## 4.4 Quantum Fields

**4.4.1 Def.**

Let $\mathcal{H}$ be a Hilbert space (complete inner product space)

* An operator in $\mathcal{H}$ is a pair $(A, D)$, consisting of a subspace $D = D_A \subset \mathcal{H}$ and a $\mathbb{C}$-linear mapping $A : D \longrightarrow \mathcal{H}$

* $A$ is densely defined whenever $D_A$ is dense in $\mathcal{H}$

Densely defined operators can be unbounded

i.e. $\sup \{ \|Af\| : f \in D, \|f\| \leq 1 \} = \infty$

many relevant operators (position, momentum) are unbounded

* Every densely defined operator $A$ in $\mathcal{H}$ has an adjoint operator $(A^*, D_{A^*})$

  - $D_{A^*} := \{ f \in \mathcal{H} \mid \exists h \in \mathcal{H} \ \forall g \in D_A : \langle h, g \rangle = \langle f, Ag \rangle \}$

  - $A^*$ defined by

$$\langle A^* f, g \rangle = \langle f, Ag \rangle, \quad f \in D_{A^*}, \quad g \in D_A$$

$A^* f$ for $f \in D_{A^*}$ is thus the uniquely determined

$h = A^* f \in \mathcal{H}$ with $\langle h, g \rangle = \langle f, Ag \rangle \quad \forall g \in D_A$.

- A <u>self-adjoint</u> operator $A$ is an operator which agrees with its adjoint $A^*$ i.e. $D_A = D_{A^*}$, $A^* f = Af$ $\forall f \in D_A$.

- For a self-adjoint operator $A$ there exists a unique representation $U : \mathbb{R} \to U(\mathcal{H})$ (unitary group)

satisfying

$$\lim_{t \to 0} \frac{U(t) f - f}{t} = -i A f$$

for each $f \in D_A$. $U$ is denoted by $U(t) = e^{-itA}$ and $A$ $(-iA)$ is called the infinitesimal generator of $U(t)$.

<u>Thm 4.4.2.</u> Thm of Stone. Let $U(t)$ be a one parameter group of unitary operators in the complex Hilbert space $\mathcal{H}$, that is $U$ is a unitary representation of $\mathbb{R}$

Then the operator $A$, defined by

$$A f := \lim_{t \to 0} i \frac{U(t) f - f}{t}$$

in the domain in which this limit exists w.r.t. the norm of $\mathcal{H}$ is self-adjoint and generates $U(t)$.

$U(t) = e^{-itA}$, $t \in \mathbb{R}$.

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- Let $SO = SO(\mathcal{H})$ denote the set of self-adjoint operators in $\mathcal{H}$.
- $\mathcal{O} = \mathcal{O}(\mathcal{H})$ the set of all densely def. operator. in $\mathcal{H}$.

<u>Def. 4.4.3</u>: A field operator on a <u>quantum field</u> is an operator valued distribution (on $\mathbb{R}^n$), that is a map.

$$\Phi : \mathcal{S}(\mathbb{R}^n) \to \mathcal{O}$$ s.t. there exists a dense subspace $D \subset \mathcal{H}$ satisfying

1. For each $f \in \mathcal{S}$ the domain of def. $D_{\Phi(f)}$ contains $D$.
2. The induced map $\mathcal{S} \to \mathrm{End}(D)$, $f \mapsto \Phi(f)|_D$ is linear.

3. For each $v \in D$ and $w \in \mathcal{H}$, the assignment $f \mapsto \langle w, \Phi(f)(v) \rangle$ is a tempered distribution

**Remark.** operator-valued distribution different from the notion of a field as a quantity defined at each point of spacetime

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## 4.5 Relativistic invariance:

Let $M = \mathbb{R}^{1,D-1}$ D-dim Minkowski space

with metric $x^2 = \langle x, x \rangle = x^0 x^0 - \sum_{i=1}^{D-1} x^j x^j$

Two subsets $X, Y \subset M$ are called space-like separated if for any $x \in X$ and $y \in Y$ the condition $(x-y)^2 < 0$ is satisfied

i.e. $$(x^0 - y^0)^2 < \sum_{j=1}^{D-1} (x^j - y^j)^2$$

The forward cone is $C_+ := \{x \in M : x^2 = \langle x, x\rangle \geq 0,\ x^0 \geq 0\}$ and the causal order is given by $x \geq y \Leftrightarrow x - y \in C_+$

Relativistic invariance of classical point particles or classical field theory is described by the Poincaré group $P$.

The Poincaré group acts on $\mathcal{S} = \mathcal{S}(\mathbb{R}^D)$ from the left by

$$h.f(x) := f(h^{-1}x)$$

it is written in the form

$$(\Lambda, a)\, f(x) = f(\Lambda^{-1}(x-a))$$

where $(\Lambda, a) \in L \ltimes M$, $\Lambda \in L$ Lorentz group, $a \in M$

The relativistic invariance of the quantum system w.r.t. to Minkowski space $M = \mathbb{R}^{1,D-1}$ is given in terms of a projective representation

$$P \longrightarrow U(\mathbb{P}(\mathcal{H}))$$ of the Poincaré group

we thus have $\quad \mathcal{U} : \mathcal{P} \longrightarrow \mathcal{U}(\mathcal{H})$

$$(\Lambda, a) \longmapsto \mathcal{U}(\Lambda, a)$$

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## 4.6 Wightman Axioms

A Wightman QFT in dim $D$ consists of the following data:

- The space of states : $\mathbb{P}(\mathcal{H})$ projective space of a separable complex Hilbert space $\mathcal{H}$
- The vacuum vector $\Omega \in \mathcal{H}$ of norm $1$
- A unitary rep $\mathcal{U} : \mathcal{P} \longrightarrow \mathcal{U}(\mathcal{H})$ of $\mathcal{P}$, the covering group of the Poincaré group.
- A collection of field operators $\quad \alpha \in I$

$$\Phi_\alpha : \mathcal{S}(\mathbb{R}^D) \to \mathcal{O}$$

with a dense $D \subset \mathcal{H}$ as their common domain

These data satisfy the following axioms:

## Axiom W1 (Covariance)

1. $\Omega$ is $\mathcal{P}$ invariant, that is

$$U(\Lambda, a)\,\Omega = \Omega \quad \text{for all } (\Lambda, a) \in \mathcal{P}$$

and $D$ is $\mathcal{P}$-invariant $\quad U(\Lambda, a)\, D \subset D$

2. The common domain $D \subset \mathcal{H}$ is invariant

$$\Phi_\alpha(f)\, D \subset D \quad \text{for all } f \in \mathcal{S},\ \alpha \in I$$

3. The actions on $\mathcal{H}$ and $\mathcal{S}$ are equivariant where $\mathcal{P}$ acts on $\mathrm{End}(D)$ by conjugation

That is on $D$ we have

$$U(\Lambda, a)\, \Phi_\alpha(f)\, U(\Lambda, a)^* = \Phi((\Lambda, a) f)$$

$$\forall f \in \mathcal{S}, \quad (\Lambda, a) \in \mathcal{P}$$

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## Axiom 2 (Locality)

$\Phi_a(f)$ and $\Phi_b(g)$ commute on $D$ if the supports of $f, g \in \mathcal{S}$ are space-like separated

that is on $D$

$$\Phi_\alpha(f)\,\Phi_\beta(g) - \Phi_\beta(g)\,\Phi_\alpha(f)$$

$$= [\,\Phi_\alpha(f), \Phi_\beta(g)] = 0 .$$

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**Axiom 3** (Spectrum condition)

The joint spectrum of the operators $P_j$ is contained in the forward con $C_+$

where

$$U(\mathbb{1}, a) = \exp(i\, a P)$$
$$= \exp i\Big(a^0 P_0 - \sum_{j=1}^{D-1} a^j P_j\Big)$$

$P_0 = H$ energy, $P_j$, $j>0$ momentum components.

This axiom ensures that there are no negative energy states

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**(Axiom 4)** Uniqueness of the vacuum

The only vectors in $\mathcal{H}$ left invariant by the translations $U(\mathbb{1}, a)$, $a \in M$ are the scalar multiples of the vacuum $\Omega$.

**Rem:** • Although the axioms seem natural, it is very hard to construct interacting Wightman QFT in $D=4$, $D=2$ WQFTs exist, many of these are conformal field theories.

• For the example of free bosonic QFT see the book of Schottenloher.

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