12}$$
$$\mathrm{tr}(\Gamma^{\mu\nu\rho\sigma}) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{13}$$
$$\mathrm{tr}(\Gamma^{\mu\nu\rho\sigma}{}_5) = -4i\epsilon^{\mu\nu\rho\sigma} \tag{14}$$
$$[\gamma^\mu, \Sigma^{\nu\rho}] = i(g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu) \tag{15}$$
$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = i\left(g^{\nu\rho}\Sigma^{\mu\sigma} - g^{\mu\rho}\Sigma^{\nu\sigma} - g^{\nu\sigma}\Sigma^{\mu\rho} + g^{\mu\sigma}\Sigma^{\nu\rho}\right). \tag{16}$$

The last equation shows that $\Sigma^{\mu\nu}$ satisfies the Lorentz algebra.

2. Show that

$$\Lambda_D = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right) \qquad \text{with} \qquad \Sigma_{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{17}$$

generates Lorentz transformations on Dirac spinors.
Hint: *you can do this in (at least) two ways, either starting from the explicit form of the γ^μ matrices in the Chiral representation and computing Λ_D explicitly, or in a representation independent way. Both are allowed and instructive!*

Exercise 2 - Majorana Mass

In this exercise, we would like to study Majorana fermions and, in particular, how to write a Lagrangian for a massive Majorana field. Remember that a Majorana field ψ_M is defined as a Dirac field, with the extra condition that it is equal to its charge conjugate, i.e.

$$\psi_M^c = \psi_M$$

where $\psi^c = -i\gamma^2\psi^*$. As we have seen, thanks to this definition, a Majorana field effectively depends on a single Weyl field, say a left-handed field ψ_L, such that we can write

$$\psi_M = \begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^* \end{pmatrix} . \tag{18}$$

1. Show that the Dirac equation for a Majorana fermion,

$$(i\partial\!\!\!/ - m)\psi_M = 0,$$

becomes

$$\bar{\sigma}^\mu i\partial_\mu\psi_L = im\sigma^2\psi_L^*.$$

2. Show that the equation above implies

$$(\Box + m^2)\psi_L = 0\,.$$

3. For a Dirac fermion, the equation of motion (Dirac equation) can be obtained by variation of the action

$$S_{\text{Dirac}} = \int d^4x\bar{\psi}_{\text{D}}(i\partial\!\!\!/ - m)\psi_{\text{D}}.$$

Can the massive Majorana equation be derived from the same Lagrangian with the substitution $\psi_{\text{D}} \to \psi_M$? Which condition should our classical Majorana field satisfy for this to be possible?

4. Show that the Majorana equation is not $U(1)$ invariant. What is the physical implication of this?

Exercise 3 - Covariant Derivative

Consider a real scalar field ϕ that transforms under a local $U(1)$ tranformation $\phi \to e^{i\alpha(x)}\phi$. Since the x-dependence of $\alpha(x)$ spoils the right transformation behavior of $\partial_\mu\phi$, one introduces the covariant derivative

$$D_\mu\phi = (\partial_\mu + B_\mu)\phi,$$

where we impose that the real vector-field B_μ transforms as

$$B_\mu(x) \to B_\mu(x) - i\partial_\mu\alpha(x). \tag{19}$$

1. Consider a scalar field ψ that transforms as $\psi \to e^{-i\alpha(x)}\psi$. How does B_μ need to transform to make the covariant derivative transform in the right way?
2. Now consider a theory with both scalar fields ϕ and ψ, transforming under the same $U(1)$ symmetry as $\phi \to e^{i\alpha(x)}\phi$ and $\psi \to e^{-i\alpha(x)}\psi$. From part 1., $D_\mu\phi$ and $D_\mu\psi$ seem to require inconsistent transformation laws for B_μ. What is a way out to build invariant kinetic terms for both ϕ and ψ, with a single gauge field B_μ?
3. Now redefine $B_\mu(x) = iqA_\mu(x)$ and go back to the theory of a complex scalar field that we studied in class. Discuss the implications of your findings.

Exercise 4 - Total Derivatives and Noether Theorem

The free Lagrangian of a left-handed Weyl-spinor may be written as

$$\mathcal{L}_1 = i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L.$$

1. Show that

   $$\mathcal{L}_2 = \frac{i}{2}\psi_L^\dagger\bar{\sigma}^\mu\overleftrightarrow{\partial}_\mu\psi_L$$

   differs from $\mathcal{L}_1$ only by a total derivative, and check explicitly that both lead to the same equations of motion.
2. Compare the energy-momentum tensor and the corresponding conserved charges derived from $\mathcal{L}_1$ and $\mathcal{L}_2$ and discuss your findings.
3. Both Lagrangians are invariant under a global $U(1)$ transformation

   $$\psi_L \to e^{i\theta}\psi_L.$$

   As for the previous point, derive the conserved currents and charges from both Lagrangians and discuss your result.