Quantum Field Theory WS 2025/26

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Sheet 03: Representations of the Lorentz Group

Please hand in your solutions on Moodle by **Friday, 07.11.25, 8am**

Exercise 1 - Lorentz algebra, rotations and boosts

1.1 Show that

$$L^i = \frac{1}{2}\varepsilon^{ijk}J^{jk} \quad \text{and} \quad K^i = J^{i0}, \tag{1}$$

defined in terms of Lorentz generators $J^{\mu\nu}$, transform as vectors under 3-dimensional rotations.

1.2 Using the Lorentz algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right), \tag{2}$$

derive the commutation relations between L^i and K^i:

$$[L^i, L^j] = i\,\varepsilon^{ijk}L^k, \tag{3}$$

$$[L^i, K^j] = i\,\varepsilon^{ijk}K^k, \tag{4}$$

$$[K^i, K^j] = -\,i\,\varepsilon^{ijk}L^k. \tag{5}$$

1.3 Starting from an arbitrary Lorentz transformation

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}, \tag{6}$$

with parameters $\omega_{\mu\nu}$, and defining the parameters

$$\theta^i = \frac{1}{2}\varepsilon^{ijk}\omega^{jk}, \quad \eta^i = \omega^{i0}, \tag{7}$$

show that a general Lorentz transformation in vector representation can be written as

$$\Lambda = e^{\left[-i\,\vec{\theta}\cdot\vec{L} + i\,\vec{\eta}\cdot\vec{K}\right]}. \tag{8}$$

Exercise 2 - Lorentz vectors built from Weyl spinors

Prove that, if ψ_R and ξ_R are right-handed Weyl spinors, $V^\mu \equiv \xi_R^\dagger\sigma^\mu\psi_R$ is a four-vector, and similarly for $W^\mu \equiv \xi_L^\dagger\bar{\sigma}^\mu\psi_L$, where ξ_L and ψ_L are left-handed Weyl spinors. Recall that $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$ *and* $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$.

Exercise 3 - Tensor representations and angular momentum

Part 1 — Tensor representations

By definition, a tensor $T^{\mu\nu}$ with two contravariant (i.e. upper) indices transforms as

$$T^{\mu\nu} \to \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta T^{\alpha\beta}. \tag{9}$$

Tensors provide examples of representations of the Lorentz group. For instance, a generic tensor $T^{\mu\nu}$ has 16 components, and Eq.9 shows that these components transform among themselves, so they form a basis for a 16-dimensional representation. However, this representation is reducible.

3.1 Show that:

- if $A^{\mu\nu}$ is antisymmetric, it remains antisymmetric under Lorentz transformations;
- if $S^{\mu\nu}$ is symmetric, it remains symmetric under Lorentz transformations,

3.2 This allows us to conclude that the antisymmetric and symmetric parts of $T^{\mu\nu}$ do not mix under Lorentz transformations. The 16-dimensional representation therefore decomposes into an antisymmetric representation

$$A^{\mu\nu} = \tfrac{1}{2}(T^{\mu\nu} - T^{\nu\mu}) \tag{10}$$

and a symmetric representation

$$S^{\mu\nu} = \tfrac{1}{2}(T^{\mu\nu} + T^{\nu\mu}). \tag{11}$$

What are the dimensions (i.e. independent components) of these two representations?

3.3 Let's focus on the symmetric part: show that the trace of a symmetric tensor,

$$S \equiv g_{\mu\nu} S^{\mu\nu}, \tag{12}$$

is invariant under Lorentz transformations as well.
This means that a traceless symmetric tensor remains traceless after a Lorentz transformation.

3.4 From the previous consideration, we can conclude that the symmetric representation decomposes further into a traceless symmetric part $S^{\mu\nu} - \frac{1}{n} g^{\mu\nu} S$ and a part proportional to the metric $\frac{1}{n} g^{\mu\nu} S$, carrying the scalar trace S. What are the dimensions (i.e. independent components) of these two further parts?

Part 2 — Decomposition of Lorentz tensors under $SO(3)$

Since we know how a tensor behaves under a generic Lorentz transformation, we can in particular study its behaviour under the $SO(3)$ rotation Lorentz subgroup. This allows us to determine the angular momentum j associated with different tensor representations.
Recall that the representations of $SO(3)$ are labeled by an integer $j = 0, 1, 2, \ldots$, and that the dimension of the representation with angular momentum j is $2j + 1$.
In this notation,

- a Lorentz scalar is clearly invariant under rotations, so it has $j = 0$;
- a four-vector $v^\mu = (v^0, \vec{v})$ is an irreducible representation of the Lorentz group but *reducible* under $SO(3)$ rotations: spatial rotations do not mix v^0 with $\vec{v}$. v_0 is invariant under spatial rotations, so it has $j = 0$, while the three spatial components v_i form an irreducible three-dimensional representation of $SO(3)$, so they have $j = 1$. In group theory language we say that, from the point of view of spatial rotations, a four-vector decomposes into the direct sum of a scalar and a $j = 1$ representation:

$$v^\mu \in 0 \oplus 1, \tag{13}$$

We now want to understand what angular momenta appear in a generic tensor $T^{\mu\nu}$ with two indices. By definition, $T^{\mu\nu}$ transforms as the tensor product of two four-vector representations.

3.5 Using the above result, expand the following product:

$$T^{\mu\nu} \in (0 \oplus 1) \otimes (0 \oplus 1), \tag{14}$$

then use the usual rule of composition of angular momenta (that adding j_1 and j_2 gives all j between $|j_1 - j_2|$ and $j_1 + j_2$) to show that

$$(0 \oplus 1) \otimes (0 \oplus 1) = 0 \oplus 1 \oplus 1 \oplus (0 \oplus 1 \oplus 2). \tag{15}$$

Thus, in the decomposition of a generic tensor $T^{\mu\nu}$ in representations of the rotation group, the $j = 0$ representation appears twice, the $j = 1$ representation appears three times, and the $j = 2$ once.

3.6 It is interesting to see how these representations are shared between the symmetric traceless, the trace and the antisymmetric part of the tensor $T^{\mu\nu}$, since these are the irreducible Lorentz representations.
Consider the following objects:

- T^{00}: time-time component;
- T^{0i}: time-space component;
- T^{i0}: space-time component;
- $T^{[ij]} = \frac{1}{2}(T^{ij} - T^{ji})$: antisymmetric part;
- $\frac{1}{3}\delta^{ij}T^{kk}$: spacial trace part;
- $T^{(ij)}_{\text{traceless}} = \frac{1}{2}(T^{ij} + T^{ji}) - \frac{1}{3}\delta^{ij}T^{kk}$: traceless symmetric part.

For each of them, indicate whether it corresponds to a scalar, a vector, or a tensor representation. Specify the number of independent components, the associated angular momentum value j, and identify the correspondence between these six tensor components and the six terms in Eq. 15. Mention some physically relevant examples.

Part 3 — Generalization

3.7 Show that a totally symmetric and traceless spatial tensor $S^{i_1,i_2,\ldots,i_N}$ with N spatial indices has angular momentum $j = N$.

Exercise 4 - The isomorphism $SL(2,\mathbb{C})/\{\pm I\} \cong SO_+(1,3)$ [Math Problem]

Let H be the real vector space of Hermitian 2×2 matrices and identify $\mathbb{R}^4 \ni x = (x^0, x^1, x^2, x^3)$ with

$$\widehat{x} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = x^0\,\mathbf{1} + \sum_{j=1}^{3} x^j \sigma_j,$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices.

Define the Minkowski bilinear form on $\mathbb{R}^4$ by

$$(x, y) = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3, \qquad \text{and} \quad \|x\|_M^2 = (x, x).$$

4.1 Show that $\det(\widehat{x}) = \|x\|_M^2$. (Conclude that x is lightlike iff $\widehat{x}$ is singular.)

4.2 For $A \in SL(2, \mathbb{C})$ define $\Phi(A) : H \to H$ by $\Phi(A)(X) = AXA^*$. Prove that $\Phi(A)$ preserves determinants on H, i.e. $\det(\Phi(A)X) = \det(X)$ for all $X \in H$.

4.3 Using the identification $x \leftrightarrow \widehat{x}$, prove that there exists a unique linear map $\phi(A) : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$A\,\widehat{x}\,A^* = \widehat{\phi(A)\,x} \quad \text{for all } x \in \mathbb{R}^4.$$

Show that $\phi : SL(2, \mathbb{C}) \to O(1, 3)$ is a group homomorphism.

4.4 Prove that $\phi(SL(2, \mathbb{C})) \subset SO_+(1, 3)$, i.e. $\det(\phi(A)) = 1$ and $\phi(A)$ is orthochronous (it maps the future light cone into itself).
Hint: Use that $SL(2, \mathbb{C})$ is connected and ϕ is continuous; check the claim at $A = I$.

4.5 Determine the kernel of ϕ.
Hint: If $A\widehat{x}A^* = \widehat{x}$ for all x, first show $A \in U(2)$, then use linear independence of $\{\mathbf{1}, \sigma_1, \sigma_2, \sigma_3\}$ to conclude $A = \lambda I$; combine with $\det A = 1$.

4.6 Show that ϕ is surjective.
One route: Every $\Lambda \in SO_+(1, 3)$ factorizes as $\Lambda = R_1\,L(\chi)\,R_2$ with $R_1, R_2 \in SO(3)$ and a boost $L(\chi)$ along the z-axis. Exhibit preimages:

$$\phi\left(\begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix}\right) = L(2\theta), \qquad \phi\big(U\big) = \text{the rotation in } SO(3) \text{ corresponding to } U \in SU(2).$$

Conclude that every element of $SO_+(1, 3)$ lies in the image of ϕ.

4.7 Deduce that ϕ descends to a bijective homomorphism

$$\overline{\phi} : SL(2, \mathbb{C})/\{\pm I\} \longrightarrow SO_+(1, 3),$$

hence is an isomorphism of Lie groups.