Quantum Field Theory WS 2025/26

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Sheet 01: QFT from the continuum limit of a discrete system

Please hand in your solutions on Moodle by **Friday, 24.10.25, 8am**

Exercise 1 - One-dimensional chain of particles connected by springs

In this problem, we will see how the formalism of quantum field theory naturally emerges from a known, discrete system, once the continuum limit is taken. To this end, we will consider the case of a one-dimensional chain of N particles connected to each other by springs, see fig. 1. We will quantize this system for finite values of N and then consider the continuum limit $N \to \infty$.

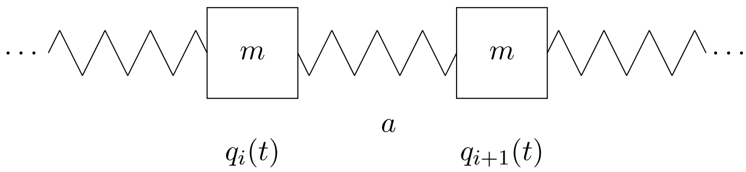

Figure 1: One-dimensional array of springs

Let us begin by reviewing a very well-known system, the one-dimensional harmonic oscillator. Its Lagrangian L is given by

$$L = \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right), \qquad \text{with} \quad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}, \ \text{and} \ \ m, \omega > 0\,. \tag{1}$$

1. Derive the expressions for the momentum p conjugate to x and for the Hamiltonian H.

2. Show that the system can be quantized by introducing creation and annihilation operators $a^\dagger, a$ defined as

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right), \quad p = -i\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right). \tag{2}$$

   To do so, give the commutation relation between $a, a^\dagger$ and check that it leads to the correct canonical commutation relation for x and p. Subsequently, express the Hamiltonian H in terms of $a, a^\dagger$ and show that all its (physically relevant) eigenvalues are quantized.

Let us consider now the system in fig. 1. We assume that all N particles have the same mass m and that the displacement of particle j from its equilibrium position is denoted as $q_j/\sqrt{m}$. Furthermore, we take the chain to be closed (forming a ring) and impose that each particle can only move along the chain, i.e. transverse displacements are not allowed. Moreover, we take a *harmonic approximation* and assume that every particle interacts only with its two neighbors and that the displacements are small compared to their equilibrium separation. The Lagrangian of the system can then be written as

$$L = \frac{1}{2}\sum_{j=1}^{N}\dot{q}_j^2 - \frac{\nu^2}{2}\sum_{j=1}^{N}\left(q_{j+1} - q_j\right)^2, \tag{3}$$

with $\nu > 0$ denoting the stiffness of the springs and we identify $q_{N+1}(t) \equiv q_1(t)$.

3. Show that the classical equations of motion are given by

$$\ddot{q}_j(t) = \nu^2 \left[q_{j+1}(t) - 2\, q_j(t) + q_{j-1}(t)\right] \,, \quad \forall j = 1, \ldots, N \,. \tag{4}$$

The so-called *normal mode* solutions to eq. (4), i.e. solutions that are harmonic functions of time can be written as

$$q_j(t) = \Re\left\{A\, e^{-i(K\, j - \omega t)}\right\} \,, \tag{5}$$

where A is a normalization constant. The frequency ω does not depend on j, and K is a boundary constant.

4. Imposing *periodic* boundary conditions, $q_j(t) = q_{j+N}(t)$, show that K can take only a discrete set of values which can be chosen as

$$K = \frac{2\pi\alpha}{N} \,, \quad \text{with} \quad \alpha = 0, 1, ..., N-1 \,.$$

5. Derive the *dispersion relation*, i.e. show that for every given value of α, the corresponding frequency ω_α is given by

$$\omega_\alpha = 2\nu \left|\sin\left(\frac{\pi\alpha}{N}\right)\right| \,. \tag{6}$$

We now write the normal mode solutions for a given value of $\alpha \in \{0, 1, ..., N-1\}$ as

$$q_j^\alpha(t) = \Re\{a_j^\alpha\, e^{i\omega_\alpha t}\} \,, \quad \text{with} \quad a_j^\alpha = A\, e^{-i2\pi\alpha\, j/N} \,. \tag{7}$$

6. Show that the coefficients a_j^α satisfy the orthogonality relation

$$\sum_{j=1}^{N} a_j^{\alpha *} a_j^{\beta} = |A|^2\, N\, \delta_{\alpha\beta} \,, \tag{8}$$

as well as the completeness relation

$$\sum_{\alpha=0}^{N-1} a_j^{\alpha *} a_k^{\alpha} = |A|^2\, N\, \delta_{jk} \,. \tag{9}$$

To render the coefficients a_j^α properly orthonormal, we, from now on, fix $A = 1/\sqrt{N}$.

7. Show that by introducing the normal coordinates

$$Q_\alpha(t) = \sum_{j=1}^{N} a_j^\alpha q_j(t) \tag{10}$$

the Lagrangian (3) becomes similar to the Lagrangian of N independent harmonic oscillators,

$$L = \frac{1}{2} \sum_{\alpha=0}^{N-1} \left(\dot{Q}_\alpha^* \dot{Q}_\alpha - \omega_\alpha^2 Q_\alpha^* Q_\alpha\right) \,. \tag{11}$$

While the original coordinates q_j were real, the normal coordinates Q_α are complex. In principle, N complex degrees of freedom correspond to $2N$ real degrees of freedom. However, as we started from N real degrees of freedom, only half of them can be linearly independent.

8. Use that $Q_{\alpha+N} = Q_\alpha$ as well as $Q^*_\alpha = Q_{-\alpha}$ to show that it is possible to write the Lagrangian in terms of a single set of N real coordinates that is now equivalent to the Lagrangian of N independent harmonic oscillators. Concretely, show that it can be written in the form

$$L = \frac{1}{2}\sum_{\alpha=0}^{N-1}\left(\dot{\tilde{Q}}^2_\alpha - \omega^2_\alpha \tilde{Q}^2_\alpha\right), \tag{12}$$

where all the $\tilde{Q}_\alpha$ are real coordinates.

Hint 1: Distinguish the cases of N odd and even.

Hint 2: Split the Q_α into their real and imaginary part according to

$$Q_\alpha = \frac{1}{\sqrt{2}}\left(Q^c_\alpha - iQ^s_\alpha\right) \tag{13}$$

and examine the special cases $\alpha = 0$ and $\alpha = N/2$ explicitly.

We can now quantize each of the N harmonic oscillators individually in the same fashion as reviewed at the beginning of this exercise.

9. Give the commutation relation between the coordinates $\tilde{Q}_\alpha$ and their conjugate momenta $\tilde{\Pi}_\beta$.

10. Show that the commutation relation of the original coordinates and their conjugate momenta are given by

$$[q_j, p_k] = i\,\hbar\, m\, \delta_{jk}\,. \tag{14}$$

Finally, let us consider now the continuum limit $N \to \infty$ for this system. Given a particular normal mode of motion (value of α), the phase difference between adjacent particles is $2\pi\alpha/N$. If we denote the equilibrium separation between two neighboring particles as d, particles separated along the chain by a multiple of the distance Nd/α oscillate with the same phase. This lets us introduce the notion of a wavelength

$$\lambda = Nd/\alpha$$

to the motion of the particles along the entire chain. One wavelength contains N/α particles, and in the limit when the wavelength λ is much larger than the relative spacing d (corresponding to each wavelength consisting of many particles), one can imagine to describe the system as a continuum. To this end, we introduce the wave number

$$k = \frac{2\pi}{\lambda} = \frac{2\pi\alpha}{Nd}\,,$$

and the equilibrium position of a given particle can be written as $x_j = j\,d$. With these notations, our normal mode solutions and corresponding normal coordinates can now be written as

$$a_{jk} = \frac{1}{\sqrt{N}} e^{-ikx_j}\,, \quad q(x_j,t) \equiv q_j(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} Q_k(t)\, e^{ikx_j}\,, \quad Q_k(t) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} q(x_j,t)\, e^{-ikx_j}\,, \tag{15}$$

where we have traded the index α for an index k denoting the wave number corresponding to the particular value of α under consideration. The continuum limit can be taken when $kd \ll 1$. Keeping the length $L = Nd$ of the chain fixed, $q(x_j,t)$ and $q(x_{j+1},t)$ become nearly equal as x_j and x_{j+1} get closer and closer to each other. This allows us to view the displacements as a continuous function of the position x on the line, namely

$$q(x_j,t) \;\to\; \sqrt{m}\,\phi(x,t)\,,$$

where the factor $\sqrt{m}$ is needed to account for the normalization we have chosen to define the q_j in eq. (3).

11. Using the formal replacement

$$\sum_{j=1}^{N}(\ldots) \to \frac{N}{L}\int_0^L \mathrm{d}x\,(\ldots)\,,$$

show that the Lagrangian in eq. (3) can be written as

$$L = \int_0^L \mathrm{d}x\,\mathcal{L}\,, \qquad \text{with} \;\; \mathcal{L} = \frac{\rho}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{\rho c^2}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 \tag{16}$$

where we introduced the mass density $\rho = m/d$ and the constant $c = \nu\,d$.

12. Write down the Euler-Lagrange equation for the field ϕ that follows from the Lagrangian in eq. (16) and use it to give a physical interpretation of the constant c.

13. Suppose you want to quantize the continuous system described by the Lagrangian eq. (16). Given the relation between the field $\phi(x)$ and the original discrete coordinate q_j, as well as the commutation relation in eq. (14), can you guess the commutation relation for the field $\phi(x)$ and its conjugate momentum $\pi(x)$ that needs to be applied in the continuum limit?