

Mathematical Supplement for Quantum Field Theory

Lecture Notes

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Chapter 1

Groups

An important starting point for studying Quantum Field theory are familiarity with the notions of representation theory of the Poincaré group. We will therefore start by recalling some key concepts from group and representation theory. We briefly list the definitions and basic concepts of group theory. A special role for us is played by continuous groups, or Lie groups, where we restrict ourselves to matrix Lie groups. These notes are based on a translation of the notes of the course *Mathematische Methoden der Physik II* ©2018 by Giovanni Felder. Further reference include [1, 2, 3, 4].

1.1 Basic concepts and examples

A *group* G is a set with a product (multiplication) $G \times G \rightarrow G$, $(g, h) \mapsto gh$, such that

1. $(gh)k = g(hk)$ for all $g, h, k \in G$,
2. there exists a neutral element (identity element) $1 \in G$ with $1g = g1 = g$ for all $g \in G$,
3. for every $g \in G$ there exists an inverse $g^{-1} \in G$ with $gg^{-1} = g^{-1}g = 1$.

The identity element is unique, because from $1_01 = 1$ and $1_01 = 1_0$ it follows that $1 = 1_0$. The inverse of an element is also unique. Indeed, if g', g'' are both inverses of g , then $g' = g'1 = g'(gg'') = (g'g)g'' = 1g'' = g''$. The inverse of 1 is 1.

The *order* of a group G is the number of elements $|G|$ of G . It may be infinite.

A group is called *abelian* if $gh = hg$ for all $g, h \in G$. In this case one often writes G additively: instead of gh write $g + h$, instead of 1 write 0, and instead of g^{-1} write $-g$. One also writes $g - h$ instead of $g + (-h)$. Familiar examples are the groups $\mathbb{Z}, \mathbb{R}, \mathbb{C}$.

A *subgroup* H of a group G is a nonempty subset of G such that $h_1, h_2 \in H \Rightarrow h_1h_2 \in H$ and $h \in H \Rightarrow h^{-1} \in H$. A subgroup of a group is itself a group.

The *direct product* $G_1 \times G_2$ of two groups G_1, G_2 is the Cartesian product with multiplication $(g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2)$. It is a group with identity $(1, 1)$ and inverse $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1})$.

Example 1.1 (Cyclic group of order n). $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ (with $+$) is an abelian group with n elements.

Example 1.2 (Symmetric group). A permutation of n elements is a bijection $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$. The permutations of n elements form a group with composition of maps as product: $(\pi_1\pi_2)(i) = (\pi_1 \circ \pi_2)(i) = \pi_1(\pi_2(i))$. The identity element is the identity map, and the inverse of π is the inverse map π^{-1} . This group is called the symmetric group S_n . It has order $n!$.

Example 1.3 (General linear groups).

$$\begin{aligned}\mathrm{GL}(n, \mathbb{R}) &= \{\text{invertible real } n \times n \text{ matrices}\}, \\ \mathrm{GL}(n, \mathbb{C}) &= \{\text{invertible complex } n \times n \text{ matrices}\}.\end{aligned}$$

The product is matrix multiplication, and the identity element is the identity matrix. More abstractly, for any vector space V (over $\mathbb{R}$ or $\mathbb{C}$) one defines

$$\mathrm{GL}(V) = \{\text{invertible linear maps } V \rightarrow V\},$$

where the product is composition of maps. If $\dim V < \infty$, then by choosing a basis V is identified with $\mathbb{R}^n$ or $\mathbb{C}^n$, and $\mathrm{GL}(V)$ is then $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$, respectively.

Example 1.4 (Orthogonal group $O(n)$). An $n \times n$ matrix A is called orthogonal if $A^T A = 1$. Every orthogonal matrix is invertible, since

$$1 = \det(A^T A) = \det(A^T) \det(A) = (\det A)^2,$$

hence $\det A \neq 0$ and therefore $A^{-1} = A^T$. If A, B are orthogonal, then $(AB)^T(AB) = B^T A^T AB = 1$, so AB is orthogonal. If A is orthogonal, then $(A^T)^T A^T = AA^T = 1$, hence $A^{-1} = A^T$ is orthogonal. Thus the orthogonal $n \times n$ matrices form a subgroup $O(n)$ of $\mathrm{GL}(n, \mathbb{R})$,

$$O(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = 1\}.$$

Of particular importance in physics is the group $O(3)$ of orthogonal transformations of physical space $\mathbb{R}^3$.

Example 1.5. $O(n)$ is the group of linear maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$ that leave the scalar product

$$(x, y) = \sum_{i=1}^n x_i y_i$$

invariant: $O(n) = \{A \mid (Ax, Ay) = (x, y) \forall x, y \in \mathbb{R}^n\}$. More generally, consider the symmetric bilinear form on $\mathbb{R}^{p+q}$

$$(x, y)_{p,q} = \sum_{i=1}^p x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i,$$

and define

$$O(p, q) = \{A \in \mathrm{GL}(p+q, \mathbb{R}) \mid (Ax, Ay)_{p,q} = (x, y)_{p,q}\}.$$

In particular, $O(n, 0) = O(n) = O(0, n)$. The group $O(1, 3)$ is the *Lorentz group*. It preserves the *Minkowski metric*

$$(x, y)_{1,3} = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

on spacetime $\mathbb{R}^4$.

Example 1.6 (Unitary group $U(n)$).

$$\begin{aligned}U(n) &= \{A \in \mathrm{GL}(n, \mathbb{C}) \mid A^* A = 1\} \\ &= \{A \in \mathrm{GL}(n, \mathbb{C}) \mid (Az, Aw) = (z, w), \forall z, w \in \mathbb{C}^n\},\end{aligned}$$

where $(z, w) = \sum \bar{z}_i w_i$ is the scalar product on $\mathbb{C}^n$.

Example 1.7 (Subgroups of special linear groups). Let G be a subgroup of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$. Define

$$\mathrm{SG} = \{A \in G \mid \det A = 1\}.$$

Then SG is nonempty (since $1 \in \mathrm{SG}$), and if $A, B \in \mathrm{SG}$, then $\det(AB) = \det A \cdot \det B = 1$ and $\det(A^{-1}) = (\det A)^{-1} = 1$. Thus SG is a group. Examples ($K = \mathbb{R}$ or $\mathbb{C}$):

$$\mathrm{SL}(n, K) = \mathrm{SGL}(n, K) = \{A \in \mathrm{GL}(n, K) \mid \det A = 1\},$$

$$\mathrm{SO}(n) = \{A \in \mathrm{SL}(n, \mathbb{R}) \mid A^T A = 1\},$$

$$\mathrm{SU}(n) = \{A \in \mathrm{SL}(n, \mathbb{C}) \mid A^* A = 1\}.$$

These are called the special linear, special orthogonal, and special unitary groups, respectively.

We now introduce some further important concepts. A group G *acts* on a set M if there is a map $G \times M \rightarrow M$, $(g, x) \mapsto gx$, satisfying $g_1(g_2x) = (g_1g_2)x$ for all $g_1, g_2 \in G$, $x \in M$.

Every group acts on itself by group multiplication; $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by applying matrices to vectors; $O(n)$ acts on $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$ since for $A \in O(n)$ we have $|Ax| = |x|$.

A *homomorphism* $\varphi : G \rightarrow H$ of a group G to a group H is a map with $\varphi(gh) = \varphi(g)\varphi(h)$. If φ is also bijective, then it is called an *isomorphism*. If $\varphi : G \rightarrow H$, $\psi : H \rightarrow K$ are homomorphisms, then $\psi \circ \varphi : G \rightarrow K$ is a homomorphism. The inverse map of an isomorphism is also an isomorphism, since

$$\varphi^{-1}(gh) = \varphi^{-1}(\varphi(\varphi^{-1}(g))\varphi(\varphi^{-1}(h))) = \varphi^{-1}\varphi(\varphi^{-1}(g)\varphi^{-1}(h)) = \varphi^{-1}(g)\varphi^{-1}(h).$$

Isomorphisms $G \rightarrow G$ thus form a group $\mathrm{Aut}(G)$, the group of automorphisms of G . The kernel of φ is the set

$$\ker \varphi = \{g \in G \mid \varphi(g) = 1\} \subseteq G,$$

and the image of φ is the set

$$\mathrm{Im} \varphi = \{\varphi(g) \mid g \in G\} \subseteq H.$$

Theorem 1.8. *Let $\varphi : G \rightarrow H$ be a homomorphism.*

1. $\varphi(1) = 1$, and $\varphi(g)^{-1} = \varphi(g^{-1})$.
2. $\ker \varphi$ is a subgroup of G , and $\mathrm{Im} \varphi$ is a subgroup of H .
3. φ is injective if and only if $\ker \varphi = \{1\}$.

Proof. Exercise □

Corollary 1.9. *A homomorphism $\varphi : G \rightarrow H$ is an isomorphism if and only if $\ker \varphi = \{1\}$ and $\mathrm{Im} \varphi = H$.*

Definition 1.10. Two groups G, H are called *isomorphic* ($G \simeq H$) if there exists an isomorphism $G \rightarrow H$.

Definition 1.11. Let H be a subgroup of a group G . The set G/H of (left) cosets of H in G is the set of equivalence classes with respect to the equivalence relation

$$g_1 \sim g_2 \iff \exists h \in H \text{ with } g_2 = g_1 h.$$

Definition 1.12. A *normal subgroup* of G is a subgroup H with the property that $ghg^{-1} \in H$ for all $g \in G$, $h \in H$.

For general subgroups H , G/H has no natural group structure.

Example 1.13. $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ (every subgroup of an abelian group is normal).

Example 1.14. Let $\mathbb{I}$ denote the identity matrix. The subset $\{\pm\mathbb{I}\} \subset \mathrm{SL}(2, \mathbb{C})$ is a subgroup. For all $A \in \mathrm{SL}(2, \mathbb{C})$ we have $A(\pm\mathbb{I})A^{-1} = \pm\mathbb{I}$, thus $\{\pm\mathbb{I}\}$ is normal. The group $\mathrm{SL}(2, \mathbb{C})/\{\pm\mathbb{I}\}$ is isomorphic to the group of Möbius transformations of the Riemann sphere. The isomorphism maps the class $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to the Möbius transformation $z \mapsto \frac{az + b}{cz + d}$.

For any homomorphism $\varphi : G \rightarrow H$, $\ker \varphi$ is a normal subgroup of G , because from $\varphi(h) = 1$ it follows that $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g)\varphi(g)^{-1} = 1$. We have the isomorphism theorem:

Theorem 1.15 (Isomorphism Theorem). *Let $\varphi : G \rightarrow H$ be a homomorphism of groups. Then*

$$G/\ker \varphi \simeq \mathrm{Im} \varphi.$$

The isomorphism is $[g] \mapsto \varphi(g)$ for any choice of representative g .

The notion of the *semidirect product* of groups is first motivated by two examples.

Example 1.16 (Group of motions of $\mathbb{R}^3$). This is the group of affine transformations of $\mathbb{R}^3$ of the form

$$x \mapsto Ax + b, \quad A \in O(3), \quad b \in \mathbb{R}^3.$$

As a set this group is $O(3) \times \mathbb{R}^3$. The product is composition of maps:

$$(A_1, b_1)(A_2, b_2) = (A_1A_2, A_1b_2 + b_1), \quad (A, b)^{-1} = (A^{-1}, -A^{-1}b), \quad e = (1, 0).$$

This group has $O(3) \cong O(3) \times \{0\}$ as a subgroup and $\mathbb{R}^3 \cong \{1\} \times \mathbb{R}^3$ as a normal subgroup. It is called the inhomogeneous orthogonal group $IO(3)$.

Example 1.17 (Poincaré group). The Poincaré group $IO(1, 3)$ is the group of affine transformations of $\mathbb{R}^4$ of the form

$$x \mapsto Ax + b, \quad A \in O(1, 3), \quad b \in \mathbb{R}^4.$$

As above, this group is $O(1, 3) \times \mathbb{R}^4$ with multiplication $(A_1, b_1)(A_2, b_2) = (A_1A_2, A_1b_2 + b_1)$. It arises in special relativity as the group of transformations that preserve the form of the wave equation: a function u satisfies the wave equation $u_{x^0x^0} - u_{x^1x^1} - u_{x^2x^2} - u_{x^3x^3} = 0$ if and only if $U(x) = u(Ax + b)$ satisfies the wave equation.

The general formulation of this notion is as follows: let $\mathrm{Aut}(H)$ be the group of isomorphisms $H \rightarrow H$ ("automorphisms of H "), where the multiplication is composition of isomorphisms.

Theorem 1.18. *Let G and H be groups and let $\rho : G \rightarrow \mathrm{Aut}(H)$, $g \mapsto \rho_g$, be a homomorphism. Then $G \times H$ with multiplication*

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1 \rho_{g_1}(h_2))$$

is a group, the semidirect product $G \ltimes_\rho H$.

Proof. Exercise □

1.2 Lie groups

A *Lie group* is a group that is at the same time a C^∞ -manifold such that multiplication and inversion are C^∞ -maps. Examples of Lie groups are $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$, $O(p, q)$, $U(p, q)$, $SO(p, q)$, $SU(p, q)$, $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{C})$.

Bibliography

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