Lemma 3.12. *The map*

$$\exp : \mathrm{Mat}(n, \mathbb{K}) \to GL(n, \mathbb{K})$$

is invertible in a neighborhood of 0*. That is, there exists a neighborhood* U *of* 0 *such that* $\exp : U \to \exp(U)$ *is bijective, and the inverse is given by an absolutely convergent power series.*

Proof. The series

$$\log X = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(X - \mathbf{1})^n}{n}$$

converges absolutely for $\|X - \mathbf{1}\| < 1$, since

$$\sum_{n=1}^{\infty} \left\| (-1)^{n+1} \frac{(X - \mathbf{1})^n}{n} \right\| \le \sum_{n=1}^{\infty} \frac{\|X - \mathbf{1}\|^n}{n} = -\log\left(1 - \|X - \mathbf{1}\|\right) < \infty.$$

Then $\exp(\log X) = X$ for $\|X - \mathbf{1}\| < 1$, and $\log(\exp X) = X$ whenever $\|\exp X - \mathbf{1}\| < 1$. □

3.3 One-Parameter Groups

Definition 3.13. A map $X : \mathbb{R} \to GL(n, \mathbb{K})$, $t \mapsto X(t)$, is called a *one-parameter group* if it is continuously differentiable, $X(0) = \mathbf{1}$, and

$$X(s + t) = X(s)X(t) \quad \text{for all } s, t \in \mathbb{R}.$$

The image of such a map is a subgroup, with $X(t)^{-1} = X(-t)$.

Theorem 3.14. *(i) For any* $X \in \mathrm{Mat}(n, \mathbb{K})$*, the map* $t \mapsto \exp(tX)$ *is a one-parameter group.*

(ii) Every one-parameter group is of this form.

Definition 3.15. For the one-parameter group $t \mapsto \exp(tX)$, the matrix X is called its *infinitesimal generator*.

Proof of Theorem 3.14. (i) Since $tXsX = sXtX$, Lemma 3.7(i) gives $\exp(tX)\exp(sX) = \exp((t + s)X)$. Differentiability and $\frac{d}{dt}\exp(tX) = \exp(tX)X$ are standard.

(ii) If $X(t)$ is a one-parameter group, then

$$\frac{d}{dt}X(t) = \lim_{h\to 0} \frac{X(t + h) - X(t)}{h} = \lim_{h\to 0} \frac{X(t)X(h) - X(t)}{h} = X(t) \lim_{h\to 0} \frac{X(h) - \mathbf{1}}{h} = X(t)\dot{X}(0),$$

with $X(0) = \mathbf{1}$. Uniqueness for first-order linear matrix ODEs yields $X(t) = \exp\left(t\dot{X}(0)\right)$. □

3.4 Matrix Lie Groups

Let $G \subset GL(n, \mathbb{K})$ be a closed subgroup of $GL(n, \mathbb{K})$ (closed means: if (g_j) is a sequence in G converging in $GL(n, \mathbb{K})$, then $\lim_{j\to\infty} g_j \in G$). Define

$$\mathrm{Lie}(G) = \{\, X \in \mathrm{Mat}(n, \mathbb{K}) \mid \exp(tX) \in G \text{ for all } t \in \mathbb{R} \,\}.$$

The space $\mathrm{Lie}(G)$ is called the *Lie algebra* of the Lie group G.

Theorem 3.16. *Let* G *be a closed subgroup of* $GL(n, \mathbb{K})$*, where* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*. Then* $\mathrm{Lie}(G)$ *is a real vector space, and for all* $X, Y \in \mathrm{Lie}(G)$ *we have* $[X, Y] = XY - YX \in \mathrm{Lie}(G)$*.*

We will prove this later using the Campbell–Baker–Hausdorff (CBH) formula. Since we are only interested in these groups, we adopt:

Definition 3.17. A (matrix) *Lie group* is a closed subgroup of $GL(n, \mathbb{K})$.

It turns out that closed subgroups of $GL(n, \mathbb{K})$ are Lie groups in the sense of Section 1.2 (see Theorem 3.27 below). The Lie algebra of a Lie group G can then alternatively be defined as the tangent space $T_1 G$ at the identity $1 \in G$:

Lemma 3.18. $\mathrm{Lie}(G)$ *consists of all tangent vectors* $\dot{X}(0) = \frac{d}{dt} X(t)\big|_{t=0}$ *of smooth curves* $X : (-\epsilon, \epsilon) \to G$ *with* $X(0) = \mathbf{1}$ *(for some* $\epsilon > 0$*).*

Proof. One-parameter subgroups in G are smooth curves through 1, so their infinitesimal generators are tangent vectors of the desired form. Conversely, let $X : (-\epsilon, \epsilon) \to G$ be a smooth curve with $X(0) = \mathbf{1}$. Fix $t \in \mathbb{R}$ and consider $X(t/n)^n \in G$. For n large, $X(t/n)$ is defined and near $\mathbf{1}$ (so that log is defined). Then

$$\begin{aligned} X(t/n)^n &= \exp(n \log(X(t/n))) = \exp\Big(n \log\Big(\mathbf{1} + \dot{X}(0)\,\frac{t}{n} + O(\tfrac{1}{n^2})\Big)\Big) \\ &= \exp\Big(n\Big(\dot{X}(0)\,\frac{t}{n} + O(\tfrac{1}{n^2})\Big)\Big) = \exp\big(t\dot{X}(0) + O(\tfrac{1}{n})\big) \xrightarrow[n\to\infty]{} \exp\big(t\dot{X}(0)\big). \end{aligned}$$

Closedness of G implies $\exp(t\dot{X}(0)) \in G$ for all t, hence $\dot{X}(0) \in \mathrm{Lie}(G)$. □

Typical examples are $(S)U(n,m)$, $(S)O(n,m)$, $GL(n,\mathbb{K})$, $SL(n,\mathbb{K})$, $Sp(n)$, etc. Indeed, they can be defined as common zero sets of continuous functions $f : GL(n, \mathbb{K}) \to \mathbb{K}$, hence are closed. For instance, $SL(n, \mathbb{K}) = \{A \mid \det(A) - 1 = 0\}$. A counterexample is $GL(n, \mathbb{Q}) \subset GL(n, \mathbb{R})$, which is not closed.

Define the *commutator* of $X, Y \in \mathrm{Mat}(n, \mathbb{K})$ by

$$[X, Y] = XY - YX.$$

It satisfies:

(i) $[\lambda X + \mu Y, Z] = \lambda[X, Z] + \mu[Y, Z]$ for $\lambda, \mu \in \mathbb{K}$ (bilinearity);

(ii) $[X, Y] = -[Y, X]$ (antisymmetry);

(iii) $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$ (Jacobi identity).

Definition 3.19. A real or complex vector space $\mathfrak{g}$ equipped with a bilinear map (the *Lie bracket*) $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying (i)–(iii) is called a (real or complex) *Lie algebra*. A *homomorphism* of Lie algebras $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is linear and satisfies $[\varphi(X), \varphi(Y)] = \varphi([X, Y])$. Bijective homomorphisms are *isomorphisms*.

Thus Theorem 3.16 asserts that $\mathrm{Lie}(G)$ carries the structure of a real Lie algebra.

Example 3.20. $\mathrm{Lie}(GL(n, \mathbb{K})) = \mathrm{Mat}(n, \mathbb{K})$ viewed as a real vector space, denoted $\mathfrak{gl}(n, \mathbb{K})$. A basis of $\mathfrak{gl}(n, \mathbb{R})$ is given by the matrices E_{ij}, $i, j = 1, \ldots, n$, with entries $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The Lie bracket reads

$$[E_{ij}, E_{kl}] = E_{il}\,\delta_{jk} - E_{jk}\,\delta_{il},$$

and $\dim \mathfrak{gl}(n, \mathbb{R}) = n^2$. As a real Lie algebra, $\mathfrak{gl}(n, \mathbb{C})$ has basis $\{E_{ij}, \sqrt{-1}\,E_{ij}\}_{i,j}$ and $\dim \mathfrak{gl}(n, \mathbb{C}) = 2n^2$.

Example 3.21. $\mathfrak{u}(n) := \mathrm{Lie}(U(n))$, $\quad \mathfrak{su}(n) := \mathrm{Lie}(SU(n))$.

Lemma 3.22.

$$\mathfrak{u}(n) = \{X \in \mathrm{Mat}(n,\mathbb{C}) \mid X^* = -X\}, \qquad \mathfrak{su}(n) = \{X \in \mathrm{Mat}(n,\mathbb{C}) \mid X^* = -X,\ \mathrm{tr}\, X = 0\}.$$

Proof. If $X \in \mathfrak{u}(n)$, then $(\exp(tX))^* \exp(tX) = \mathbf{1}$ for all $t \in \mathbb{R}$. Since $(\exp(tX))^* = \exp(tX^*)$, differentiating at $t = 0$ gives $X + X^* = 0$. If $X \in \mathfrak{su}(n)$, additionally $1 = \det(\exp(tX)) = \exp(t\ \mathrm{tr}\, X)$ for all t, hence $\mathrm{tr}\, X = 0$. Conversely, $X^* = -X$ implies $(\exp(tX))^* \exp(tX) = \exp(-tX)\exp(tX) = \mathbf{1}$ and if $\mathrm{tr}\, X = 0$ then $\det(\exp(tX)) = 1$. □

The real dimensions are $\dim_{\mathbb{R}} \mathfrak{u}(n) = n^2$ and $\dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$.

Example 3.23. $\mathfrak{o}(n) := \mathrm{Lie}(O(n)), \quad \mathfrak{so}(n) := \mathrm{Lie}(SO(n))$.

Lemma 3.24.

$$\mathfrak{o}(n) = \mathfrak{so}(n) = \{X \in \mathrm{Mat}(n,\mathbb{R}) \mid X^T = -X\}.$$

Proof. Analogous to Lemma 3.22. Note that $X^T = -X$ forces zero diagonal, hence $\mathrm{tr}\, X = 0$ automatically. Thus $\dim \mathfrak{so}(n) = \frac{1}{2}(n^2 - n)$. □

Example 3.25. $\mathfrak{su}(2) = \{X \in \mathrm{Mat}(2,\mathbb{C}) \mid X + X^* = 0,\ \mathrm{tr}\, X = 0\}$. A basis is given by the Pauli matrices:

$$t_1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad t_2 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad t_3 = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

with commutation relations

$$[t_j, t_k] = -2\sum_{l=1}^{3} \epsilon_{jkl}\, t_l,$$

where $\epsilon_{123} = 1$ and ϵ_{ijk} is totally antisymmetric.

3.4.1 The Campbell–Baker–Hausdorff Formula

We will prove Theorem 3.16 using CBH.

Theorem 3.26 (CBH)**.** *Let $X, Y \in \mathrm{Mat}(n,\mathbb{K})$. For t sufficiently small,*

$$\exp(tX)\exp(tY) = \exp\Big(tX + tY + \frac{t^2}{2}[X,Y] + O(t^3)\Big).$$

Proof. For small t, $\|\exp(tX)\exp(tY) - \mathbf{1}\| < 1$. By Lemma 3.12, there is $Z(t) = \sum_{n=1}^{\infty} t^n Z_n$ with $\exp(tX)\exp(tY) = \exp Z(t)$. Comparing coefficients:

$$\exp(tX)\exp(tY) = \mathbf{1} + t(X+Y) + \frac{t^2}{2}(X^2 + 2XY + Y^2) + \cdots,$$

$$\exp Z(t) = \mathbf{1} + tZ_1 + \Big(\tfrac{1}{2}Z_1^2 + Z_2\Big)t^2 + \cdots,$$

so $Z_1 = X + Y$ and

$$Z_2 = \tfrac{1}{2}(X^2 + 2XY + Y^2) - \tfrac{1}{2}(X+Y)^2 = \tfrac{1}{2}[X,Y].$$

□

Remark. One can show (full CBH) that for small t,

$$\exp(tX)\exp(tY) = \exp\Big(\sum_{k=1}^{\infty} t^k Z_k\Big),$$

where Z_k is a linear combination of k-fold commutators (i.e. expressions obtained from X, Y by $(k-1)$ applications of $[X,\cdot]$ and $[Y,\cdot]$). For example,

$$Z_3 = \tfrac{1}{12}([X,[X,Y]] + [Y,[Y,X]]), \qquad Z_4 = -\tfrac{1}{24}[X,[Y,[X,Y]]].$$

Thus the Lie algebra determines the group multiplication near the identity.

Proof of Theorem 3.16. (1) $\mathrm{Lie}(G)$ is a real vector space. If $X \in \mathrm{Lie}(G)$, then $\lambda X \in \mathrm{Lie}(G)$ for all $\lambda \in \mathbb{R}$ by definition. If $X, Y \in \mathrm{Lie}(G)$, we show $X + Y \in \mathrm{Lie}(G)$. As $n \to \infty$,

$$\exp\Big(\tfrac{t}{n}X\Big)\exp\Big(\tfrac{t}{n}Y\Big) = \exp\Big(\tfrac{t}{n}(X+Y) + O(\tfrac{1}{n^2})\Big).$$

Hence

$$A_n = \Big[\exp\Big(\tfrac{t}{n}X\Big)\exp\Big(\tfrac{t}{n}Y\Big)\Big]^n = \exp\Big(t(X+Y) + O(\tfrac{1}{n})\Big) \xrightarrow[n\to\infty]{} \exp\big(t(X+Y)\big).$$

Since $A_n \in G$ and G is closed, $\exp\big(t(X+Y)\big) \in G$.

(2) If $X, Y \in \mathrm{Lie}(G)$, then $[X,Y] \in \mathrm{Lie}(G)$. First, show $\exp([X,Y]) \in G$. For large n,

$$\exp(\pm\tfrac{1}{n}X)\exp(\pm\tfrac{1}{n}Y) = \exp\Big(\pm\tfrac{1}{n}(X+Y) + \tfrac{1}{2n^2}[X,Y] + O(\tfrac{1}{n^3})\Big),$$

hence

$$\begin{aligned} B_n &= \Big[\exp(\tfrac{1}{n}X)\exp(\tfrac{1}{n}Y)\exp(-\tfrac{1}{n}X)\exp(-\tfrac{1}{n}Y)\Big]^{n^2} \\ &= \Big[\exp\Big(\tfrac{1}{n}(X+Y) + \tfrac{1}{2n^2}[X,Y] - \tfrac{1}{n}(X+Y) + \tfrac{1}{2n^2}[X,Y] + O(\tfrac{1}{n^3})\Big)\Big]^{n^2} \\ &= \exp\Big([X,Y] + O(\tfrac{1}{n})\Big) \xrightarrow[n\to\infty]{} \exp([X,Y]). \end{aligned}$$

As $B_n \in G$ for all n, closedness gives $\exp([X,Y]) \in G$. Replacing X by tX shows $\exp\big(t[X,Y]\big) \in G$ for all t, hence $[X,Y] \in \mathrm{Lie}(G)$. □

Theorem 3.27. (Without proof) *Let $G \subset GL(n,\mathbb{K})$ be a matrix Lie group with Lie algebra $\mathfrak{g}$ and exponential map $\exp : \mathfrak{g} \to G$. There exist open neighborhoods $0 \in U \subset \mathfrak{g}$ and $\mathbf{1} \in V \subset G$ such that $\exp : U \to G$ is a smooth embedding with $\exp(U) = V$.*

A smooth embedding is an injective smooth map whose differential is injective at every point. For each $g \in G$, the map $\varphi_g : U \to G$, $X \mapsto g\exp(X)$, is a homeomorphism $U \to gV$. The inverse maps φ_g^{-1} serve as charts, endowing G with a manifold structure.

Theorem 3.28. *Let $G \subset GL(n,\mathbb{K})$ be a Lie group with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{K})$. The subgroup generated by finite products*

$$\exp(X_1)\cdots\exp(X_k), \qquad X_1,\dots,X_k \in \mathfrak{g},\ k \geq 1,$$

is precisely the identity component of G.

Proof. Such a product lies on the curve $t \mapsto \exp(tX_1)\cdots\exp(tX_k)$, which passes through $\mathbf{1}$ at $t = 0$, hence is in the identity component. Conversely, let g be in the identity component, and choose U, V as in Theorem 3.27. There is a continuous path $g : [0,1] \to G$ with $g(0) = \mathbf{1}$ and $g(1) = g$. For each $t \in [0,1]$, pick an open interval $U_t \subset [0,1]$ around t so small that $g(U_t) \subset g(t)V$. Compactness yields a finite subcover $U_{t_0} \cup \cdots \cup U_{t_k} \supset [0,1]$. Refining if needed, assume $0 = t_0 < t_1 < \cdots < t_k = 1$ with $t_i \in U_{t_{i+1}}$. Then $g = g(1) \in g(t_{k-1})V$, i.e. $g = g(t_{k-1})\exp(X_k)$ for some $X_k \in \mathfrak{g}$. Similarly, $g(t_i) = g(t_{i-1})\exp(X_i)$ for $i = 1, \ldots, k$, with $g(t_0) = \mathbf{1}$. Hence $g = \exp(X_1)\cdots\exp(X_k)$. □

Example 3.29. $G = U(n)$. Every unitary matrix is unitarily diagonalizable:

$$U = A\,\mathrm{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_n})\,A^{-1}, \quad A \in U(n).$$

Thus $U = \exp(X)$ with $X = A\,\mathrm{diag}(i\varphi_1, \ldots, i\varphi_n)\,A^{-1}$, and $t \mapsto \exp(tX)$ is a one-parameter subgroup in $U(n)$. Therefore $\exp : \mathfrak{u}(n) \to U(n)$ is surjective.

Remark. We have discussed matrix Lie groups, but the results hold for general Lie groups as well. The Lie algebra $\mathfrak{g}$ of a Lie group G is, as a vector space, the tangent space T_1G at the identity. To define the Lie algebra structure, identify T_1G with the Lie algebra of left-invariant vector fields, as follows.

To a vector field X on G associate its directional derivative operator $X : C^\infty(G) \to C^\infty(G)$ (denoted by the same letter). If $X(g) = \dot\gamma(0) \in T_gG$ is the tangent vector of a curve $\gamma(t)$ with $\gamma(0) = g$, then $Xf(g) = \frac{d}{dt}f(\gamma(t))\big|_{t=0}$. The Lie bracket of vector fields X_1, X_2 is defined via their derivations by $[X_1, X_2]f = X_1X_2f - X_2X_1f$, and one verifies that $[X_1, X_2]$ is again a vector field.

For $g \in G$, let $L_g : G \to G$ be left multiplication, $L_g(h) = gh$. A vector field X is *left-invariant* if $(L_g)_*X(h) = X(gh)$ for all $g, h \in G$ (here $(L_g)_* : T_hG \to T_{gh}G$ is the differential). Left-invariant vector fields form a Lie algebra $\mathfrak{g}$; the map $\mathfrak{g} \to T_1G$, $X \mapsto X(1)$ is a linear isomorphism with inverse $a \mapsto (g \mapsto (L_g)_*a)$.

Given a vector field X on a manifold M, its flow $\varphi_t^X : M \to M$ is defined by $\frac{d}{dt}\varphi_t^X(x) = X(\varphi_t^X(x))$ with $\varphi_0^X(x) = x$, when defined. The exponential map $\exp : \mathfrak{g} \to G$ is given by $\exp(tX) = \varphi_t^X(1)$; it is a diffeomorphism from a neighborhood of 0 onto a neighborhood of $1 \in G$ and satisfies CBH.

Every compact Lie group is isomorphic to a matrix Lie group (the isomorphism is both a diffeomorphism and a group homomorphism). In general, any Lie group is at least locally isomorphic to a matrix Lie group: there exists a smooth surjective group homomorphism $G \to G_0 \subset GL(n, \mathbb{K})$ which is a local diffeomorphism.