Quantum Field Theory WS 2025/26

Lecturers: Prof. Lorenzo Tancredi, Prof. Murad Alim
Assistants: Camilla Forgione, Felix Forner, Leonardo Sartori, Dr. Chiara Savoini, Fabian Wagner, Dr. Denis Werth

Sheet 04: Classical Field Theory

Please hand in your solutions on Moodle by **Friday, 14.11.25, 8am**

Exercise 1 - The Pauli-Lubanski Vector and the Poincaré Group

1.1 Consider the Pauli-Lubanski vector

$$W^{\mu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_{\sigma}\,, \tag{1}$$

where P^{μ} is the momentum operator and $J^{\mu\nu}$ are the generators of the Lorentz group. Prove that

$$[W^{\mu}, P^{\nu}] = 0\,, \qquad [W^{\mu}W_{\mu}, J^{\nu\rho}] = 0\,. \tag{2}$$

1.2 Consider a massless momentum $k^{\mu} = (k^0, k^1, k^2, k^3)$ with $k^2 = 0$. By introducing light-cone coordinates

$$k^{\pm} = \frac{k^0 \pm k^3}{\sqrt{2}}\,, \tag{3}$$

prove that, without any extra assumptions, the little group for massless particles in 4 space-time dimensions is $ISO(2)$, i.e. the Euclidean Group of Rotations and Translations in the plane.

Hint: Pay attention to the upper and lower indices in the spatial derivatives!

We consider now the Poincaré group in a different number of dimensions.

1.3 How do the commutation relations of the Poincaré algebra depend on the number of space-time dimensions?

1.4 Identify the boost and rotation generators in three space-time dimensions. How many of each are there?

1.5 What are the invariant (Casimir) operators in three space-time dimensions?

1.6 What is the little group for the massive and massless representations of the Poincarè group in three space-time dimensions? What does this imply for the allowed spins of massive and massless particles?

Exercise 2 - Charge conservation with complex scalar fields

Consider the Lagrangian density

$$\mathcal{L} = (\partial_{\mu}\phi^*)(\partial^{\mu}\phi) - m^2\phi^*\phi \tag{4}$$

of a free complex scalar field ϕ.

2.1 Show that the global $U(1)$ transformation

$$\phi(x) \to \phi'(x) = e^{i\alpha}\phi(x)\,, \tag{5}$$

where α is a real parameter, leaves the Lagrangian density invariant.

2.2 Find the conserved current associated with this symmetry.

Consider now two complex scalar fields, described by

$$\mathcal{L} = (\partial_\mu \phi_a^*)(\partial^\mu \phi^a) - m^2 \phi_a^* \phi^a, \qquad a = 1, 2 \,. \tag{6}$$

2.3 Show that

$$\phi^a(x) \to \phi'^a(x) = M^a{}_b \phi^b(x) \,, \tag{7}$$

with $M \in U(2) = \{A \in \mathbb{C}^{2\times 2} : A^{-1} = A^\dagger \equiv (A^*)^T\}$, is a symmetry transformation.

2.4 Find the four conserved charges, one given by the generalization of part 2.2, and the other three given by

$$Q_j = \frac{i}{2} \int \mathrm{d}^3\mathbf{x} \left(\phi_a^* (\sigma^j)^a{}_b \pi^{b\,*} - \pi_a (\sigma^j)^a{}_b \phi^b \right) , \tag{8}$$

where σ^j are the Pauli matrices and π_a is the conjugate field of ϕ_a.

Exercise 3 - Dilatation symmetry of the massless KG action

Consider the Klein-Gordon (KG) action for a real, massless, free field

$$S = \frac{1}{2} \int d^4x \, g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi \quad , \tag{9}$$

and a *dilatation* transformation described by the parameter α,

$$x^\mu \to x'^\mu = e^\alpha \, x^\mu, \tag{10}$$

$$\phi(x) \to \phi'(x') = \phi(x) \; e^{-d_\phi \, \alpha} \,. \tag{11}$$

3.1 Show that this transformation is a global symmetry, for a specific choice of the parameter d_ϕ.

3.2 Compute the Noether current j^μ associated with this symmetry and verify that it is conserved. Find the conserved charge Q.

3.3 Although we have not yet discussed field quantization, following the discussion in exercise sheet 1, let us promote $\phi(x)$ to be a quantum field, satisfying the following (equal-time) commutation relations

$$[\phi(\mathbf{x},t), \phi(\mathbf{y},t)] = [\pi(\mathbf{x},t), \pi(\mathbf{y},t)] = 0 \;, \qquad [\phi(\mathbf{x},t), \pi(\mathbf{y},t)] = i\hbar\delta^{(3)}(\mathbf{x}-\mathbf{y}) \,. \tag{12}$$

Show that the charge operator Q satisfies the commutation relation

$$[\phi(\mathbf{x},t), Q(t)] = i\hbar \mathcal{D}\phi, \tag{13}$$

where

$$\mathcal{D}\phi(x) \equiv \left. \frac{d\phi(x)}{d\alpha} \right|_{\alpha=0} = (1 + x\cdot\partial)\phi(x) \qquad \text{with} \quad x = (\mathbf{x}, t) \,. \tag{14}$$

This implies that Q is the generator of the infinitesimal dilatation transformation.

3.4 Show that the above transformation is not a symmetry anymore if a non-vanishing mass term is added to the KG action.

3.5 Show that, instead, a term $V(\phi) = \lambda\,\phi^4$ does not spoil the dilatation symmetry. What is the dimension of λ?

Exercise 4 - Symmetries of a non-relativistic action

Consider a non-relativistic field theory defined by the Lagrangian

$$\mathcal{L} = i\phi^*\partial_0\phi + c\nabla\phi^* \cdot \nabla\phi\,, \tag{15}$$

where c is a real parameter. Note that, even though this Lagrangian density is not real, the corresponding action integral is real.

4.1 Derive the Euler-Lagrange equations for the fields ϕ and ϕ^*.

4.2 Consider the plane-wave solution $\phi(\mathbf{x}, t) = e^{i(\mathbf{p}\cdot\mathbf{x}-\omega t)}$ and find the frequency ω as a function of $\mathbf{p}$.

4.3 The Lagrangian density in Eq.(15) is not Lorentz-invariant. However, it is invariant under *space-time translations* and the *internal symmetry* transformation

$$\phi \to e^{-i\alpha}\phi\,, \quad \phi^* \to e^{i\alpha}\phi^*\,, \tag{16}$$

where α is a real parameter. Derive the conserved currents associated with these two symmetries. Compute the conserved charges, i.e. an energy, a linear momentum and a charge associated with the internal symmetry, as integrals of the fields and their derivatives.

4.4 Fix the sign of the constant c by demanding that the energy is bounded from below.

Exercise 5 - Scalar electrodynamics and minimal coupling

Goal: understand how promoting a global symmetry to a local one forces the introduction of a gauge field. Determines its transformation law and the form of the interaction.

In Exercise 2 we considered a global $U(1)$ symmetry of the Lagrangian in Eq.(4) describing a complex scalar field. We now promote the parameter α to be space-time dependent, i.e.

$$\phi(x) \to e^{i\alpha(x)}\phi(x)\,. \tag{17}$$

5.1 Show explicitly that $\mathcal{L}$ is no longer invariant, and compute the variation $\delta\mathcal{L}$.

5.2 Introduce a vector field $A_\mu(x)$ and define the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu\,. \tag{18}$$

Find how $A_\mu(x)$ must transform under the local $U(1)$ symmetry so that

$$D_\mu\phi \to e^{i\alpha(x)}D_\mu\phi\,. \tag{19}$$

5.3 Show that the modified Lagrangian

$$\mathcal{L}_{\text{gauge}} = (D_\mu \phi)^*(D^\mu \phi) - m^2 \phi^* \phi \tag{20}$$

is invariant under the local $U(1)$ transformation. The coupling to the electromagnetic field A_μ obtained via the replacement $\partial_\mu \to D_\mu$ in the free Lagrangian is called *minimal coupling.*

The Lagrangian in Eq.(20) is the one of scalar electrodynamics.

Beyond the local symmetry, the modified Lagrangian density is invariant under a $U(1)$ global symmetry.

5.4 Derive the Noether's current

$$j^\mu = i[\phi^*(D^\mu \phi) - (D^\mu \phi)^* \phi] \tag{21}$$

associated with an infinitesimal global transformation in the gauged Lagrangian. Show that this current is gauge-invariant and conserved, i.e. $\partial_\mu j^\mu = 0$.

5.5 Verify that the current

$$j_A^\mu = i[\phi^*(\partial^\mu \phi) - (\partial^\mu \phi)^* \phi]\,, \tag{22}$$

coupled to the vector field A_μ, is not conserved (differently from what you found in Exercise 2).

Add the kinetic term for the gauge field

$$\mathcal{L}_A = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}, \qquad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{23}$$

5.6 Show that the field strength $F_{\mu\nu}$ is gauge invariant.

5.7 Write the Euler-Lagrange equation for the gauge field A_μ and show that the scalar field ϕ acts as a source, i.e.

$$\partial_\mu F^{\mu\nu} = e\, j^\nu. \tag{24}$$

This can be interpreted as the analogue of Maxwell's equations with a charged scalar source.

Consider adding a gauge-invariant, higher-dimension operator

$$\mathcal{L}_\eta = -\frac{\eta}{2\Lambda^2}(\phi^*\phi) F^{\mu\nu} F_{\mu\nu}\,, \tag{25}$$

where Λ is some energy scale. In this case the coupling is not minimal.

5.8 Use dimensional analysis to determine the dimension of the coupling η.

5.9 Explain how this operator modifies the equations of motion for A_μ and how its effect scales with the energy $E \ll \Lambda$.

Exercise 6 - Kaluza-Klein modes from a 5D scalar field

Consider a real scalar field $\phi(t, \mathbf{x}, y)$ in a five-dimensional spacetime, where

$$(t, \mathbf{x}) \in \mathbb{R}^{1,3}, \qquad \text{and} \qquad y \in \left[-\frac{R}{2}, \frac{R}{2}\right] \tag{26}$$

is an extra spatial dimension compactified on a circle of circumference R, so that $y \sim y + R$.
The field ϕ satisfies the five-dimensional KG equation

$$\left(\Box_5 + m^2\right)\phi = 0, \qquad \text{where} \quad \Box_5 = \partial_t^2 - \nabla^2 - \frac{\partial^2}{\partial y^2}\,. \tag{27}$$

6.1 Using the periodicity in y, expand ϕ as a Fourier series

$$\phi(t, \mathbf{x}, y) = \sum_{n=-\infty}^{+\infty} \phi_n(t, \mathbf{x})\, e^{i2\pi n y/R}. \tag{28}$$

Insert this expansion into the equation of motion and show that it can be rewritten as an infinite set of four-dimensional KG equations for the modes ϕ_n.

Determine the effective four-dimensional masses m_n of these modes called *Kaluza–Klein modes*, and discuss their physical interpretation from a four-dimensional point of view.

6.2 If Standard Model particles are allowed to propagate in the extra dimension y, which would be the lightest new Kaluza-Klein mode?

Collider experiments have not observed new particles lighter than about 300 GeV. What corresponding upper bound does this imply for R? Express the result in meters, bearing in mind that $\hbar c \simeq 197\,\mathrm{MeV} \cdot \mathrm{fm}$.