Chapter 2

Representations of Groups

When a group acts on a vector space by linear maps, one speaks of a *representation*. The fundamental properties of representations are discussed in this chapter.

2.1 Definitions and Examples

Definition 2.1. A (real or complex) representation of a group G on an $\mathbb{R}$- (resp. $\mathbb{C}$-) vector space $V \neq 0$ is a homomorphism

$$\rho : G \to GL(V).$$

The vector space V is called the *representation space* of ρ.

Thus, a representation ρ assigns to each element $g \in G$ an invertible linear map $\rho(g) : V \to V$ such that, for all $g, h \in G$,

$$\rho(gh) = \rho(g)\rho(h).$$

A representation ρ on V is denoted (ρ, V). When there is no ambiguity, one often simply refers to "the representation ρ" or "the representation V". The dimension of (ρ, V) is the dimension of the representation space V.

Example 2.2 (Trivial Representation)**.** $V = \mathbb{C}$ and $\rho(g) = 1$ for all $g \in G$.

Example 2.3. Let $G = S_n$, $V = \mathbb{C}^n$ with basis $e_1, \ldots, e_n$, and define $\rho(g)e_i = e_{g(i)}$ for $g \in S_n$.

Example 2.4. Let $G = O(3)$, $V = C^\infty(\mathbb{R}^3)$, and define $(\rho(g)f)(x) = f(g^{-1}x)$.

Example 2.5. Let $G = \mathbb{Z}_n$, $V = \mathbb{C}$, and $\rho(m) = e^{\frac{2\pi i}{n} m}$, where $GL(\mathbb{C})$ is identified with $\mathbb{C} \setminus \{0\}$.

Lemma 2.6. *Let $\rho : G \to GL(V)$ be a representation. Then $\rho(1) = 1_V$, the identity map $V \to V$, and $\rho(g)^{-1} = \rho(g^{-1})$.*

Proof. We have $\rho(g) = \rho(g1) = \rho(g)\rho(1)$. Since $\rho(g)$ is invertible, $\rho(1)$ must be the identity. Moreover,

$$1_V = \rho(1) = \rho(gg^{-1}) = \rho(g)\rho(g^{-1}),$$

hence $\rho(g)^{-1} = \rho(g^{-1})$. $\square$

Definition 2.7. A *homomorphism of representations* $(\rho_1, V_1) \to (\rho_2, V_2)$ is a linear map $\varphi : V_1 \to V_2$ such that

$$\varphi\rho_1(g) = \rho_2(g)\varphi \quad \forall\, g \in G.$$

Two representations (ρ_1, V_1) and (ρ_2, V_2) are *equivalent* (or isomorphic) if there exists a bijective homomorphism of representations $\varphi : V_1 \to V_2$. The vector space of all homomorphisms $(\rho_1, V_1) \to (\rho_2, V_2)$ is denoted $\mathrm{Hom}_G(V_1, V_2)$.

Definition 2.8. An *invariant subspace* of a representation (ρ, V) is a subspace $W \subset V$ such that $\rho(g)W \subset W$ for all $g \in G$. A representation (ρ, V) is called *irreducible* if it has no invariant subspaces other than V and $\{0\}$; otherwise it is *reducible*. If $W \neq \{0\}$ is invariant, then the restriction $\rho|_W : G \to GL(W)$ defines a *subrepresentation*.

Definition 2.9. A representation (ρ, V) is *completely reducible* if there exist invariant subspaces $V_1, \dots, V_n$ such that

$$V = V_1 \oplus \cdots \oplus V_n$$

and each $(\rho|_{V_i}, V_i)$ is irreducible. Such a decomposition is called a *decomposition into irreducible representations*.

Remark. Not every reducible representation is completely reducible. For example, the representation of $\mathbb{Z}$ on $\mathbb{C}^2$ given by

$$n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

is reducible (with invariant subspace $\mathbb{C}\binom{1}{0}$) but not completely reducible, since the matrices are not diagonalizable for $n \neq 0$.

Lemma 2.10. *Let (ρ, V) be a finite-dimensional representation such that for every invariant subspace $W \subset V$ there exists an invariant subspace W' with $V = W \oplus W'$. Then (ρ, V) is completely reducible.*

Proof. Proceed by induction on $\dim V$. For $\dim V = 1$ there is nothing to prove. Suppose $\dim V = d + 1$. Either V is irreducible, or there exists an invariant subspace W with $1 \leq \dim W \leq d$. Let W' be invariant with $V = W \oplus W'$. Then $1 \leq \dim W' \leq d$. By the induction hypothesis, W and W' are completely reducible, hence so is V. □

2.2 Unitary Representations

Definition 2.11. A representation ρ on an inner-product space V is *unitary* if $\rho(g)$ is unitary for all $g \in G$, i.e.

$$\rho(g)^* = \rho(g)^{-1}.$$

Theorem 2.12. *Unitary representations are completely reducible.*

Proof. Let W be an invariant subspace of a unitary representation (ρ, V) and let

$$W^\perp = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}.$$

Then $W^\perp$ is invariant because for $v \in W^\perp$ and $g \in G$,

$$(\rho(g)v, w) = (v, \rho(g)^* w) = (v, \rho(g)^{-1} w) = 0$$

for all $w \in W$. Hence $\rho(g)v \in W^\perp$. By the previous lemma, (ρ, V) is completely reducible. □

Theorem 2.13. *Let (ρ, V) be a representation of a finite group G. Then there exists an inner product $(\cdot,\cdot)$ on V such that (ρ, V) is unitary.*

Proof. Let $(\cdot,\cdot)_0$ be any inner product on V and define

$$(v, w) = \sum_{g \in G} (\rho(g)v, \rho(g)w)_0.$$

This defines an inner product since it is sesquilinear, Hermitian, and positive definite. Moreover,

$$(\rho(g)v, \rho(g)w) = \sum_{h \in G} (\rho(hg)v, \rho(hg)w)_0 = \sum_{h \in G} (\rho(h)v, \rho(h)w)_0 = (v, w).$$

Hence $\rho(g)$ is unitary for all $g \in G$. $\square$

Corollary 2.14. *Every representation of a finite group is completely reducible.*

Chapter 3

Lie Groups and Lie Algebras

3.1 Lie groups

A *Lie group* is a group that is at the same time a C^∞-manifold such that multiplication and inversion are C^∞-maps. Examples of Lie groups are $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$, $O(p,q)$, $U(p,q)$, $SO(p,q)$, $SU(p,q)$, $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$. We view $G \subset \mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$ as a subset of $\mathbb{K}^{n^2}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) by writing the matrix entries of a matrix $A \in G$ as a point $(A_{11}, A_{12}, \ldots, A_{nn})$ in $\mathbb{K}^{n^2}$. This identification defines the structure of a metric space on G. The distance $d(A,B)$ between two matrices in G is

$$d(A,B)^2 = \sum_{i,j=1}^{n} |A_{ij} - B_{ij}|^2 = \mathrm{tr}\,(A-B)^*(A-B).$$

Theorem 3.1. *Let G be a subgroup of $\mathrm{GL}(n,\mathbb{K})$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then multiplication $G \times G \to G$, $(A,B) \mapsto AB$ and inversion $G \to G$, $A \mapsto A^{-1}$ are continuous maps.*

Proof. The matrix entries of AB are polynomials in the matrix entries of A and B and thus continuous. The matrix entries of A^{-1} are, by Cramer's rule, rational functions of the matrix entries of A. The denominator is the polynomial $\det A$, which never vanishes in $\mathrm{GL}(n,\mathbb{K})$. Hence $A \mapsto A^{-1}$ is continuous. □

Continuity properties are important in the theory of Lie groups. Let $G \subset \mathrm{GL}(n,\mathbb{K})$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A *path* in G is a continuous map $w : [0,1] \to G$. Two matrices A_1, A_2 in G are said to be *path-connected* ($A_1 \sim A_2$) if there exists a path w in G with $w(0) = A_1$ and $w(1) = A_2$. Every $A \in G$ is connected to itself (with the path $w(t) = A$, $t \in [0,1]$). Moreover, from $A_1 \sim A_2$ via a path w, it follows that $A_2 \sim A_1$ via the path $\bar{w}(t) = w(1-t)$. If A_1 is connected to A_2 by a path w_1 and A_2 to A_3 by a path w_2, then the concatenated path

$$(w_1 * w_2)(t) = \begin{cases} w_1(2t), & 0 \le t \le \frac{1}{2}, \\ w_2(2t-1), & \frac{1}{2} \le t \le 1, \end{cases}$$

connects A_1 to A_3. Thus $\sim$ is an equivalence relation.

Definition 3.2. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The (path) *connected components* of $G \subset \mathrm{GL}(n,\mathbb{K})$ are the equivalence classes with respect to $\sim$. If G consists of a single connected component, then G is said to be (path) connected.

Theorem 3.3. *Let $G \subset \mathrm{GL}(n,\mathbb{K})$ be a subgroup, and let G^0 be the connected component containing* 1*. Then G^0 is a normal subgroup of G and $G/G^0 = \{$connected components of $G\}$.*

Proof. Exercise. □

As an example we examine the connectedness properties of unitary and orthogonal groups.

Theorem 3.4. *1.* $\mathrm{SO}(n)$*,* $\mathrm{SU}(n)$*, and* $U(n)$ *are connected.*

2. $O(n)$ *consists of two connected components:* $\{A \in O(n) \mid \det A = 1\}$ *and* $\{A \in O(n) \mid \det A = -1\}$.

Proof. Unitary case: the normal form theorem for unitary matrices from linear algebra states that for every matrix $A \in U(n)$ there exists a matrix $U \in U(n)$ with $A = UDU^{-1}$, where $D = \mathrm{diag}(e^{i\varphi_1}, \dots, e^{i\varphi_n})$, $\varphi_j \in \mathbb{R}$. The map $w : [0,1] \to U(n)$, $t \mapsto U\mathrm{diag}(e^{it\varphi_1}, \dots, e^{it\varphi_n})U^{-1}$ is continuous, since multiplication by U and U^{-1} is continuous (by the previous theorem) and $t \mapsto e^{it\varphi}$ is continuous. The path w connects 1 to the arbitrary matrix $A \in U(n)$. If A has determinant 1, then $\det D = 1$ and $\sum_i \varphi_i \in 2\pi\mathbb{Z}$; without loss of generality $\sum_i \varphi_i = 0$. It follows that $\det(w(t)) = 1$, and w is a path in $\mathrm{SU}(n)$. Hence $U(n)$ and $\mathrm{SU}(n)$ are connected.

$SO(n)$ and $O(n)$, exercise.

□

3.2 Exponential Map

Let $\mathrm{Mat}(n, \mathbb{K})$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be the vector space of all $n \times n$ matrices with entries in $\mathbb{K}$. Identifying $\mathrm{Mat}(n, \mathbb{K})$ with $\mathbb{K}^{n^2}$ via $X = (x_{ij}) \mapsto (x_{11}, x_{12}, \dots, x_{nn})$, we endow it with the norm

$$\|X\| = \Big(\sum_{i,j=1}^{n} |x_{ij}|^2 \Big)^{1/2} = \big(\mathrm{tr}(X^* X) \big)^{1/2}.$$

Lemma 3.5. *For all* $X, Y \in \mathrm{Mat}(n, \mathbb{K})$,

$$\|XY\| \le \|X\| \, \|Y\|.$$

Proof.

$$\|XY\|^2 = \sum_{i,j=1}^{n} \Big| \sum_{k=1}^{n} X_{ik} Y_{kj} \Big|^2 \le \sum_{i,j=1}^{n} \Big(\sum_{l=1}^{n} |\,\overline{X}_{il}\,|^2 \Big) \Big(\sum_{m=1}^{n} |Y_{mj}|^2 \Big) = \|X\|^2 \|Y\|^2,$$

where we used the Cauchy–Schwarz inequality with $u_k = \overline{X}_{ik}$ and $v_k = Y_{kj}$ for fixed i, j. □

Lemma 3.6. *The series*

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

converges absolutely for every $X \in \mathrm{Mat}(n, \mathbb{K})$.

Proof. By Lemma 3.5, inductively $\|X^k\| \le \|X\|^k$. Hence

$$\Big\| \tfrac{1}{k!} X^k \Big\| \le \frac{\|X\|^k}{k!} \quad \text{and} \quad \sum_{k=0}^{\infty} \Big\| \tfrac{1}{k!} X^k \Big\| \le e^{\|X\|} < \infty.$$

□

It follows that the matrix entries of $\exp(X)$ are absolutely convergent power series in the entries x_{ij} of X, hence analytic in the x_{ij}.

Lemma 3.7. *Let* $X, Y \in \mathrm{Mat}(n, \mathbb{K})$, $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. *Then:*

(i) If $XY = YX$*, then* $\exp(X)\exp(Y) = \exp(X+Y)$.

(ii) $\exp(X)$ *is invertible and* $\exp(X)^{-1} = \exp(-X)$.

(iii) For $A \in GL(n,\mathbb{K})$*, one has* $A\exp(X)\,A^{-1} = \exp(AXA^{-1})$.

(iv) $\det(\exp(X)) = \exp(\operatorname{tr} X)$.

(v) $\exp(X^*) = (\exp X)^*$ *and* $\exp(X^T) = (\exp X)^T$.

Proof. (i) Using absolute convergence,

$$\exp(X)\exp(Y) = \sum_{j=0}^{\infty}\frac{1}{j!}X^j\sum_{k=0}^{\infty}\frac{1}{k!}Y^k = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}X^jY^{n-j} = \sum_{n=0}^{\infty}\frac{(X+Y)^n}{n!} = \exp(X+Y),$$

where commutativity ensures $(X+Y)^n = \sum_j \binom{n}{j}X^jY^{n-j}$.

(ii) Apply (i) with $Y = -X$.

(iii) Conjugation moves inside the series:

$$A\Big(\sum_{j=0}^{\infty}\frac{X^j}{j!}\Big)A^{-1} = \sum_{j=0}^{\infty}\frac{AX^jA^{-1}}{j!} = \sum_{j=0}^{\infty}\frac{(AXA^{-1})^j}{j!}.$$

(iv) Put $X = AYA^{-1}$ with Y upper triangular (Schur decomposition over $\mathbb{C}$). If $y_1,\dots,y_n$ are the diagonal entries of Y, then Y^k is upper triangular with diagonal entries $y_1^k,\dots,y_n^k$. Hence $\exp(Y)$ is upper triangular with diagonal $e^{y_1},\dots,e^{y_n}$, so

$$\det(\exp Y) = e^{y_1}\cdots e^{y_n} = e^{y_1+\cdots+y_n} = e^{\operatorname{tr} Y}.$$

Use (iii) and invariance of determinant and trace under conjugation.

(v) $(X^k)^* = (X^*)^k$ and likewise for T; continuity of $X \mapsto X^*$ (resp. T) yields the claim. $\square$

Definition 3.8. The map $\exp : \mathrm{Mat}(n,\mathbb{K}) \to GL(n,\mathbb{K})$, $X \mapsto \exp(X)$, is called the *exponential map*.

Example 3.9.

$$\exp\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix} = \mathbf{1} + \begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}.$$

More generally, if N is nilpotent then

$$\exp(N) = \mathbf{1} + N + \frac{N^2}{2!} + \cdots + \frac{N^k}{k!}$$

is a polynomial in N (for k large enough so that $N^{k+1} = 0$).

Example 3.10. Let $X = \left(\begin{smallmatrix}0 & -\varphi\\ \varphi & 0\end{smallmatrix}\right)$. Since $X^2 = \varphi^2(-\mathbf{1})$, one finds

$$\exp(X) = \cos\varphi\,\mathbf{1} + \sin\varphi\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix} \in SO(2).$$

Example 3.11. Let $X = i\sum_{j=1}^{3} n_j\sigma_j$ with $|n| = 1$ (Pauli matrices). Then $X^2 = -\mathbf{1}$ and

$$\exp(\vartheta X) = \cos\vartheta\,\mathbf{1} + \sin\vartheta\,X.$$

Lemma 3.12. *The map*

$$\exp : \mathrm{Mat}(n,\mathbb{K}) \to GL(n,\mathbb{K})$$

is invertible in a neighborhood of 0. *That is, there exists a neighborhood* U *of* 0 *such that* $\exp : U \to \exp(U)$ *is bijective, and the inverse is given by an absolutely convergent power series.*

Proof. The series

$$\log X = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(X - \mathbf{1})^n}{n}$$

converges absolutely for $\|X - \mathbf{1}\| < 1$, since

$$\sum_{n=1}^{\infty} \left\| (-1)^{n+1} \frac{(X-\mathbf{1})^n}{n} \right\| \le \sum_{n=1}^{\infty} \frac{\|X - \mathbf{1}\|^n}{n} = -\log\left(1 - \|X - \mathbf{1}\|\right) < \infty.$$

Then $\exp(\log X) = X$ for $\|X - \mathbf{1}\| < 1$, and $\log(\exp X) = X$ whenever $\|\exp X - \mathbf{1}\| < 1$. □

3.3 One-Parameter Groups

Definition 3.13. A map $X : \mathbb{R} \to GL(n,\mathbb{K})$, $t \mapsto X(t)$, is called a *one-parameter group* if it is continuously differentiable, $X(0) = \mathbf{1}$, and

$$X(s+t) = X(s)X(t) \quad \text{for all } s,t \in \mathbb{R}.$$

The image of such a map is a subgroup, with $X(t)^{-1} = X(-t)$.

Theorem 3.14. *(i) For any* $X \in \mathrm{Mat}(n,\mathbb{K})$*, the map* $t \mapsto \exp(tX)$ *is a one-parameter group.*

(ii) Every one-parameter group is of this form.

Definition 3.15. For the one-parameter group $t \mapsto \exp(tX)$, the matrix X is called its *infinitesimal generator.*

Proof of Theorem 3.14. (i) Since $tXsX = sXtX$, Lemma 3.7(i) gives $\exp(tX)\exp(sX) = \exp((t+s)X)$. Differentiability and $\frac{d}{dt}\exp(tX) = \exp(tX)X$ are standard.

(ii) If $X(t)$ is a one-parameter group, then

$$\frac{d}{dt}X(t) = \lim_{h\to 0} \frac{X(t+h) - X(t)}{h} = \lim_{h\to 0} \frac{X(t)X(h) - X(t)}{h} = X(t) \lim_{h\to 0} \frac{X(h) - \mathbf{1}}{h} = X(t)\dot{X}(0),$$

with $X(0) = \mathbf{1}$. Uniqueness for first-order linear matrix ODEs yields $X(t) = \exp\left(t\dot{X}(0)\right)$. □

3.4 Matrix Lie Groups

Let $G \subset GL(n,\mathbb{K})$ be a closed subgroup of $GL(n,\mathbb{K})$ (closed means: if (g_j) is a sequence in G converging in $GL(n,\mathbb{K})$, then $\lim_{j\to\infty} g_j \in G$). Define

$$\mathrm{Lie}(G) = \{\, X \in \mathrm{Mat}(n,\mathbb{K}) \mid \exp(tX) \in G \text{ for all } t \in \mathbb{R} \,\}.$$

The space $\mathrm{Lie}(G)$ is called the *Lie algebra* of the Lie group G.

Theorem 3.16. *Let* G *be a closed subgroup of* $GL(n,\mathbb{K})$*, where* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*. Then* $\mathrm{Lie}(G)$ *is a real vector space, and for all* $X, Y \in \mathrm{Lie}(G)$ *we have* $[X,Y] = XY - YX \in \mathrm{Lie}(G)$.

We will prove this later using the Campbell–Baker–Hausdorff (CBH) formula. Since we are only interested in these groups, we adopt:

Definition 3.17. A (matrix) *Lie group* is a closed subgroup of $GL(n, \mathbb{K})$.

It turns out that closed subgroups of $GL(n, \mathbb{K})$ are Lie groups in the sense of Section 1.2 (see Theorem 3.28 below). The Lie algebra of a Lie group G can then alternatively be defined as the tangent space T_1G at the identity $1 \in G$:

Lemma 3.18. $\mathrm{Lie}(G)$ *consists of all tangent vectors* $\dot{X}(0) = \frac{d}{dt}X(t)\big|_{t=0}$ *of smooth curves* $X : (-\epsilon, \epsilon) \to G$ *with* $X(0) = \mathbf{1}$ *(for some* $\epsilon > 0$*).*

Proof. One-parameter subgroups in G are smooth curves through 1, so their infinitesimal generators are tangent vectors of the desired form. Conversely, let $X : (-\epsilon, \epsilon) \to G$ be a smooth curve with $X(0) = \mathbf{1}$. Fix $t \in \mathbb{R}$ and consider $X(t/n)^n \in G$. For n large, $X(t/n)$ is defined and near $\mathbf{1}$ (so that log is defined). Then

$$\begin{aligned} X(t/n)^n &= \exp(n \log(X(t/n))) = \exp\Big(n \log\Big(\mathbf{1} + \dot{X}(0)\,\frac{t}{n} + O(\tfrac{1}{n^2})\Big)\Big) \\ &= \exp\Big(n\Big(\dot{X}(0)\,\frac{t}{n} + O(\tfrac{1}{n^2})\Big)\Big) = \exp\big(t\dot{X}(0) + O(\tfrac{1}{n})\big) \xrightarrow[n\to\infty]{} \exp\big(t\dot{X}(0)\big). \end{aligned}$$

Closedness of G implies $\exp\big(t\dot{X}(0)\big) \in G$ for all t, hence $\dot{X}(0) \in \mathrm{Lie}(G)$. □

Typical examples are $(S)U(n, m)$, $(S)O(n, m)$, $GL(n, \mathbb{K})$, $SL(n, \mathbb{K})$, $Sp(n)$, etc. Indeed, they can be defined as common zero sets of continuous functions $f : GL(n, \mathbb{K}) \to \mathbb{K}$, hence are closed. For instance, $SL(n, \mathbb{K}) = \{A \mid \det(A) - 1 = 0\}$. A counterexample is $GL(n, \mathbb{Q}) \subset GL(n, \mathbb{R})$, which is not closed.

Define the *commutator* of $X, Y \in \mathrm{Mat}(n, \mathbb{K})$ by

$$[X, Y] = XY - YX.$$

It satisfies:

(i) $[\lambda X + \mu Y, Z] = \lambda[X, Z] + \mu[Y, Z]$ for $\lambda, \mu \in \mathbb{K}$ (bilinearity);

(ii) $[X, Y] = -[Y, X]$ (antisymmetry);

(iii) $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$ (Jacobi identity).

Definition 3.19. A real or complex vector space $\mathfrak{g}$ equipped with a bilinear map (the *Lie bracket*) $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying (i)–(iii) is called a (real or complex) *Lie algebra*. A *homomorphism* of Lie algebras $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is linear and satisfies $[\varphi(X), \varphi(Y)] = \varphi([X, Y])$. Bijective homomorphisms are *isomorphisms*.

Thus Theorem 3.16 asserts that $\mathrm{Lie}(G)$ carries the structure of a real Lie algebra.

Example 3.20. $\mathrm{Lie}(GL(n, \mathbb{K})) = \mathrm{Mat}(n, \mathbb{K})$ viewed as a real vector space, denoted $\mathfrak{gl}(n, \mathbb{K})$. A basis of $\mathfrak{gl}(n, \mathbb{R})$ is given by the matrices E_{ij}, $i, j = 1, \ldots, n$, with entries $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The Lie bracket reads

$$[E_{ij}, E_{kl}] = E_{il}\,\delta_{jk} - E_{jk}\,\delta_{il},$$

and $\dim \mathfrak{gl}(n, \mathbb{R}) = n^2$. As a real Lie algebra, $\mathfrak{gl}(n, \mathbb{C})$ has basis $\{E_{ij}, \sqrt{-1}\,E_{ij}\}_{i,j}$ and $\dim \mathfrak{gl}(n, \mathbb{C}) = 2n^2$.

Example 3.21. $\mathfrak{u}(n) := \mathrm{Lie}(U(n))$, $\quad \mathfrak{su}(n) := \mathrm{Lie}(SU(n))$.