§4 Axioms of relativistic Quantum Field Theory

4.1 Tempered Distributions

4.1.1 Definition:

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions is defined by

$$\mathcal{S}(\mathbb{R}^n) = \{ f : \mathbb{R}^n \longrightarrow \mathbb{C} \mid f \in C^\infty(\mathbb{R}^n) \text{ and } p_{\alpha,\beta}(f) < \infty \text{ for every pair of multi-indices } \alpha, \beta \in \mathbb{N}^n \}$$

where the semi-norms (not necessarily positive def.)

$$p_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}^n} | x^\alpha \partial_\beta f(x) |$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

$$|\alpha| = \alpha_1 + \dots + \alpha_n$$

$$\partial_\beta := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$$

$p_{\alpha,\beta}(f) < \infty$ means that even if you multiply a derivative of f by any power of x, the result stays bounded

So the condition encodes both smoothness (all derivatives exist and are continuous) and rapid decay

(all derivatives vanish faster than any polynomial grows

<u>Examples:</u> $f(x) = e^{-|x|^2}$, Any $P(x)e^{-|x|^2}$ with P pol.

<u>Not Schwartz</u>: $f(x) = e^{-|x|}$ decays exponentially but its derivatives don't decay rapidly enough, $\overset{\text{der.}}{\text{not}}$ smooth at $x=0$

$f(x) = \dfrac{1}{1+x^2}$ decays only polynomially

<u>4.1.2) Definition</u> Tempered distribution

The continuous dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions. A linear functional

$T: \mathcal{S} \longrightarrow \mathbb{C}$ is tempered iff it is continuous for the seminorm topology for all $p_{\alpha,\beta}$

The elements of $\mathcal{S}(\mathbb{R}^n)$ are called test functions.

Some distributions are represented by functions for example for an arbitrary measurable and bounded function g on $\mathbb{R}^n$ the functional

$$T_g(f) := \int_{\mathbb{R}^n} g(x)\, f(x)\, dx, \qquad f \in \mathcal{S}$$

defines a distribution.

A well-known distribution which cannot be represented as a distribution of the form T_g for a fct. g on $\mathbb{R}^n$ is the delta distribution,

$$\delta_y : \mathcal{S} \longrightarrow \mathbb{C}, \quad f \longmapsto f(y)$$

the evaluation at $y \in \mathbb{R}^n$.

Nevertheless δ_y is frequently called the delta function at y and one writes $\delta_y = \delta(x-y)$ in order to use the formal integral

$$\delta_y(f) = f(y) = \int_{\mathbb{R}^n} \delta(x-y)\, f(x)\, dx$$

here the r.h.s of the equation is defined by the L.H.S

Distributions have derivatives. For example

$$\frac{\partial}{\partial q_j} T(f) := -T\left(\frac{\partial}{\partial q_j} f\right)$$

and $\partial_\alpha T(f) := (-1)^{|\alpha|} \; T(\partial^\alpha f)$, $f \in \mathcal{S}$

if T is of the form T_g

one obtains $\partial_\alpha T_g = T_{\partial_\alpha g}$

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Ex: Let $H = \chi_{[0,\infty)}$ on $\mathbb{R}$ and define

$T_H(f) = \int_0^\infty f(t)\, dt$ then $T_H'(f) = -\int_0^\infty f'(t)\, dt$
$= f(0) = \delta_0(f)$ so $\delta_0 = H'$
in the distributional sense.

4.13 Proposition:

Every tempered distribution $T \in \mathcal{S}'$ has a representation as a finite sum of derivatives of continuous functions of polynomial growth, that is there exist $g_\alpha : \mathbb{R}^n \longrightarrow \mathbb{C}$ s.t.

$$T = \sum_{0 \le |\alpha| < k} \partial_\alpha T_{g_\alpha}$$

4.2| Fourier transform on $\mathcal{S}$ and $\mathcal{S}'$

4.2.1|Df Fix a non-degenerate bilinear form $x \cdot p$ on $\mathbb{R}^n$ (Euclidean or Minkowski). For $u \in \mathcal{S}(\mathbb{R}^n)$ define

$$\mathcal{F} u(p) = \hat{u}(p) := \int_{\mathbb{R}^n} u(x)\, e^{ixp}\, dx$$

$$\mathcal{F}^{-1} v(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} v(p)\, e^{-ixp}\, dp$$

for $u \in \mathcal{S}$ $\quad \mathcal{F} u$ is also $\in \mathcal{S}$

4.2.2| Proposition: The Fourier transform is a linear continuous map $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S}$

4.2.3 | Definition: Fourier transform of a distribution

For $T \in \mathcal{S}'$ define $\mathcal{F}T \in \mathcal{S}'$ by

$$(\mathcal{F}T)(\varphi) = T(\hat{\varphi}), \quad \varphi \in \mathcal{S}$$

Note that for a function g

$$(\mathcal{F}T_g)(\varphi) = T_g(\hat{\varphi}) = \int_{\mathbb{R}^D} \int_{(\mathbb{R}^D)'} g(x)\, \varphi(p)\, e^{ixp}\, dp\, dx$$

$$= T_{\mathcal{F}g}(\varphi)$$

Examples:

$$\mathcal{F}(\delta_0) = \int_{\mathbb{R}^D} \delta_0(x)\, e^{ixp}\, dx = 1$$

$$\mathcal{F}^{-1}(e^{ip\cdot y}) = 2\pi^{-D} \int_{\mathbb{R}^D} e^{ip(y-x)}\, dp = \delta(x-y)$$

$$\mathcal{F}(\partial_k u) = -i\, p_k\, \hat{u}$$

$$\mathcal{F}(\partial_\alpha u) = -i p_\alpha\, \hat{u}$$

4.2.4) Application to PDE

Let $P(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial

The PDE $P(-i\partial) u = v$ becomes

$P(p)\,\hat{u} = \hat{v}$ in Fourier space.

A fundamental solution is $G \in \mathcal{S}'$ with

$P(-i\partial) G = \delta$. If $T \in \mathcal{S}'$ solves

$P T = 1$, then $G = \mathcal{F}^{-1} T$ is a

fundamental solution and $u = G * v$ solves

$P(-i\partial) u = v$

where the convolution is defined by

$$u * v\,(x) := \int_{\mathbb{R}^n} u(y)\, v(x-y)\, dy$$

for $u, v \in \mathcal{S}$.

4.3 The Klein-Gordon Equations

Let $\mathbb{R}^{1,D-1}$ denote Minkowski space with quadratic form

$$x^2 = \langle x, x \rangle = x_0^2 - \sum_{j=1}^{D-1} x_j^2$$

The d'Alembert operator is defined by

$$\Box = \partial_0^2 - \sum_{j=1}^{D-1} \partial_j^2$$

4.3.1 For $m>0$ the (inhomogeneous) Klein-Gordon equation is $(\Box + m^2)u = v$. In Fourier space it reads $(-p^2+m^2)\hat{u} = \hat{v}$, $p^2 = p_0^2 - \vec{p}^{\,2}$

A fundamental solution can be determined by solving

$$(-p^2+m^2)T = 1$$

so $T = (m^2-p^2)^{-1}$

As a distribution given by

$$T(v) = \int_{\mathbb{R}^{D-1}} \left(PV \int_{\mathbb{R}} \frac{v(p)}{w(p) - p_0^2} \, dp_0 \right) d^{D-1}p$$

$$w(p) = m^2 + \vec{p}^2$$

where PV is the principal value of the integral

The corresponding fundamental solution is called the propagator and is given by

$$G(x) = (2\pi)^{-D} \int_{\mathbb{R}^D} (m^2 - p^2)^{-1} e^{-ixp} \, dp$$

Free fields

$$(\Box + m^2) \phi = 0$$

$$(p^2 - m^2) \hat{\phi} = 0$$

therefore $\hat{\phi}$ has its support on the mass shell

$$\{ p \in (\mathbb{R}^{1,D-1})' : p^2 = m^2 \}$$

Consequently $\hat{\phi}$ is proportional to $\delta(p^2 - m^2)$

as a distribution, that is $\hat{\phi} = g(p)\,\delta(p^2 - m^2)$

and we get ϕ by the inverse Fourier

transform

$$\phi(x) = (2\pi)^{-D} \int_{\mathbb{R}^D} g(p)\ \delta(p^2 - m^2)\ e^{-ipx}\, dp$$

manipulating this further and denoting by $x^0 = t$ we arrive at (See Schottenloher for details)

Prop. 4.3.2

Any (tempered) solution of $(\Box + m^2)\phi = 0$

has Fourier support on $p^2 = m^2$ and can be written for suitable $a, a^* \in \mathcal{S}(\mathbb{R}^{D-1})$

as

$$\phi(t, x) := (2\pi)^{-D+1} \int_{\mathbb{R}^{D-1}} a(p)\, e^{i(\vec{p}\cdot\vec{x} - w(p)t)} + a^*(p)\, e^{-i(\vec{p}\cdot\vec{x} - w(p)t)}\ \frac{d\vec{p}}{2w(p)}$$