Quantum Field Theory WS 2025/26

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Sheet 02: The Poincaré group

Please hand in your solutions on Moodle by **Friday, 31.10.25, 8am**

Exercise 1 - Lorentz transformations

Consider a Lorentz transformation infinitesimally close to the identity, $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$, with $\omega^{\mu}{}_{\nu}$ infinitesimal.

1. What condition should ω satisfy to make Λ a Lorenz transformation?
2. Find the form of ω that corresponds to rotations around the i-th axis. Give an explicit matrix for an infinitesimal rotation around the z-axis.
3. Find the form of ω that corresponds to boosts along the i-th axis. Give an explicit matrix for a boost in the x-direction.
4. Show that any ω satisfying the condition derived in the first point can be written as a linear combination of boosts and rotations.

Consider now a generic Lorentz transformation $\Lambda^{\mu}{}_{\nu}$.

5. Prove that the totally antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ is invariant under proper Lorentz transformations, and that it changes sign under a parity transformation.
6. Prove that the measure

$$\frac{d^3\vec{k}}{(2\pi)^3 2E_k} \qquad \text{with} \quad E_k = \sqrt{\vec{k}^2 + m^2} \tag{1}$$

   is Lorentz invariant using two different ways:

   (a) Apply an explicit Lorentz transformation.
   <u>Hint</u>: *There is a subset of Lorentz transformations under which the measure is trivially invariant, so you only need to demonstrate it for the remaining one(s).*

   (b) Alternatively, find a way to express the integration measure in a manifestly Lorentz-invariant form by introducing a suitable combination of δ and Θ functions and extending the integration to include the energy component.

Exercise 2 - The Lorentz Algebra

Consider again an infinitesimal Lorentz transformation acting on a 4-vector V^{μ} as $V^{\mu} \to V'^{\mu} = \Lambda^{\mu}{}_{\nu} V^{\nu}$ with

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu} \,. \tag{2}$$

1. Prove that the form of the generators $J^{\mu\nu}$ of the Lorentz group acting on 4-vectors, i.e. in the so-called *vector representation*, reads

$$[J^{\mu\nu}]^{\rho}{}_{\sigma} = i(g^{\mu\rho}\delta^{\nu}{}_{\sigma} - g^{\nu\rho}\delta^{\mu}{}_{\sigma})\,. \tag{3}$$

   Hint: *The generators are defined such that for infinitesimal ω, we have*

$$\left[e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\right]^{\rho}{}_{\sigma} V^{\sigma} = V^{\rho} + \omega^{\rho}{}_{\sigma}V^{\sigma}\,. \tag{4}$$

2. Use the explicit form of the generators $[J^{\mu\nu}]^{\rho}{}_{\sigma}$ derived above to derive the commutation relations of the Lorentz group,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})\,. \tag{5}$$

Consider now a scalar function $\phi(x)$ (a "classical field") depending on the space-time coordinate x^{μ}, and a (proper orthochronous) Lorentz transformation Λ acting on the coordinates as $x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$. The fact that $\phi(x)$ is a scalar under Lorentz transformations can be phrased as follows:

$$\phi'(x') = \phi(x) \tag{6}$$

or alternatively, considering the transformed field at the original space-time point, we may write

$$\phi'(x) = \phi(\Lambda^{-1}x)\,. \tag{7}$$

3. Take now Λ to be an infinitesimal Lorentz transformation. Show that to first order in ω, we have

$$\phi'(x) - \phi(x) = -\frac{i}{2}\omega^{\mu\nu}L_{\mu\nu}\phi(x) \tag{8}$$

   where

$$L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\,, \qquad \text{with} \quad \partial_{\mu} = \frac{\partial}{\partial x^{\mu}}\,. \tag{9}$$

4. Prove that the $L^{\mu\nu}$ just defined, satisfy the Lorentz algebra in eq. (5).

5. Even if we have not quantized anything yet, let us make the further step of thinking of $\phi(x)$ as a *quantum field*, i.e. an operator-valued distribution acting on the Hilbert space of the quantum theory. Let us then define $U(\Lambda)$ as the (unitary) operator generating Lorentz transformations on the Hilbert space of the quantum theory. If we label a state by its momentum $|p\rangle$, we write

$$|p\rangle \to |\Lambda p\rangle = U(\Lambda)|p\rangle\,. \tag{10}$$

   With this, one can easily show that equation (7) can also be written as[1]

$$U(\Lambda)\phi(x)U(\Lambda)^{-1} = \phi(\Lambda^{-1}x)\,. \tag{11}$$

   By writing

$$U(\Lambda) = \exp\left[-\frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}\right]$$

   and using the results you derived in the previous points, show that

$$[\mathcal{M}^{\mu\nu}, \phi(x)] = L^{\mu\nu}\phi(x)\,. \tag{12}$$

[1] We will say more about this in the lecture once we consider quantum fields, for now you can take this transformation law as given.

Exercise 3 - Parity and Time Reversal

In this problem, we investigate parity and time reversal and the need of *antiunitary* operators.

Part 1 - Antiunitary Operators

A theorem by Wigner from the 1930s states that in order to represent a symmetry transformation as an operator acting on the Hilbert space of physical states, there are fundamentally *two* options. The first option is that the operator is linear and unitary. In your quantum mechanics courses, you have likely encountered many examples of such operators, as this is the most common case in physics. The reason is that most symmetries are continuous and connected to the identity, which is unitary. However, there is a second option: the operator could also be *antilinear* and *antiunitary*.
Let H be a Hilbert space with scalar product $\langle \cdot\,, \cdot \rangle$. Let $\psi_1, \psi_2 \in H$, then an *antiunitary* operator $A : H \to H$ satisfies the property

$$\langle A\psi_1, A\psi_2 \rangle = \langle \psi_1, \psi_2 \rangle^* \,. \tag{13}$$

1. Show that eq. (13) is incompatible with A being a linear operator and that instead, A must be *antilinear*,

$$A\left(a_1\psi_1 + a_2\psi_2\right) = a_1^* A\psi_1 + a_2^* A\psi_2\,, \qquad a_1, a_2 \in \mathbb{C}\,. \tag{14}$$

   Hint: *You may safely assume A to be invertible.*

2. Give a brief explanation of the jargon "antiunitary operators anticommute with i (the imaginary unit)".

Part 2 - Concatenations of Poincaré Transformations

Relativistic quantum field theories are required to be invariant not only under Lorentz transformations, but also under space-time translations. The corresponding symmetry transformation is given by

$$x^\mu \quad \to \quad x'^\mu \;=\; \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{15}$$

with Λ a Lorentz transformation and a a constant 4-vector. Transformations of this type are called *inhomogeneous* Lorentz transformations or *Poincaré* transformations. We denote such a transformation by the tuple (Λ, a).

3. Verify that the transformations in eq. (15) form a group, i.e. show that two sequential Poincaré transformations, say (Λ_1, a_1) followed by (Λ_2, a_2), amount to a single Poincaré transformation $(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$.

4. Given a Poincaré transformation (Λ, a), derive its inverse.

The transformation from eq. (15) corresponds to the 4-dimensional *vector representation* of the Poincaré group. Let us now study the Poincaré group as an abstract group, with the tuple (Λ, a) now denoting an abstract element of the group. We can then understand your findings from point 3. as the group multiplication law, which can be generalized to *any* representation. Concretely, if we denote the abstract group element (Λ, a) in some representation as $R(\Lambda, a)$, then we must have

$$R(\Lambda_2, a_2)\, R(\Lambda_1, a_1) = R(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)\,. \tag{16}$$

If one wants to study infinitesimal Poincaré transformations, it is conventional to write them as

$$R(1+\omega, \epsilon) = 1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + i\epsilon_\mu P^\mu\,, \tag{17}$$

where $J_{\mu\nu}$ are the generators of the Lorentz group, P_μ the generators of translations in the form corresponding to the particular representation, and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and ϵ^μ are taken infinitesimal.

5. Consider the product of an infinitesimal Poincaré transformation sandwiched between a general Poincaré transformation and its inverse,

$$R(\Lambda, a)R(1+\omega, \epsilon)R^{-1}(\Lambda, a)\,. \tag{18}$$

Use eq. (16) to write this product as a single Poincaré transformation $R(\bar{\Lambda}, \bar{a})$.

6. While allowing the representation under consideration to be either linear or antilinear, expand both $R(\bar{\Lambda}, \bar{a})$ and eq. (18) to first order in ω and ϵ to show that

$$R(\Lambda, a)\ (i\,J^{\mu\nu})\ R^{-1}(\Lambda, a) = i\,\Lambda_\rho{}^\mu\Lambda_\sigma{}^\nu\,(J^{\rho\sigma} - a^\rho P^\sigma + a^\sigma P^\rho)\ , \tag{19}$$
$$R(\Lambda, a)\ (i\,P^\mu)\ R^{-1}(\Lambda, a) = i\,\Lambda_\nu{}^\mu P^\nu\,, \tag{20}$$

where on the right-hand-sides we have Λ and a appearing in the vector representation.

Part 3 - Parity and Time Reversal

Let's specify now to the cases where (Λ, a) is either parity or time reversal,

$$\mathbf{P} \equiv R(\mathcal{P}, 0)\,, \qquad \mathbf{T} \equiv R(\mathcal{T}, 0)\,, \tag{21}$$

$$\mathcal{P}^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \mathcal{T}^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{22}$$

Furthermore, let us assume that parity and time reversal are symmetries of our theory and that we want to act with $\mathbf{P}$ and $\mathbf{T}$ on physical states in the corresponding Hilbert space. According to Wigner's theorem, this means that we want their representations to be either *linear* and *unitary* or *antilinear* and *antiunitary*.

7. Specifying eq. (20) to the case $\mu = 0$ and applying it to some physical state Ψ, show that $\mathbf{P}$ should be represented in a linear, unitary representation, while $\mathbf{T}$ should be represented in an antilinear, antiunitary representation.

   Hint 1: *In the lecture, you will see that the generator of translations in time corresponds to the Hamiltonian $\mathcal{H}$ of the theory, $P^0 = \mathcal{H}$.*

   Hint 2: *A physical theory should have a lowest energy state (a* vacuum*).*

Exercise 4 - Normal Subgroups [Math problem 1]

Let G, H be groups and $\varphi : G \to H$ a homomorphism. Prove the following:

1. $\varphi(1) = 1$, and $\varphi(g)^{-1} = \varphi(g^{-1})$.
2. $\ker \varphi$ is a subgroup of G, and $\operatorname{Im} \varphi$ is a subgroup of H.
3. φ is injective if and only if $\ker \varphi = \{1\}$.
4. Prove that $\ker \varphi$ is a normal subgroup of G.

Let $\varphi : \mathbb{R} \to SO(2)$ be defined by

$$\varphi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

5. Prove that φ is a group homomorphism.
6. Determine $\ker \varphi$ and $\operatorname{Im} \varphi$.

Exercise 5 - Semidirect Product [Math problem 2]

Let G and H be groups and let $\rho : G \to \operatorname{Aut}(H)$, $g \mapsto \rho_g$, be a homomorphism. Prove the following: Then $G \times H$ with multiplication

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2,\ h_1\, \rho_{g_1}(h_2))$$

is a group, the semidirect product $G \ltimes_\rho H$.