SCATTERING AMPLITUDES IN QUANTUM FIELD THEORY

An introduction to modern methods in perturbative Quantum Field Theory

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Preamble

These notes have grown out of a series of lectures that I had the opportunity to hold for the Technical University of Munich, starting in the Winter Semester 2021/2022. The material has been approximately covered in two semesters, with two hours of lectures per week. I claim no originality for most of the material presented here, while I have put some effort in either improving some derivation, or filling the gaps where something conceptually important was given for granted. All in all, the main effort was put in organizing this material in what I believe would be a logical way of presenting some of the important developments in this fascinating field. Moreover, I tried to include, even if at an elementary level, most developments which are relevant for research today, starting from tree-level on-shell methods, and getting to the theory of iterated integrals and special functions, including a simple introduction to elliptic polylogarithms.

As it is often the case, the original version of these notes was only hand-written, and I am extremely thankful to Sara Ditsch, Julian Piribauer and Davide Maria Tagliabue for having invested substantial time and effort to go through the material, fix obvious typos, improve the presentation, and for having put together the first Latex version of this manuscript.

While I will refrain from citing most of the standard books and review articles in the main text, I would like to provide here a list of the excellent sources on the theory of scattering amplitudes which I actively used in preparing these lectures:

- *Matthew Robinson*, Symmetry and the Standard Model
- *Steven Weinberg*, The Quantum Theory of Fields, Volume 1
- *Matthew Schwartz*, Quantum Field Theory and the Standard Model
- *Henriette Elvang and Yu-Tin Huang*, Scattering Amplitudes in Gauge Theory and Gravity
- *Johannes Henn and Jan Plefka*, Scattering Amplitudes in Gauge Theories
- *Lance Dixon*, Calculating Scattering Amplitudes Efficiently
- *Stephen Parke and Michelangelo Mangano*, Multi-Parton Amplitudes in Gauge Theories
- *Michael Peskin*, Simplifying Muli-Jet QCD Computation
- *Kirill Melnikov*, Modern methods for perturbative computations in QFT
- *Keith Ellis, Zoltan Kunszt, Kirill Melnikov, Giulia Zanderighi*, One-loop calculations in quantum field theory: from Feynman diagrams to unitarity cuts

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1 Introduction

Scattering Amplitudes play a fundamental role in Quantum Field Theory (QFT) for many reasons. First of all, they provide us with one of the most useful ways to extract phenomenological predictions for particle scattering from the complex formalism of QFT. In fact, the cross-section for a given process σ can be loosely computed by integrating the scattering amplitude squared $|A|^2$ on the phase-space of the produced particles $d\phi$

$$\sigma \propto \int |A|^2 d\phi \,.$$

Moreover, scattering amplitudes turn out to be extremely fascinating mathematical quantities, which in particular exhibit unexpected symmetries and an apparently enhanced simplicity, compared to what one would naively expect from the Lagrangian formulation of the corresponding QFT. There are many reasons for this, the most important ones possibly being that scattering amplitudes are *on-shell*, *gauge invariant* objects, which do not suffer from the ambiguities inherent to the off-shell expansion in Feynman diagrams. In fact, in traditional QFT lectures you learn to compute scattering amplitude order by order in perturbation theory, using Feynman diagrams with increasingly large numbers of loops to enumerate all relevant contributions. The expansion in Feynman diagrams has the advantage of making locality of interactions manifest, through the concatenation of local interaction vertices and off-shell propagators to build the corresponding scattering probabilities. While locality is an extremely useful property to have, we have to pay a price: as discussed above, Feynman diagrams are off-shell quantities and, individually, they do not preserve the symmetries of the underlying theory (for example Ward identities are violated when evaluated for the individual diagrams). Moreover, out of mere combinatorical arguments, it is easy to see that the number of Feynman diagrams grows factorially with the number of loops and/or particles involved in the scattering process. On top of that, for every order in perturbation theory, we typically need to compute an additional loop integral: not only is computing integrals extremely complicated, but also these are not the nicest integrals you could think of, they are divergent, both in the Ultra-Violet (UV) and in the Infra-Red (IR), which makes it necessary to introduce a regularization to make sense of them.

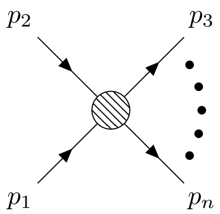


Figure 1: Feynman Diagram for the scattering of n particles with momenta p_i.

Considering all of this, it is then even more astonishing to realize that summing together hundreds or thousands of extremely complicated Feynman diagrams, the final result for the scattering amplitude often turns out to be much simpler than the individual ingredients required to obtain it. The purpose of these lectures will be to explore the underlying mathematics of the scattering amplitude and to develop techniques that allow us to calculate amplitudes trying to make their symmetries as manifest as possible, at each step of the calculation. We cannot claim that this problem is solved in full generality, on the contrary we are quite far from it. Nevertheless, the past three decades have witnessed an impressive jump forward in our understanding of Scattering Amplitudes. Interestingly, this program is largely building upon ideas that had been developed in the different context of the (non-perturbative) analytic S-matrix program, which received much attention in the 60s.

The first part of this course will focus on computation methods for tree-level amplitudes. First, we will review some well-known properties of the Lorentz group, focusing on the spinor representation and the little group. Next, we will have another look at spinor indices and introduce the powerful spinor helicity formalism. We will then use the latter to compute tree-level amplitudes in QED and QCD and show that it allows to obtain extremely compact expressions for scattering amplitudes, completely bypassing the Feynman diagrams expansion

The second part is instead dedicated on the generalization of these techniques to one-loop amplitudes. We present first the idea of integrand reduction, which allows to decompose one-loop amplitudes in terms of so-called master integrals. We then introduce the very powerful technique of one-loop unitarity, which instead attempts to generalize tree-level techniques to avoid the use of Feynman diagrams all together.

The third and last part deals finally with describing modern techniques to deal with multiloop amplitudes. While a fully general approach to compute amplitudes without resorting to Feynman diagrams is not available in this case, a lot of interesting techniques have been developed, on the one hand to handle efficiently the expansion in Feynman diagrams, and on the other to address the computation of multiloop Feynman integrals. We will in particular describe in detail tensor decomposition, integration by parts identities and the method of differential equations. From there, we will introduce iterated integrals and focus on the important class of multiple polylogarithms. We will also briefly introduce the symbol map and outline the generalization of multiple polylogarithms beyond genus zero.

Conventions

We work in four dimensional Minkowski space-time $\mathbb{R}^{1,3}$ with the mainly minus metric $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$.
We will usually work with all momenta incoming. External momenta are denoted p^μ. For loop momenta, we usually use l^μ or k^μ.

Part I

On-shell Methods for Tree-level Amplitudes

2 Symmetry Groups, Representations and Spinors

To begin our analysis of properties of scattering amplitudes, the most natural place to start is the Poincaré group. In fact, scattering amplitudes have definite transformation properties under Poincaré transformations, which are often hidden when one uses the standard formalism of Feynman diagrams and Feynman rules. Our first goal will therefore be to find a notation that makes them manifest. For massless particles, a lot can be gained using the so-called spinor helicity formalism. Extension to the massive case also exists, but as of the time of writing, their use remains limited. We will see an example of this in one of the exercises.

2.1 The Poincaré Group

As far as we know, the Poincaré group P is the symmetry group of our world. It contains Lorentz transformations and space-time translations. A general element of P can be written as

$$g(\Lambda, b) = \begin{bmatrix} & & & & b^0 \\ & & \Lambda^\mu{}_\nu & & b^1 \\ & & & & b^2 \\ & & & & b^3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} , \tag{2.1}$$

where Λ is an element of the Lorentz group and b^μ is an arbitrary four-vector, generating space-time translations. A space-time vector transforms under P as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + b^\mu , \tag{2.2}$$

where $\Lambda^\mu{}_\nu$ is a 4×4 matrix with $\det \Lambda = \pm 1$.

Irreducible Representations and the Little Group

As it is well known, one-particle states in Quantum Field Theory are classified by the irreducible representations (irreps) of the Poincaré group. More precisely, one-particle states can be defined as a set of states

$$|\psi\rangle \in S , \tag{2.3}$$

which transform into themselves under the action of the Poincaré group P,

$$|\psi_i\rangle \xrightarrow{\text{P}} P_{ij} |\psi_j\rangle . \tag{2.4}$$

The condition of *irreducibility* implies that there should be *no proper subset* of states

$$|\psi'\rangle \in S' \subset S , \tag{2.5}$$

which transform only among themselves. Physical states are actually represented by *rays* in the corresponding Hilbert space. With a somewhat imprecise notation

$$|\psi^R\rangle = e^{i\phi} |\psi\rangle , \quad \forall \phi \in \mathbb{R} . \tag{2.6}$$

What this means is that states are always defined up to an *unphysical complex phase*. The probability to find a state $|\psi_1^R\rangle$ in a different state represented by $|\psi_2^R\rangle$ is given by the scalar product of any two states in the corresponding rays

$$P(\psi_1^R \to \psi_2^R) = |\langle\psi_2|\psi_1\rangle|^2 . \tag{2.7}$$

Imagine now to perform a Poincaré transformation P on the system. All states are transformed to $\psi_i \to \psi'_i = P\psi_i$. Being P a symmetry of our world, we expect that probabilities measured in the transformed frame will remain the same

$$P(\psi_1'^R \to \psi_2'^R) = P(\psi_1^R \to \psi_2^R)\,. \tag{2.8}$$

Quite in general, Wigner proved that Eq. (2.8) implies important constraints on the corresponding operator U_P that implements the transformation on the Hilbert space of the physical spaces, i.e. on its representation on the Hilbert space of physical spaces. The operator acts moving a state in a given ray, to a state in another ray

$$\text{if } |\psi_i\rangle \in \psi_i^R \quad \Longrightarrow \quad U_P\,|\psi_i\rangle \in \psi_i'^R\,, \tag{2.9}$$

and one can prove that for probabilities to be conserved, U_P must be either *unitary* and *linear* or *anti-unitary* and *anti-linear*. On the other hand, any transformation continuously connected to the identity transformation must be represented by unitary and linear operators, since acting with the identity operator is by construction a unitary operation. The anti-unitary ones represent instead discrete transformations as *parity* or *time-reversal*. This is the reason why we are mainly interested in considering *unitary* irreducible representations of the Poincaré group (irreps). We recall in passing that, since the Poincaré group acts on rays, its representations on the physical states turn out to be *projective representations*, i.e. the composition rule is true up to a phase

$$U_{P_1}U_{P_2}\,|\psi\rangle = e^{i\phi_{12}}U_{P_1P_2}\,|\psi\rangle\ . \tag{2.10}$$

Wigner showed that the irreps of the Poincaré group can be classified through the irreps of the so-called *Little group* of the particle's momentum p^μ, i.e. the group that leaves the momentum invariant

$$W^{\mu\nu}p_\nu = p^\mu\,. \tag{2.11}$$

One then finds in general that the transformation of a one-particle state can be written as

$$U(\Lambda,b)\psi_{p,\sigma} = e^{-ib\cdot p}\sum_{\sigma'} D^{(j)}_{\sigma\sigma'}(W(\Lambda,p))\psi_{\Lambda p,\sigma}\,. \tag{2.12}$$

Here $e^{-ib\cdot p}$ takes care of the space-time translations and $D^{(j)}_{\sigma\sigma'}(W(\Lambda,p))$ of the Lorentz transformations. In particular, $D^{(j)}_{\sigma\sigma'}(W(\Lambda,p))$ is the corresponding representation of the Little group $W(\Lambda,p)$ associated to the momentum p^μ and j is a general label for the irreducible representation. By classifying all possible representations of $W(\Lambda,p)$, we can therefore classify all possible types of one-particle states.

It is easy to convince oneself that the irreps of the Little group depend on the nature of the momentum p^μ of the corresponding particle. There are three physically relevant cases.

Vacuum
The vacuum corresponds to $p^\mu = 0$ and is not of further interest - nothing happens.

Massive Particles
For massive particles the condition $p^2 > 0$ holds. Transforming the momentum into the rest frame, we can always write $p^\mu = (m,0,0,0)$. With this, one can easily see that the Little group is the three-dimensional rotation group SO(3). The representation theory for SO(3) is well known from non-relativistic quantum mechanics: the irreducible

representations are labelled by an index s, called the *spin* of the representation and have dimension

$$\dim(R_s) = (2s+1)\,, \quad \text{with} \quad s = \frac{n}{2}\,,\ \ n \in \mathbb{N}\,, \tag{2.13}$$

which correspond to integer or half-integer spin particles. Consequently, all usual massive particles, such as massive fermions, i.e. spin-$1/2$ particles or massive vector bosons, i.e. spin-1 particles, simply transform under SO(3) like in non-relativistic Quantum Mechanics. For spin s, one has $(2s+1)$ states and the representation matrix $D^{(s)}_{\sigma\sigma'}(W(\Lambda,p))$ is a $(2s+1)$-dimensional unitary matrix.

Massless Particles

This case is the least trivial but also the one physically more interesting for us, so let us describe it more in detail. The momentum for massless particles is light-like $p^2 = 0$ and we can always Lorentz transform to a frame where $p^\mu = (E, 0, 0, E)$. In this way we can see that this momentum is left invariant by rotations in a two dimensional plane O(2) (including reflections!). This is not all, though. This can be seen, for example, by introducing light-cone coordinates

$$x^\mu = \{x^+, x^-, x^1, x^2\}\,, \quad \text{with} \quad x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}\,. \tag{2.14}$$

In this representation, the light-like momentum becomes

$$p^\mu = \{p^+, 0, 0, 0\}\,. \tag{2.15}$$

Let us call $M^{\mu\nu} = -i\,(x^\mu\partial^\nu - x^\nu\partial^\mu)$ the generators of the Lorentz Group in standard coordinates, then we have for example

$$M^{j+} = -i\left(x^j\partial^+ - x^+\partial^j\right) = -\frac{M^{0j} + M^{3j}}{\sqrt{2}}\,, \qquad \text{for} \quad j = 1, 2\,, \tag{2.16}$$

$$M^{j-} = -i\left(x^j\partial^- - x^-\partial^j\right) = -\frac{M^{0j} - M^{3j}}{\sqrt{2}}\,, \qquad \text{for} \quad j = 1, 2\,. \tag{2.17}$$

Recall, importantly, that

$$\partial^i = \frac{\partial}{\partial x_i} = -\frac{\partial}{\partial x^i} = -\partial_i\,, \qquad \text{for} \quad i = 1, 2, 3\,, \tag{2.18}$$

such that, for example by raising all indices, we have

$$\begin{aligned} M^{j-} &= -i\left[x^j \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_3}\right) - \frac{x^0 - x^3}{\sqrt{2}}\frac{\partial}{\partial x_j}\right] \\ &= -i\left[x^j \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^3}\right) + \frac{x^0 - x^3}{\sqrt{2}}\frac{\partial}{\partial x^j}\right]\,, \qquad j = 1, 2\,. \end{aligned} \tag{2.19}$$

Clearly, the massless momentum is invariant under rotations in the $1-2$ plane, generated by M^{12} (the usual O(2) described above), but also under the two extra generators M^{1-} and M^{2-}, which obviously commute with p^+ in light-cone coordinates. One can then easily prove that the symmetry generated by these two extra generators is isomorphic to the translations on the two-dimensional plane and, as a consequence, the

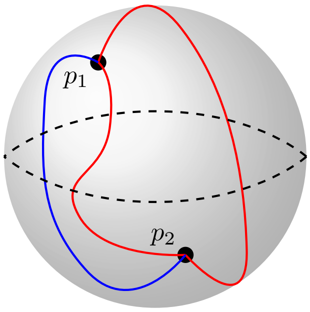


Figure 2: Illustration of the Poincaré group not being simply connected.

Little group of a massless particle is actually the Euclidean group $\mathrm{E}(2) = I\,\mathrm{SO}(2)$. In particular, it is easy to see that their commutation relations read

$$[M^{1-}, M^{2-}] = 0\,, \quad [M^{1-}, M^{12}] = -iM^{2-}\,, \quad [M^{2-}, M^{12}] = iM^{1-}\,, \tag{2.20}$$

which are exactly what we expect: M^{1-} and M^{2-} commute, as the generators of translations in the two dimensional plane. While the action on M^{12} on the latter two generators transforms them into each other, as a rotation is supposed to do.

Now, both translations and two-dimensional rotations are represented by continuous eigenvalues. This is a problem since we do not observe such continuous eigenvalues associated to any known particle. We can get rid of the eigenvalues associated to translations by *requiring* that physical particles are only those states which transform trivially under translations. These states are still distinguished by the eigenvalues of the operator M^{12} which generates rotations in the $1-2$ plane, $J_3\psi = h\psi$, where h is usually called the *helicity* of the particle and

$$D^h_{\sigma\sigma'}(W(\Lambda, p)) = e^{ih\theta(\Lambda,p)}\,. \tag{2.21}$$

As far as SO(2) goes, h can also be an arbitrary real number. Note, however, that the topology of the Poincaré group is $\mathbb{R} \times \mathbb{R}_3 \times S_3/\mathbb{Z}_2$. Focusing on $S_3/\mathbb{Z}_2$, this is a sphere whose antipodal points are identified. Let us call two such points p_1 and p_2. A curve connecting these two points is a closed curve on the Poincaré manifold, but it clearly cannot be shrunk to a point, see Figure 2. Mathematically this means that the Poincaré group is *not simply connected* and for a massless particle of helicity h it only tells us that

$$e^{ih\theta(\Lambda,p)} = e^{2\pi ih} \neq \mathbb{1}\,. \tag{2.22}$$

On the other hand, if we draw a closed curve that goes from p_1 to p_2 and then goes back to p_1 we obtain a curve that is equivalent to a point, i.e.

$$e^{ih\theta(\Lambda,p)} = e^{4\pi ih} = \mathbb{1}\,. \tag{2.23}$$

One finds therefore

$$h = \left\{n \text{ or } \frac{n}{2}\right\} \quad n \in \mathbb{N}\,, \tag{2.24}$$

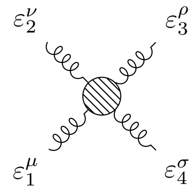
i.e. the helicity h has to be integer or half-integer. This explains why we observe only discrete helicities in nature and not continuous ones, but still does not tell us why helicities seem to *come in pairs*. The photon, for instance, can have two helicity states ± 1, similarly to the gluon, while the graviton is expected to come with the two helicities ± 2. The reason for this is *parity invariance*. Parity $\mathcal{P}$ does not only swap $\vec{p} \to -\vec{p}$, it also flips the helicity. In order to build a QFT invariant under parity, massless particles such as the photon need to come in both helicities and form a doublet under $\mathcal{P}$ in $\mathrm{O}(1,3)$. Note, that in a theory that is not parity invariant we do not have doublets, e.g. left- and right-handed neutrinos do not necessarily come in pairs, but the left-handed one could exist independently of the right-handed one.

Transformation of the Scattering Amplitude

Let us now get back to the scattering amplitude S and its transformation properties under the action of the Poincaré group. Since S is computed as a matrix element between multi-particle states

$$S = \langle \psi^{\text{out}}_{p_1\sigma_1\ldots p_m\sigma_m} | \psi^{\text{in}}_{p'_1\sigma'_1\ldots p'_n\sigma'_n} \rangle \,, \tag{2.25}$$

we expect that it should transform under the Little group according to each of the particles in the initial and final state. Now think about how S is usually computed: we draw all Feynman diagrams, substitute Feynman rules and sum them together. Consider for example the case of the scattering of 4 gluons. Then this way of constructing the scattering amplitude will produce an expression of the form



$$= S = \varepsilon_1^\mu \varepsilon_2^\nu \varepsilon_3^\rho \varepsilon_4^\sigma \mathcal{F}_{\mu\nu\rho\sigma} \,, \tag{2.26}$$

where ε_i^μ are the polarization vectors of the external gluons, and $\mathcal{F}_{\mu\nu\rho\sigma}$ transforms as a (rank-4) Lorentz Tensor. Clearly, in order for this amplitude to transform properly under the Little group, the polarization vectors have the role to contract the Lorentz indices and bring back the information on the Little group covariance. What this means is that polarization vectors (spinors) transform doubly under the action of the Poincaré group, both as Lorentz vectors (or spinors for spin-$1/2$ particles) and also under the Little group.

As we are discussing this, it makes sense to connect this discussion to the transformation properties of the objects that we are used to deal with in QFT. In the standard text-book approach to QFT, we usually work with non-observable fields ($\phi(x)$, $\psi^\alpha(s)$, $A^\mu(x)$...). We should stress that these fields *do not need to transform as unitary irreducible representations of the Poincaré group* - and in fact they don't! Take for example a massless fermionic field

$$\psi^\alpha(x) = \sum_\lambda \int d\tilde{p} \Big[b(p,\lambda) u^\alpha(p,\lambda) e^{ipx} + \text{h.c.} \Big] \,. \tag{2.27}$$

The transformation of the field $\psi^\alpha(x)$ encoded in the index α shows that it transforms under a finite dimensional representation of the Lorentz group. As we already stressed, these representations are *not unitary* because the Lorentz group is non compact. On the other hand, the particle states $b(p,\lambda)$ and $b^\dagger(p,\lambda)$ are the objects that *must transform* as infinite unitary irreducible representations of the Poincaré group, which are classified by the

Little group. In order for the right- and left-hand side of the equation to make sense, the connection is provided by the wave functions $u^\alpha(p,\lambda)$, which must then transform both as finite dimensional representations of the Lorentz group and under the Little Group.

Since amplitudes are build starting from the polarization vectors, our next goal is to study the Lorentz group and its irreducible representations in order to find a convenient representation for ϵ^μ, u^α, etc, which makes their transformation properties manifest. This will be the Spinor-Helicity formalism, that we will introduce starting from a general discussion of the Lorentz group.

2.2 The Lorentz Group

A space-time vector transforms under the Lorentz group as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \,, \tag{2.28}$$

where $\Lambda^\mu{}_\nu$ is a 4×4 matrix. Demanding that the scalar product of two space-time vectors is preserved under Lorentz transformation yields:

$$\begin{aligned} & x'^\mu x'_\mu = x^\mu x_\mu = x_0^2 - \vec{x}^2 \,, \\ \Rightarrow \quad & \Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma g_{\mu\rho} x^\nu x^\sigma = x^\mu x_\mu \,, \\ \Rightarrow \quad & \Lambda^\mu{}_\nu g_{\mu\rho} \Lambda^\rho{}_\sigma = g_{\nu\sigma} \,, \end{aligned} \tag{2.29}$$

or in terms of matrices

$$\Lambda^T g \Lambda = g \,, \tag{2.30}$$

which is the defining property of the Lorentz group, denoted as $\mathrm{O}(1,3)$. The Lorentz group includes parity transformations $\mathcal{P}$ and time reversal $\mathcal{T}$. We are usually only interested in the proper Lorentz group $\mathrm{SO}^+(1,3)$, i.e. the subgroup of the Lorentz group that is continuously connected to the identity transformation ($\det\Lambda = +1$) and does only contain orthochronous transformations ($\Lambda^0{}_0 > 1$). Every element of $\mathrm{O}(1,3)$ can then be written as a semi-direct product of $\mathrm{SO}^+(1,3)$ and the discrete transformations $\{\mathbb{1}, \mathcal{T}, \mathcal{P}, \mathcal{PT}\}$.

A Universal Cover: The Special Linear Group $\mathrm{SL}(2,\mathbb{C})$

The connected components of the Lorentz group, including what we called before the proper Lorentz group, are not simply connected. The special linear group $\mathrm{SL}(2,\mathbb{C})$ of 2×2 complex matrices with unit determinant turns out to be the universal cover of the proper Lorentz group. To see this, let us first consider a four-vector $x^\mu \in \mathbb{R}^{1,3}$ and build from it the 2×2 matrix

$$X = x^\mu \sigma_\mu \,, \tag{2.31}$$

where $\sigma^\mu = (\mathbb{1}_2, \vec{\sigma})$ are the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{2.32}$$

Note, that the Pauli matrices are defined with a lower index and $\sigma^\mu = \bar{\sigma}_\mu$ Inserting the Pauli matrices, we have

$$X = \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix} \tag{2.33}$$

and it is easy to see that this is the *most general* 2×2 Hermitian matrix ($X^\dagger = X$). This means that we have a one to one map

$$\forall\, x^\mu \in \mathbb{R}^{1,3} \quad \leftrightarrow \quad X \;\; \text{Hermitian}\,. \tag{2.34}$$

Notice that taking the determinant of X corresponds to computing the norm of the corresponding Minkowski vector

$$\det X = (x^0)^2 - \vec{x}^2 = x^\mu x_\mu\,, \tag{2.35}$$

hence, a Lorentz transformation on x^μ acts on the space of Hermitian matrices by preserving their determinant.

Consider now a general transformation of X under the general complex linear group $\mathrm{GL}(2,\mathbb{C})$, i.e.

$$\forall\, A \in \mathrm{GL}(2,\mathbb{C}) \quad \rightarrow \quad X' = AXA^\dagger\,. \tag{2.36}$$

Clearly, Hermiticity is automatically preserved for every choice of A. However, if we want that X' is still a Lorentz transformation, the determinant of the matrix must be preserved,

$$\det(AXA^\dagger) = |\det A| \det X \quad \rightarrow \quad |\det A| = 1\,, \quad \det A = e^{i\eta}\,. \tag{2.37}$$

So every matrix in $\mathrm{GL}(2,\mathbb{C})$ with determinant equal to a phase generates a Lorentz transformation, i.e.

$$A(x^\mu \sigma_\mu)A^\dagger = (\Lambda^\mu{}_\nu(A) x^\nu)\, \sigma_\mu\,. \tag{2.38}$$

Clearly, given matrices which differ by a phase, say A and $A' = e^{i\phi}A$, generate the same Lorentz transformation

$$X' = AXA^\dagger = A'XA'^\dagger\,, \tag{2.39}$$

and we can use this ambiguity to fix

$$\det A = 1\,. \tag{2.40}$$

This means that Eq.(2.36) is a proper Lorentz transformation for $A \in \mathrm{SL}(2,\mathbb{C})$.

Finally, if $A \in \mathrm{SL}(2,\mathbb{C})$ then so is $-A$, since

$$\det A = \det(-A)\,. \tag{2.41}$$

Consequently, A and $-A$ produce the same Lorentz transformation, i.e. to both we can associate one single Λ according to Eq. (2.38). Hence

$$\mathrm{SO}^+(1,3) = \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2\,, \tag{2.42}$$

and $\mathrm{SL}(2,\mathbb{C})$ is the universal covering group of $\mathrm{SO}^+(1,3)$.

Generators and Algebra

The Lorentz group includes boost B and rotations R. Just as a reminder, rotation around the x-axis and the boost in x-direction are given by

$$(R_x)^\mu{}_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & \sin\theta_x \\ 0 & 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix}, \quad (B_x)^\mu{}_\nu = \begin{bmatrix} \cos\theta_x & -\sin\theta_x & 0 & 0 \\ -\sin\theta_x & \cos\theta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.43}$$

and similarly for R_y, B_y, R_z, B_z. One can then read off the generators by expanding them for $\theta_i \ll 1$, $i = x, y, z$. We denote the generators for rotations R as J and the generators for boosts B as K. For example for the rotation around the x-axis we have

$$(J_x)^\mu{}_\nu = \left[-i \frac{dR_x(\theta_x)}{d\theta_x}\bigg|_{\theta_x=0} \right] . \tag{2.44}$$

One then easily finds for the six generators of the Lorentz group

$$\begin{aligned}
(J_x)^\mu{}_\nu &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{bmatrix} , \; (J_y)^\mu{}_\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} , \; (J_z)^\mu{}_\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \\
(K_x)^\mu{}_\nu &= \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \quad (K_y)^\mu{}_\nu = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \quad (K_z)^\mu{}_\nu = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} .
\end{aligned} \tag{2.45}$$

Knowing the generators, one can work out the Lorentz algebra,

$$[J_i, J_j] = i\varepsilon_{ijk} J_k , \qquad [J_i, K_j] = i\varepsilon_{ijk} K_k , \qquad [K_i, K_j] = -i\varepsilon_{ijk} J_k . \tag{2.46}$$

A general Lorentz transformation can therefore be parameterized as

$$\Lambda = \exp\left\{ i\vec{J} \cdot \vec{\theta} + i\vec{K} \cdot \vec{\theta} \right\} . \tag{2.47}$$

Note that by exponentiation of the algebra we do not generate the full Lorentz group $\mathrm{O}(1,3)$, but only the proper orthochronous group $\mathrm{SO}^+(1,3)$, which is continuously connected to the identity. To study the Lorentz algebra, we can decouple the two sets of generators by introducing

$$N_i^\pm = \frac{1}{2} \left(J_i \pm i K_i \right) . \tag{2.48}$$

We find

$$\left[N_i^+, N_j^+ \right] = i\varepsilon_{ijk} N_k^+ , \qquad \left[N_i^-, N_j^- \right] = i\varepsilon_{ijk} N_k^- , \qquad \left[N_i^+, N_j^- \right] = 0 , \tag{2.49}$$

which are two decoupled copies of the Lie algebra of SU(2). Hence, we find for the Lie algebra of the Lorentz group

$$\mathfrak{so}^+(1,3) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) . \tag{2.50}$$

Consequently, any representation of the proper Lorentz group $\mathrm{SO}^+(1,3)$ is specified by two labels (j, j'). The corresponding number of degrees of freedom is $(2j+1)(2j'+1)$.

Clearly, the regular "physical" rotation generators can be recovered as $\vec{J} = \vec{N}^+ + \vec{N}^-$. The eigenvalue of this operator provides the "spin" of the representation. If A and B are the eigenvalues of N^+ and N^- respectively, by the usual rules to sum angular momenta a representation (A, B) of the Lorentz group can be associated to the representations of SU(2) with spins

$$j = \{A + B, A + B - 1, \ldots, |A - B|\} . \tag{2.51}$$

Some most common representations of the Lorentz group and their corresponding spins are collected in Table 1. Note that while the representations in the Lorentz group are irreducible,

Lorentz group $SU(2) \times SU(2)$	Rotation group $SO(3)$
$(0,0)$	0
$(\frac{1}{2},0)$	$\frac{1}{2}$
$(0,\frac{1}{2})$	$\frac{1}{2}$
$(\frac{1}{2},\frac{1}{2})$	$1 \oplus 0$
$(1,0)$	1
$(1,1)$	$2 \oplus 1 \oplus 0$

Table 1: Representations of the Lorentz group and of the Rotation group

those of the Rotation group can be reducible, see for example $(\frac{1}{2},\frac{1}{2})$ and $(1,1)$. Moreover, remember that upon exponentiating a Lie algebra, we get the universal covering group. Thus, by exponentiating $\mathfrak{su}(2)$, we obtain the universal covering group of SO(3), i.e. SU(2). Equivalently, from $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, one finds that the universal covering group of $SO^+(1,3)$ is $SL(2,\mathbb{C})$.

2.3 Representations of the Lorentz group: Spinors

We will now discuss some important representations of the Lorentz group in more detail.

Scalar Representation

$(0,0)$ is the scalar, i.e. trivial, representation. This is how scalar fields transform.

Left-Handed Spinors

$(\frac{1}{2},0)$ is called the left-handed spinor representation of $SO^+(1,3)$. Here we are choosing the trivial representation for N_i^- and the 1/2 representation for N_i^+. This implies

$$N_i^- = 0\,, \quad \Longrightarrow \quad J_i = iK_i\,, \quad N_i^+ = \frac{1}{2}\sigma_i\,. \tag{2.52}$$

Using

$$N_i^+ = \frac{1}{2}\left(J_i + iK_i\right) = iK_i = \frac{1}{2}\sigma_i \tag{2.53}$$

we get

$$J_i = \frac{1}{2}\sigma_i\,, \qquad K_i = -\frac{i}{2}\sigma_i\,. \tag{2.54}$$

This gives for the rotation and boost matrices

$$\begin{aligned} \vec{R}(\vec{\theta}) &= e^{i\vec{\theta}\cdot\vec{J}} = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\,, \\ \vec{B}(\vec{\phi}) &= e^{i\vec{\phi}\cdot\vec{K}} = e^{\vec{\phi}\cdot\frac{\vec{\sigma}}{2}}\,. \end{aligned} \tag{2.55}$$

Explicitly, this corresponds to the following boost and rotation matrices:

$$
\begin{aligned}
R_x^L(\theta_x) &= \begin{bmatrix} \cos\frac{\theta_x}{2} & i\sin\frac{\theta_x}{2} \\ i\sin\frac{\theta_x}{2} & \cos\frac{\theta_x}{2} \end{bmatrix}, \qquad R_y^L(\theta_y) = \begin{bmatrix} \cos\frac{\theta_y}{2} & \sin\frac{\theta_y}{2} \\ -\sin\frac{\theta_y}{2} & \cos\frac{\theta_y}{2} \end{bmatrix}, \\
R_z^L(\theta_z) &= \begin{bmatrix} e^{i\frac{\theta_z}{2}} & 0 \\ 0 & e^{-i\frac{\theta_z}{2}} \end{bmatrix}, \qquad B_x^L(\phi_x) = \begin{bmatrix} \cosh\frac{\phi_x}{2} & \sinh\frac{\phi_x}{2} \\ \sinh\frac{\phi_x}{2} & \cosh\frac{\phi_x}{2} \end{bmatrix}, \\
B_y^L(\phi_y) &= \begin{bmatrix} \cosh\frac{\phi_y}{2} & -i\sinh\frac{\phi_y}{2} \\ i\sinh\frac{\phi_y}{2} & \cosh\frac{\phi_y}{2} \end{bmatrix}, \qquad B_z^L(\phi_z) = \begin{bmatrix} e^{\frac{\phi_z}{2}} & 0 \\ 0 & e^{-\frac{\phi_z}{2}} \end{bmatrix}.
\end{aligned}
\tag{2.56}
$$

Right-Handed Spinors

$(0, \frac{1}{2})$ is the right-handed spinor representation of the Lorentz group. Arguing as before, this time with $N_i^+ = 0$ and $J_i = -iK_i$, we find

$$
J_i = \frac{1}{2}\sigma_i \,, \qquad K_i = \frac{i}{2}\sigma_i \,. \tag{2.57}
$$

Note that the generator of rotations J_i is the same for the right-handed and left-handed representation, while the generator of boosts K_i differs by the sign. Thus, right and left-handed spinors differ in their transformation under boosts, but transform equally under rotations.

From Right- to Left-Handed Spinors

Let ψ_L be a left-handed spinor, meaning that it transforms as following under rotations and boosts:

$$
\text{rotation}: \ \psi_L' = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\psi_L \,, \qquad \text{boost}: \ \psi_L' = e^{\vec{\phi}\cdot\frac{\vec{\sigma}}{2}}\psi_L \,. \tag{2.58}
$$

We will now show that one can transform right- and left-handed spinors into each other by multiplying with

$$
\pm i\sigma_2 = \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.59}
$$

and complex conjugating. For that purpose, we consider the transformation properties of the object

$$
\overline{\psi}_L = i\sigma_2 \left(\psi_L\right)^* \,. \tag{2.60}
$$

Under rotations, it transforms as

$$
\begin{aligned}
\overline{\psi}_L' &= i\sigma_2 \left(\psi_L'\right)^* \\
&= i\sigma_2 e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2}} \left(\psi_L\right)^* \\
&= i\sigma_2 e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2}} \left[-i\sigma_2 i\sigma_2\right] \left(\psi_L\right)^* \\
&= e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\overline{\psi}_L \,,
\end{aligned}
\tag{2.61}
$$

where we inserted $\mathbb{1} = -i\sigma_2(i\sigma_2)$ in the third line and used the identity $(i\sigma_2)\vec{\sigma}^*(-i\sigma_2) = -\vec{\sigma}$ in the fourth line. Consequently, $\overline{\psi}_L$ transforms as a spinor under rotations. Let us now consider the transformation under boosts:

$$\begin{aligned}\overline{\psi}'_L &= i\sigma_2\,(\psi'_L)^* \\ &= i\sigma_2 e^{\vec{\phi}\cdot\frac{\vec{\sigma}^*}{2}}\,(\psi_L)^* \\ &= e^{-\vec{\phi}\cdot\frac{\vec{\sigma}}{2}}\overline{\psi}_L\,,\end{aligned} \tag{2.62}$$

where we performed identical manipulations as above. We see that $\overline{\psi}_L$ transforms as a right-handed spinor.

Notice that we can now go back to the same left-handed spinor as follows:

$$-i\sigma_2\left(\overline{\psi}_L\right)^* = -i\sigma_2\big(i\sigma_2\,(\psi_L)^*\big)^* = \sigma_2^2\psi_L = \psi_L\,. \tag{2.63}$$

To summarize, one can transform a left-handed into a right-handed spinor by complex conjugation and multiplying by $i\sigma_2$. To go from a right-handed to a left-handed spinor, one has to complex conjugate and multiply by $-i\sigma_2$. Notice, that the position of the minus sign is conventional but that we need a different sign in the $L \to R$ and $R \to L$ transformation in order to guarantee that the two transformations in sequence lead us back to the spinor we started from.

Vector-Representation

The $(\frac{1}{2}, \frac{1}{2})$-representation of the Lorentz group contains spin 0 and 1 particles, according to Table 1. In fact, it corresponds to the vector representation through the same identification we used to identify the covering group of $\mathrm{SO}^+(1,3)$

$$X^{\dot{a}b} = x^\mu(\sigma_\mu)^{\dot{a}b}\,. \tag{2.64}$$

Here we used a common notation for spinor indices, denoting indices in the left-handed representation as undotted b and indices in the right-handed representation as dotted indices $\dot{a}$. For now this is only a notation, we will describe it in more detail in the next chapter and connect it to the spinor helicity formalism.

Parity and Handness

We have now considered several representations of the Lorentz group $\mathrm{SO}^+(1,3)$. Notice that most of the times we will consider physical theories that are invariant under parity transformations, such as QED or QCD. We have seen than one consequence of parity invariance is that the photon has to have two helicities. For spin-$\frac{1}{2}$ particles, parity swaps left- and right-handed spinors, since $\phi \leftrightarrow -\phi$ and thus $B_R(\phi) \leftrightarrow B_L(\phi)$. To get a parity invariant representation we therefore must consider both left- and right-handed representations at once, which we can combine in so-called Dirac spinors Ψ

$$\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right) \quad \rightarrow \quad \Psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}\,. \tag{2.65}$$

We recall finally that Dirac spinors take this explicit block-diagonal form in the so-called chiral representation for the Dirac algebra.

3 Spinor Helicity Formalism

3.1 Spinor Indices

Let us now come back to the dotted and undotted index notation, that we just briefly introduced for the vector representation of the Lorentz group, see Eq. (2.64). There we wrote a momentum transforming under $(\frac{1}{2}, \frac{1}{2})$ as

$$p^{\dot{a}b} = (\sigma_\mu)^{\dot{a}b} p^\mu \,, \tag{3.1}$$

and (conventionally) we decided that $\dot{a}$ denotes a right-handed index, while b is left-handed. Note, that $p^\mu \in \mathrm{SO}^+(1,3)$ transforms as a Lorentz vector, while $p^{\dot{a}b} \in \mathrm{SL}(2,\mathbb{C})$ transforms in the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group. Moreover, if p^μ is a light-like vector, we have

$$\det\left(p^{\dot{a}b}\right) = p^2 = 0 \,, \tag{3.2}$$

which implies that $p^{\dot{a}b}$ is not full-rank, but instead it is a 2×2 matrix of rank 1. Since any rank-one 2×2 matrix can be written as the outer product of two vectors, we can define two spinors $\tilde{\lambda}^{\dot{a}}$ and λ^b such that

$$p^{\dot{a}b} = \tilde{\lambda}^{\dot{a}} \lambda^b \,. \tag{3.3}$$

Eq. (3.3) is one possible starting point to introduce the spinor-helicity formalism, where the fundamental objects of interests are indeed the (two-dimensional) spinors $\tilde{\lambda}^{\dot{a}} = |p\rangle^{\dot{a}}$ and $\lambda^b = |p]^b$.

Instead on jumping directly into the formalism, we will do a bit of a detour, which allows us to get more comfortable with manipulating dotted and undotted indices. We start from Eq. (2.64) and see how to generalize this index notation for physical Dirac spinors in $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$. For that purpose, let us explore some properties of the spinors.

We stress that here, as everywhere in these lectures except when clearly stated otherwise, we will manipulate set of *massless* momenta assuming that they all correspond to *incoming particles*. This implies that if we have N momenta p_i, all corresponding scalar invariants are positive, i.e.

$$2 p_i \cdot p_j = s_{ij} > 0 \,, \qquad \text{with} \;\; p_i^2 = 0 \,, \; \forall \; i = 1, \ldots, N \,. \tag{3.4}$$

Transformation under Complex Conjugation

First, let us examine how dotted and undotted indices behave under complex conjugation. Given

$$p^{\dot{a}b} = \begin{pmatrix} p^0 + p^3 & p^1 - i p^2 \\ p^1 + i p^2 & p^0 - p^3 \end{pmatrix} , \tag{3.5}$$

we see that upon complex conjugation one finds

$$(p^{\dot{a}b})^* = \begin{pmatrix} p^0 + p^3 & p^1 + i p^2 \\ p^1 - i p^2 & p^0 - p^3 \end{pmatrix} = (p^{\dot{a}b})^T = p^{b\dot{a}} \,. \tag{3.6}$$

Since the indices are just dummy, this shows that complex conjugation exchanges dotted and undotted indices.

Raising and Lowering Indices: The Spinor Metric

As a second observation, recall that to go from left- to right-handed spinors (or the other way around), one has to complex conjugate the spinors and multiply by $\pm i\sigma_2$,

$$\begin{aligned} i\sigma_2\psi_L^* &= \psi_R\,, \\ -i\sigma_2\psi_R^* &= \psi_L\,. \end{aligned} \tag{3.7}$$

The matrix $i\sigma_2$ that we used to go from left- to right-handed spinors is special. Recalling the definition of the Pauli matrices, one finds that

$$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (i\sigma_2)(-i\sigma_2) = \mathbb{1}_2\,. \tag{3.8}$$

Let us have a look at how this matrix transforms under left- and right-handed Lorentz transformations. Using the matrices for the left-handed Lorentz transformations from Eq.(2.56), one easily finds

$$R_j^L(i\sigma_2)(R_j^L)^T = B_j^L(i\sigma_2)(B_j^L)^T = i\sigma_2\,.$$

Similarly, for right-handed Lorentz boosts and rotations one finds

$$R_j^R(i\sigma_2)(R_j^R)^T = B_j^R(i\sigma_2)(B_j^R)^T = i\sigma_2\,.$$

Hence, $i\sigma_2$ is invariant under left- and right-handed Lorentz transformations, similarly to the metric $g^{\mu\nu}$ in standard Minkowski space. Indeed, $i\sigma_2$ can be used to raise and lower spinor indices:

$$\psi^a = (i\sigma_2)^{ab}\psi_b\,, \qquad \psi^{\dot a} = (i\sigma_2)^{\dot a\dot b}\psi_{\dot b}\,.$$

Let us then introduce the following convention, consistent with our choice in Eq. (2.64). We start by saying that left-handed spinors have *lower undotted* indices, which gives

$$\psi_L = \psi_a \quad \to \quad (\psi_L)^* = \psi_{\dot a}\,, \tag{3.9}$$

where we used the fact that complex conjugation must send dotted to undotted indices and vice versa. Now we know that we can obtain a right-handed spinor by acting with $i\sigma_2$. As this is the metric in spinor space, its action will be that of raising the spinor index to give

$$\psi_R = (i\sigma_2)(\psi_L)^* = (i\sigma_2)^{\dot a\dot b}\psi_{\dot b} = \psi^{\dot a}.$$

Then one can go from right- to left-handed spinors again by another complex conjugation and by lowering the index with a new action of the metric (but remembering that we need a minus sign!)

$$\psi_L = (-i\sigma_2)(\psi_R)^* = (-i\sigma_2)_{ab}\psi^b\,. \tag{3.10}$$

From this, we can conclude that $i\sigma_2$ acts as the metric in the space of dotted indices, while $-i\sigma_2$ acts in the space of undotted indices. Now remember what we know:

i. complex conjugation swaps dotted and undotted indices,

ii. $i\sigma_2$ is real $\Rightarrow$ $(i\sigma_2)^* = i\sigma_2$ and $(-i\sigma_2)^* = -i\sigma_2$, and putting these equations together we get

$$(i\sigma_2)^{ab} = (i\sigma_2)^{\dot{a}\dot{b}} = \varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$
$$(-i\sigma_2)_{ab} = (-i\sigma_2)_{\dot{a}\dot{b}} = \varepsilon_{ab} = \varepsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \tag{3.11}$$

which guarantees

$$(i\sigma_2)^{ac}(-i\sigma_2)_{cb} = \varepsilon^{ac}\varepsilon_{cb} = \delta^a_b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{3.12}$$

Spinor Products

Now that we have a notation for Weyl spinors and a metric in spinor space, we can define the spinor products in the form $\psi^a\chi_a = \psi^a\chi^b\varepsilon_{ab}$. First of all, let us see how they transform under Lorentz transformations:

$$\psi^a\chi_a = \psi^a\chi^b\varepsilon_{ab} \quad \rightarrow \quad \psi'^a\chi'^b\varepsilon_{ab} = \psi^c\chi^d\Lambda^a{}_c\Lambda^b{}_d\varepsilon_{ab} = \psi^c\chi^d\varepsilon_{cd} = \psi^c\chi_c \, ,$$

where Λ is a generic rotation or boost in spinor space. So as expected, left-handed spinor products are invariant under Lorentz transformations. Similarly, one can define right-handed spinor products $\psi_{\dot{a}}\chi^{\dot{a}}$, that is also Lorentz invariant. Importantly, spinor product are *anti-symmetric*:

$$\psi^a\chi_a = \psi^a\chi^b\varepsilon_{ab} = -\psi^a\chi^b\varepsilon_{ba} = -\chi^a\psi_a \, .$$

In particular, this implies $\psi^a\psi_a = 0$. It is important to stress here that we are building spinor products of objects that are *not Grassmann numbers*! In fact, we intend to apply this formalism to the wave functions that appear in the definition of the scattering amplitudes, which also for spinors are simple c-numbers![1]

Keeping track of dotted and undotted indices is especially confusing, in particular because their nature is entirely conventional. We can simplify our life enormously by introducing a new notation, which will ultimately allow us to forget about dotted and undotted indices all together. We write

$$\psi^a = [\psi^a \, , \quad \chi_a = \chi]_a \, , \qquad \text{for left-handed spinors} \, , \tag{3.13}$$
$$\psi_{\dot{a}} = \langle\psi_{\dot{a}} \, , \quad \chi^{\dot{a}} = \chi\rangle^{\dot{a}} \qquad \text{for right-handed spinors} \, , \tag{3.14}$$

such that spinor products among Weyl spinors can be written respectively

$$\psi^a\chi_a = [\psi\chi] \, , \qquad \psi_{\dot{a}}\chi^{\dot{a}} = \langle\psi\chi\rangle \, . \tag{3.15}$$

3.2 Dirac Spinors

Since we will mainly deal with CP invariant theories, we will not use Weyl spinors very often, but work instead with Dirac spinors, in the Chiral representation. We will focus on the wave functions appearing in scattering amplitudes, i.e. the positive frequency solutions

[1] One can alternatively build spinor products of Grassmann fields at the Lagrangian level, but this is not what we are interested in doing here.

$u(p)$ for incoming particles and $\overline{u}(p)$ for outgoing particles, as well as the negative frequency solutions $v(p)$ for outgoing anti-particles and $\overline{v}(p)$ for incoming anti-particles. They satisfy the Dirac equations

$$\begin{aligned}(\not{p} - m)u(p) &= 0\,,\\ (\not{p} + m)v(p) &= 0\,.\end{aligned} \tag{3.16}$$

We will mainly work with the massless case

$$\not{p}u(p) = 0\,, \qquad \not{p}v(p) = 0\,. \tag{3.17}$$

We denote $\not{p} = p^\mu\,\gamma_\mu$ and use the Chiral representation for the Dirac matrices

$$\gamma^\mu = \begin{pmatrix} \mathbb{0}_2 & \sigma^\mu \\ \overline{\sigma}^\mu & \mathbb{0}_2 \end{pmatrix},$$

where $\sigma^\mu = (\mathbb{1}_2, \vec{\sigma})$ and $\overline{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma})$. We also define a fifth gamma matrix

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & \mathbb{1}_2 \end{pmatrix}. \tag{3.18}$$

With this we can also define the usual left and right chiral projectors

$$P_L = \frac{\mathbb{1}_4 - \gamma^5}{2} = \begin{pmatrix} \mathbb{1}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix}, \qquad P_R = \frac{\mathbb{1}_4 + \gamma^5}{2} = \begin{pmatrix} \mathbb{0}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & \mathbb{1}_2 \end{pmatrix}. \tag{3.19}$$

Using these projectors, we can separate the wave functions into a left- and right-handed part

$$u_{L,R} = P_{L,R}\, u(p)\,. \tag{3.20}$$

So the wave function for a Dirac spinor is built out of a left- and a right-handed Weyl spinor

$$u(p) = \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix},$$

which satify the Weyl equation

$$p^\mu \sigma_\mu u_L(p) = p^\mu \overline{\sigma}_\mu u_R(p) = 0\,. \tag{3.21}$$

Now one can easily prove that, given a real momenta p^μ, the object $\tilde{u}(p) = (i\sigma_2)(u_L(p))^*$ satisfies the Weyl equation $p^\mu\overline{\sigma}_\mu\tilde{u}(p) = 0$, i.e. $\tilde{u}(p)$ is a right-handed spinor. Thus, as for Lorentz spinors u_L and u_R are left- and right-handed spinors, being transformed into each other by multiplying with $\pm i\sigma_2$ and complex conjugation.

Writing out the Weyl equation in components, one finds

$$u_L(p) = -\frac{\overline{\sigma}\cdot\overline{p}}{p_0}u_L(p)\,, \qquad u_R(p) = +\frac{\overline{\sigma}\cdot\overline{p}}{p_0}u_R(p)\,, \tag{3.22}$$

i.e. $u_L(p)$ is an incoming particle with spin opposite to its momentum, while $u_R(p)$ is an incoming particle with spin along its momentum. For massless particles, $u_L(p)$ also corresponds to an outgoing right-handed anti-particle and $v_R(p)$ to an outgoing left-handed anti-particle. Outgoing particles are conjugated spinors $\overline{u}_{L,R}(p)$. Again, for massless particles $\overline{u}_{L,R}(p)$ can also denote incoming anti-particles with handedness switched $\overline{u}_{L,R}(p) = \overline{v}_{R,L}(p)$.

One can show that an explicit solution to the Weyl equation is given by

$$u_L(p) = e^{i\alpha} \begin{pmatrix} -\sin\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix} . \tag{3.23}$$

We will see soon that the overall phase is connected to the way spinors transform under the Little group.

Spinor Helicity for Dirac Spinors

We will now switch to spinor helicity notation. Following our previous conventions for dotted and undotted indices, in the chiral representation we denote $u_L(p) = u_a$ and $u_R(p) = u^{\dot{a}}$ and write

$$\begin{aligned} U_L = |p] = N_p \begin{pmatrix} u_a \\ 0 \end{pmatrix}; \quad U_R = |p\rangle = N_p \begin{pmatrix} 0 \\ u^{\dot{a}} \end{pmatrix}; \\ \overline{U}_L = \langle p| = N_p \begin{pmatrix} 0 & u_{\dot{a}} \end{pmatrix}; \quad \overline{U}_R = [p| = N_p \begin{pmatrix} u^a & 0 \end{pmatrix} . \end{aligned} \tag{3.24}$$

Note the slight abuse of notation: while we previously denoted right- and left-handed two-dimensional Weyl spinors as $u]_a$ etc., we now dropped the square and angle brackets for the Weyl spinors and used them to denote four-dimensional spinors instead. In many references, the notation $u_a = |p]_a$ is employed. In the end, the two notations can be used equivalently. One only has to keep in mind whether two- or four-dimensional spinors are meant in some particular cases.

Having introduced the spinor helicity notation for Dirac spinors, one can easily show that

$$\langle pq] = [pq\rangle = 0; \quad \langle pp\rangle = [pp] = 0 . \tag{3.25}$$

The first equation follows from simply multiplying the vectors. For the second equation, one has to use the antisymmetry of the Weyl spinors

$$\langle pp\rangle = N_p^2 u_{\dot{a}} u^{\dot{a}} = 0 . \tag{3.26}$$

Moreover, notice that for real momenta

$$\langle pq\rangle^* = (\overline{U}_L U_R)^* = \overline{U}_R U_L = [qp] \tag{3.27}$$

and consequently by antisymmetry

$$\langle pq\rangle[qp] = -|\langle pq\rangle|^2 = -|[pq]|^2 . \tag{3.28}$$

Before figuring out what $\langle pq\rangle$ and $[qp]$ are, we have to derive some other identities. We start with the completeness relation

$$\sum_\lambda u_\lambda(p)\overline{u}_\lambda(p) = \not{p} = |p\rangle[p| + |p]\langle p| , \tag{3.29}$$

which simply follows from the definition of angle and square bracket spinors. Using (3.25) and (3.29) one can write

$$\langle pq\rangle[qp] = \langle p\big(|q\rangle[q| + |q\rangle[q|\big)p] = \langle p|\not{q}|p]$$

This can be computed realizing that any string of spinors beginning with an angle(square) bracket of momentum p and ending with square(angle) bracket of the same momentum, can be rewritten as a trace. It is easy to see how this works in the case at hand

$$\begin{aligned}\langle p|\not{q}|p] =& \overline{u}_L(p)\not{q}u_L(p) = \overline{u}(p)\left(\frac{1+\gamma_5}{2}\right)\not{q}\left(\frac{1-\gamma_5}{2}\right)u(p)\\ =& \overline{u}_a\left(\frac{1+\gamma_5}{2}\right)_{ab}\not{q}_{bc}\left(\frac{1-\gamma_5}{2}\right)_{cd}u_d\\ =& \mathrm{Tr}\left[\not{p}\not{q}\left(\frac{1-\gamma_5}{2}\right)\right] = 2p\cdot q\,,\end{aligned}$$

which makes sense for $p \cdot q > 0$. We will see below how to analytically continue the spinors for negative signs of the momenta. For real momenta, we can use (3.28) to write the spinor products as

$$\langle pq\rangle = e^{i\phi_{pq}}\sqrt{|2p\cdot q|}\,,\quad [pq] = -e^{-i\phi_{pq}}\sqrt{|2p\cdot q|}\,. \tag{3.30}$$

As we will see later on, the fact that spinor products go to zero as square-roots will be extremely convenient to efficiently parametrize the behaviour of amplitudes in so-called collinear limits.

Let us make some final remarks on the notation. We have found for Dirac spinors

$$\not{p} = |p\rangle[p| + |p]\langle p|\,. \tag{3.31}$$

The equivalent formulas for Weyl spinors clearly read

$$|p\rangle^{\dot{a}}[p|^b = p^{\dot{a}b}\,,\qquad |p]_a\langle p|_{\dot{b}} = p_{a\dot{b}}\,. \tag{3.32}$$

This is consistent with Eq. (3.3), which we repeat here for convenience

$$p^{\dot{a}b} = \tilde{\lambda}^{\dot{a}}\lambda^b\,, \tag{3.33}$$

where $\lambda^b = [p|^b$ and $\tilde{\lambda}^{\dot{a}} = |p\rangle^{\dot{a}}$.

Spinor Identities for Dirac Spinors

In order to make the best out of this formalism, we now need to prove some useful identities among spinor products. We already showed that

$$\langle pq\rangle = -\langle qp\rangle;\quad [pq] = -[qp] \qquad \text{(antisymmetry)} \tag{3.34}$$

$$\langle pp\rangle = [pp] = 0 \tag{3.35}$$

$$\langle pq\rangle = [qp]^* \qquad \text{(complex conjugation for real momenta)} \tag{3.36}$$

$$\langle pq\rangle[qp] = 2p\cdot q\,. \tag{3.37}$$

Spinor products also fulfil the following identities:

$$\langle p\gamma^\mu p] = [p\gamma^\mu p\rangle = 2p^\mu \qquad \text{(Gordon identity)} \tag{3.38}$$

$$
\begin{aligned}
&[p\gamma^\mu q\rangle = \langle q\gamma^\mu p] && \text{(Conjugation)} && (3.39)\\
&[p\gamma^\mu q\rangle[k\gamma_\mu l\rangle = 2[pk]\langle lq\rangle && \text{(Fierz identity)} && (3.40)\\
&\langle p\gamma^\mu q]\langle k\gamma_\mu l] = 2\langle pk\rangle[lq] && && \\
&\langle p\gamma^\mu q]\gamma_\mu = 2(|q]\langle p| + |p\rangle[q|) && \text{(Fierz identity)} && (3.41)\\
&|p\rangle[p| = \frac{1+\gamma_5}{2}\not{p} && \text{(projectors)} && (3.42)\\
&|p]\langle p| = \frac{1-\gamma_5}{2}\not{p} && && \\
&\langle pq\rangle\langle kl\rangle = \langle pk\rangle\langle ql\rangle + \langle pl\rangle\langle kq\rangle && \text{(Schouten identity)} && (3.43)\\
&\sum_{i=1}^{n} p_i = 0 && && \\
&\Rightarrow [p|\sum_{i=1}^{n}\not{p}_i|q\rangle = \sum_{i=1}^{n}[pi]\langle iq\rangle = 0 && \text{(n-point momentum conservation)} && (3.44)\\
&\langle p\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}q] = [q\gamma^{\mu_{2n+1}}...\gamma^{\mu_1}q\rangle && \text{(reversal for odd } \gamma^\mu) && (3.45)\\
&\langle p\gamma^{\mu_1}...\gamma^{\mu_{2n}}q\rangle = -\langle q\gamma^{\mu_{2n}}...\gamma^{\mu_1}q\rangle && \text{(reversal for even } \gamma^\mu) && (3.46)
\end{aligned}
$$

We will now prove some of these identities.

Gordon (3.38)

$$
\begin{aligned}
\langle p|\gamma^\mu|p] &= \overline{u}_L(p)\gamma^\mu u_L(p) = \overline{u}(p)\left(\frac{1+\gamma_5}{2}\right)\gamma^\mu u(p) = \sum_{\text{spins}} u_a\overline{u}_b\left(\frac{1+\gamma_5}{2}\right)_{bc}\gamma^\mu_{ca}\\
&= \mathrm{Tr}\left[\not{p}\left(\frac{1+\gamma_5}{2}\right)\gamma^\mu\right] = 2p^\mu
\end{aligned} \tag{3.47}
$$

Conjugation (3.39)

Let us calculate both sites separately. Let's start with the left-hand site:

$$
\begin{aligned}
[p|\gamma^\mu|q\rangle &= \overline{u}_R(p)\gamma^\mu u_R(q)\\
&= N_pN_q\begin{pmatrix}u^a & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ \overline{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}0\\ u^{\dot{a}}\end{pmatrix}\\
&= N_pN_q u^a(p)(\sigma^\mu)_{a\dot{a}}u^{\dot{a}}(q)
\end{aligned} \tag{3.48}
$$

For the right-hand site, we find similarly:

$$
\begin{aligned}
\langle q|\gamma^\mu|p] &= N_pN_q\begin{pmatrix}0 & u_{\dot{a}}\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ \overline{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}u_a\\ 0\end{pmatrix}\\
&= N_pN_q u_{\dot{a}}(q)(\overline{\sigma}^\mu)^{\dot{a}a}u_a(p)\\
&= N_pN_q u^b(p)u^{\dot{b}}(q)\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}(\overline{\sigma}^\mu)^{\dot{a}a}
\end{aligned} \tag{3.49}
$$

Using $\varepsilon_{\dot{a}\dot{b}}(\overline{\sigma}^\mu)^{\dot{a}a}\varepsilon_{ab} = (-i\sigma^2)_{\dot{a}\dot{b}}(\overline{\sigma}^\mu)^{\dot{a}a}(-i\sigma^2)_{ab} = ((\sigma^\mu)^T)_{\dot{b}b} = (\sigma^\mu)_{b\dot{b}}$, we find that the right-hand site and left-hand site agree.

Fierz (3.40)

$$\begin{aligned}[p|\gamma^\mu|q\rangle[k|\gamma_\mu|l\rangle &= N u^a(p)(\sigma^\mu)_{a\dot a}u^{\dot a}(q)u^b(k)(\sigma_\mu)_{b\dot b}u^{\dot b}(l) \\ &= 2Nu^a(p)u^{\dot a}(q)u^b(k)u^{\dot b}(l)\varepsilon_{ab}\varepsilon_{\dot a\dot b} \\ &= 2Nu^a(p)u_a(k)u_{\dot a}(l)u^{\dot a}(q) \\ &= 2[pk]\langle lq\rangle\end{aligned} \tag{3.50}$$

Where we introduced the shorthand $N = N_p N_q N_l N_k$. In the second line, we use $(\sigma^\mu)_{a\dot a}(\sigma_\mu)_{b\dot b} = 2\varepsilon_{ab}\varepsilon_{\dot a\dot b}$, which we will show in the exercises. The other Fierz identity follows analogously.

Projectors (3.42)

We have seen that

$$|p\rangle[p| + |p]\langle p| = u_R\overline{u}_R + u_L\overline{u}_L = \not{p}\,. \tag{3.51}$$

Moreover, we have $P_R u_R = u_R, P_R u_L = 0, P_L u_R = 0, P_L u_L = u_L$, leading to

$$|p\rangle[p| = \frac{1+\gamma_5}{2}\not{p} \qquad \text{and} \qquad |p]\langle p| = \frac{1-\gamma_5}{2}\not{p}\,. \tag{3.52}$$

Schouten identity (3.43)

Since spinors are two-dimensional objects, only two spinors can in general be linearly independent. This is the statement that is usually referred to as Schouten identity. Consider now four spinors associated to the four vectors p, q, k, l. We can select two of them to be independent and write $|q\rangle = A\,|k\rangle + B\,|l\rangle$. Contracting this identity with $\langle k|$ and $\langle l|$ one gets

$$\langle kq\rangle = B\,\langle kl\rangle \quad \text{and} \quad \langle lq\rangle = B\,\langle lk\rangle\ , \tag{3.53}$$

which in turn implies

$$|p\rangle = \frac{\langle kq\rangle}{\langle kl\rangle}\,|l\rangle + \frac{\langle lq\rangle}{\langle lk\rangle}\,|k\rangle\ . \tag{3.54}$$

Multiplying this equation through with $\langle kl\rangle\,\langle p|$ and using asymmetry we find (3.43).

Analytic Continuation of the Spinors

While we mostly work in all-incoming or all-outgoing kinematics, real scattering amplitudes involve both particles in the initial and final state. Moreover, as we will see later on, once we deal with loops we will have to consider virtual particles flowing from one side to the other of a so-called *cut*, and this will require us having a way to *analytically continue* the spinors when $p^\mu \to -p^\mu$. There are different conventions we can follow, and various books use different choices. Here we use the uniform convention

$$\langle -p| = i\langle p|\,, \quad [-p| = i[p|\,, \quad |-p\rangle = i|p\rangle\,, \quad |-p] = i|p]\,. \tag{3.55}$$

Clearly this analytic continuation just consists of redefining the spinors by a phase $|-p\rangle = e^{i\pi/2}|p\rangle$ etc, and is consistent with

$$\not{p} = |p\rangle\,[p| + |p]\,\langle p| \quad \to \quad -\not{p} = |{-p}\rangle\,[-p| + |{-p}]\,\langle -p| = -\left(|p\rangle\,[p| + |p]\,\langle p|\right)\,. \tag{3.56}$$

Note that this is also consistent with the formula we found for the product of a spinor product and its complex conjugated

$$\langle pq\rangle[qp] = 2p\cdot q \quad \to \quad \langle(-p)q\rangle[q(-p)] = -\langle pq\rangle[qp] = -2p\cdot q\,. \tag{3.57}$$

Transformation under Parity

Parity acts by reversing all helicities of the external particles. Take a Dirac spinor in the Chiral representation which with our conventions reads

$$u(E,\vec{p}) = \begin{pmatrix} u_L(E,\vec{p}) \\ u_R(E,\vec{p}) \end{pmatrix} = \begin{pmatrix} u_a(E,\vec{p}) \\ u^{\dot{a}}(E,\vec{p}) \end{pmatrix}$$

Under Parity we have $\vec{p} \to -\vec{p}$ and therefore using the standard transformations of spinors we see that

$$u(E,\vec{p}) \xrightarrow{P} u(E,-\vec{p}) = \gamma^0 u(E,\vec{p}) = \begin{pmatrix} 0 & \delta^{\dot{b}}_{\dot{a}} \\ \delta^a_b & 0 \end{pmatrix} \begin{pmatrix} u_a(E,\vec{p}) \\ u^{\dot{a}}(E,\vec{p}) \end{pmatrix} = \begin{pmatrix} u^{\dot{b}}(E,\vec{p}) \\ u_b(E,\vec{p}) \, , \end{pmatrix} \tag{3.58}$$

where $\mathbb{1} = \delta^a_b = \delta^{\dot{a}}_{\dot{b}}$. In terms of spinor products introduced above, this means effectively

$$|p\rangle \xrightarrow{P} |p] \, , \qquad |p] \xrightarrow{P} |p\rangle \tag{3.59}$$

so that we see that parity swaps angle and square brackets $\langle\rangle \leftrightarrow []$

Transformation under Charge Conjugation

Charge conjugation exchanges a quark and anti-quark without changing their spin. In the massless case, this is is equivalent to a flip of the helicity of the quark line. Indeed, using the relation between antiparticles and particle we find

$$u(E,\vec{p}) \xrightarrow{C} v(E,\vec{p}) = (-i\gamma^2)(u(E,\vec{p}))^* = \begin{pmatrix} 0 & (i\sigma_2)_{ba} \\ (-i\sigma_2)^{\dot{b}\dot{a}} & 0 \end{pmatrix} \begin{pmatrix} u_{\dot{a}}(E,\vec{p}) \\ u^a(E,\vec{p}) \end{pmatrix} \tag{3.60}$$

$$= \begin{pmatrix} u^{\dot{b}}(E,\vec{p}) \\ u_b(E,\vec{p}) \end{pmatrix} = \gamma^0 u(E,\vec{p}) \tag{3.61}$$

The charge conjugation operator therefore acts on the spinor products in the same way as Parity, by swapping angle and square brackets

$$|p\rangle \xrightarrow{C} |p] \, , \qquad |p] \xrightarrow{C} |p\rangle \, . \tag{3.62}$$

Notice that this is consistent with the action of the charge conjugation operator on spinor fields which reads

$$C\psi(x)C^{-1} = -i\gamma^2\gamma^0(\bar{\psi})^T = -i\gamma^2\gamma^0(\psi^\dagger\gamma^0)^T = -i\gamma^2\psi^*$$

which is defined up to a phase.

Little Group Scaling

We started this discussion searching for a way to represent scattering amplitudes, which makes their transformation properties under the little group manifest. Let us now get back to this. Recall, that given a momentum p^μ, the little group is the set of transformations that leaves p^μ invariant. We also found that in the Dirac representation of the Lorentz group we can write a four-momentum as

$$\not{p} = |p\rangle [p| + |p] \langle p| \, . \tag{3.63}$$

Now it's easy to see that for p to be invariant, there is only one admissible transformation

$$|p\rangle \to z\,|p\rangle \, ; \quad |p] \to \frac{1}{z}|p] \, , \tag{3.64}$$

where $z \in \mathbb{C}$. Notice that both $\langle p|$ and $|p\rangle$ must transform in the same way (and similarly $[p|$ and $|p]$), which in turn implies

$$\langle p| \to z\,\langle p| \, ; \quad [p| \to \frac{1}{z}[p| \, . \tag{3.65}$$

Now, since real momenta we have $|p\rangle^* = [p|$, then it's easy to see z has to be a phase

$$|p\rangle^* = z^*\,|p\rangle^* = \frac{1}{z}[p| \, , \quad \to \quad |z|^2 = 1 \, , \quad z = e^{i\phi} \, , \tag{3.66}$$

i.e. the Little Group acts as a phase change on the spinors, as already anticipated.

3.3 Spin-1 Particles

We will be mainly interested in calculations in Yang Mills theories. We now know how to nicely represent massless spin-1/2 particles (quarks and leptons), but we still have not discussed what we can say about massless gauge bosons, as photons and gluons. It is actually relatively easy to derive a representation for the polarization vectors of massless spin-1 particles using spinor-helicity formalism.

As for spinors, we assume all particles are incoming. Since we are interested in representing on-shell physical particles, we assume we are working in a class of physical gauges usually referred to as *axial gauges*, which are specified by requiring that the polarization vector is orthogonal to a fixed direction r^μ. We will also assume that $r^2 = 0$, effectively working in a so-called light-cone gauge. In this class of gauges, polarization vectors fulfil

$$\begin{aligned} \varepsilon_\mu (\varepsilon^\mu)^* &= -1 \ \ \text{(Normalization)} \, , \\ p_\mu \varepsilon^\mu &= 0 \ \ \text{(Transversality)} \, , \\ r_\mu \varepsilon^\mu &= 0 \ \ \text{(Gauge fixing)} \, . \end{aligned} \tag{3.67}$$

Notice that the two conditions of transversality and gauge fixing impose that the polarization is confined in a 2-dimensional plane, as expected. With this choice the completeness relation for the polarizations reads

$$\sum_{\lambda \text{ pol}} \varepsilon^\mu_\lambda \varepsilon^{*\nu}_\mu = -g^{\mu\nu} + \frac{k^\mu r^\nu + k^\nu r^\mu}{k \cdot r} \, . \tag{3.68}$$

We can construct an explicit representation for the polarization vectors in this class of gauges as follows. We start by defining two four-vectors

$$\eta^\mu_1 = [r\gamma^\mu p\rangle \ \text{ and } \ \eta^\mu_2 = \langle r\gamma^\mu p], \qquad \text{with} \qquad (\eta^\mu_1)^* = \eta^\mu_2 \, , \tag{3.69}$$

which clearly satisfy

$$\eta^\mu_{1,2} r_\mu = \eta^\mu_{1,2} p_\mu = 0 \, . \tag{3.70}$$

It is then easy to see that η_1 and η_2 are orthogonal but not properly normalized, i.e.

$$(\eta_1^\mu)^* \eta_{2\mu} = 0\,, \qquad \eta_1^* \cdot \eta_1 = \langle r\gamma^\mu p][r\gamma_\mu p\rangle = 2\,\langle rp\rangle\,[rp]\,. \tag{3.71}$$

Now, this implies that $\eta_{1,2}^\mu$ span the space orthogonal to the two momenta p^μ, r^μ, which is exactly where a transverse, physical polarization vector for a massless spin-1 particle is supposed to be defined. In fact, given two massless vectors p^μ and r^μ, every four-dimensional vector can be decomposed as

$$v^\mu = \alpha p^\mu + \beta r^\mu + \gamma \eta_1^\mu + \delta \eta_2^\mu\,. \tag{3.72}$$

With this, it is the natural to define for the two polarizations vector through the two orthogonal vectors

$$\begin{aligned} \varepsilon_1^\mu(p,r) &= -\frac{[r\gamma^\mu p\rangle}{\sqrt{2}[rp]} = (\varepsilon_-^\mu)^* = \varepsilon_+^\mu\,, \\ \varepsilon_2^\mu(p,r) &= +\frac{\langle r\gamma^\mu p]}{\sqrt{2}\,\langle rp\rangle} = (\varepsilon_+^\mu)^* = \varepsilon_-^\mu\,, \end{aligned} \tag{3.73}$$

where we remind that we working in the the all-incoming convention. Together with having the expected orthogonality properties $\varepsilon_{1,2}^\mu p_\mu = \varepsilon_{1,2}^\mu r_\mu = 0$, we also easily see that they are properly normalized

$$\begin{aligned} (\varepsilon_+^\mu)^*(\varepsilon_{+\mu}) &= \frac{\langle r\gamma^\mu p]}{2\,\langle pr\rangle}\frac{\langle p\gamma_\mu r]}{[pr]} = \frac{\langle rp\rangle\,[rp]}{\langle pr\rangle\,[rp]} = -1\,, \quad \text{and} \\ (\varepsilon_+^\mu)^*(\varepsilon_{-\mu}) &= (\varepsilon_-^\mu)^*(\varepsilon_{+\mu}) = 0\,. \end{aligned} \tag{3.74}$$

Hence, $\varepsilon_{+,-}^\mu$ produces states with helicity $+,-$, respectively, as one can check explicitly by acting on these states with the helicity operator.

Let us make another comment about their gauge covariance. In particular, we would like to see explicitly that changing the vector r^μ corresponds to a gauge transformation in the class of the light-cone axial gauges which we are using. Take two polarization vectors $\varepsilon_\mu^-(p,r)$ and $\varepsilon_\mu^-(p,q)$ depending on two different gauge vectors r^μ and q^μ and compute their difference. We then have

$$\begin{aligned} \varepsilon_\mu^-(p,r) - \varepsilon_\mu^-(p,q) &= \frac{1}{\sqrt{2}}\left[\frac{\langle r\gamma^\mu p]}{\langle rp\rangle} - \frac{\langle q\gamma^\mu p]}{\langle qp\rangle}\right] \\ &= \frac{1}{\sqrt{2}}\frac{\langle r\gamma^\mu p]\,\langle qp\rangle - \langle q\gamma^\mu p]\,\langle rp\rangle}{\langle rp\rangle\,\langle qp\rangle} \\ &= -\frac{1}{\sqrt{2}}\frac{\langle r\gamma^\mu \not{p} q\rangle - \langle q\gamma^\mu \not{p} r\rangle}{\langle rp\rangle\,\langle qp\rangle} \\ &= -\frac{1}{\sqrt{2}}\frac{\langle r(\gamma^\mu \not{p} + \not{p}\gamma^\mu)q\rangle}{\langle rp\rangle\,\langle qp\rangle} \\ &= -\frac{1}{\sqrt{2}}\left[\frac{2\,\langle rq\rangle}{\langle rp\rangle\,\langle qp\rangle}\right]p^\mu\,, \end{aligned} \tag{3.75}$$

where we used $\gamma^\mu \not{p} + \not{p}\gamma^\mu = p^\nu\{\gamma^\mu,\gamma^\nu\} = 2p^\mu$. Hence, we find that the difference of the two polarization vectors is proportional to p^μ. Now recall that polarization vectors only appear in the amplitude as $\varepsilon_\mu M^\mu$ and the Ward identity states that, as long as all other gauge bosons are on-shell,

$$p_\mu M^\mu = 0\,. \tag{3.76}$$

This shows that changing the arbitrary vector r^μ corresponds to a gauge transformation and, through the Ward identities, does not affect the final result for the scattering amplitude. We stress here a very important fact: *we are allowed to choose a different gauge vector for each external gauge boson, independently!* This will be very important later on to simplify complex calculation.

Following the discussion in section 3.2 we can study the transformation of $\varepsilon^\mu_\pm$ under the little group. As the polarization depends on two momenta, we can transform both independently and we see that

$$\begin{aligned}
\varepsilon^\mu_+ &= -\frac{[r\gamma^\mu p\rangle}{\sqrt{2}[rp]} \Rightarrow \begin{cases} \text{LG}(p) & \varepsilon^\mu_+ \to z^2\varepsilon^\mu_+ \\ \text{LG}(r) & \varepsilon^\mu_+ \to \varepsilon^\mu_+ \end{cases} \\
\varepsilon^\mu_- &= +\frac{\langle r\gamma^\mu p]}{\sqrt{2}\langle rp\rangle} \Rightarrow \begin{cases} \text{LG}(p) & \varepsilon^\mu_- \to z^{-2}\varepsilon^\mu_- \\ \text{LG}(r) & \varepsilon^\mu_- \to \varepsilon^\mu_- \end{cases} \quad ,
\end{aligned} \tag{3.77}$$

where $LG(q)$ means that we have performed a little group transformation on the momentum q^μ. We see that the polarization vectors scale as twice a spin-1/2 particle under little group transformations on the momentum p^μ, which indicates these are particles of spin-1. Moreover, they are invariant under the Little group of r, another manifestation of the fact that r^μ is just a gauge momentum.

Notice, that all relations among spinor products must respect the correct scaling, which provides a very powerful way to check manipulations on spinor products and scattering amplitudes in general. In fact, we know that scattering amplitudes must transform under the little group. Consequently, this representation allows us to constraint the form of (massless) scattering amplitudes in terms of which $|p\rangle$ or $|p]$ are allowed to appear. Since the scattering amplitude for massless particles is a function of momenta and polarizations, it can be rewritten as combinations of angle and square brackets only. For each external particle, the amplitude must then scale properly under the little group:

$$A(p_1, ..., p_i, ..., p_N) \to A(p_1, ..., W^\mu p_{i\mu}, ..., p_N) = z^{+2h_i} A(p_1, ..., p_i, ..., p_N)$$

where h_i is the helicity of particle i. For a fermion of helicity $-\frac{1}{2}$, this gives $1/z$, i.e. the fermion is left-handed. Equivalently, for $h = +\frac{1}{2}$, this gives a factor of z corresponding to a right-handed particle.

The action of parity on the scattering amplitude

When studying the action of parity on spin-1/2 objects, we have seen that it swaps angle and square brackets. How can we implement its action on a scattering amplitude corresponding to a mixture of spin-1/2 and spin-1 particles? To answer this question, we first notice that by simply changing angle to square brackets, we don not fully implement parity on the bosons. Parity should swap plus and minus helicities, which is obtained by swapping brackets and with an extra minus sign, in fact:

$$\varepsilon^\mu_+ = -\frac{[r\gamma^\mu p\rangle}{\sqrt{2}[rp]} \quad \xrightarrow{\langle\rangle \leftrightarrow []} \quad -\frac{\langle r\gamma^\mu p]}{\sqrt{2}\langle rp\rangle} = -\varepsilon^\mu_- \tag{3.78}$$

$$\varepsilon_-^\mu = +\frac{\langle r\gamma^\mu p]}{\sqrt{2}\langle rp\rangle} \quad \stackrel{\langle\rangle\leftrightarrow[]}{\longrightarrow} \quad +\frac{[r\gamma^\mu p\rangle}{\sqrt{2}[rp]} = -\varepsilon_+^\mu \,. \tag{3.79}$$

So we can write

$$\varepsilon_+^\mu \stackrel{P}{\longrightarrow} -(\varepsilon_+^\mu)_{\langle\rangle\leftrightarrow[]}\,, \qquad \varepsilon_-^\mu \stackrel{P}{\longrightarrow} -(\varepsilon_-^\mu)_{\langle\rangle\leftrightarrow[]}\,.$$

With this, consider now an amplitude which involves n gluons and $N-n$ fermions and assume the amplitude is written in terms of spinor products. The rule to transform it under parity can then be written as

$$A(\{p_1,...,p_n\},\{p_{n+1},...,p_N\}) \ \stackrel{P}{\longrightarrow}\ (-1)^n \left[A(\{p_1,...,p_n\},\{p_{n+1},...,p_N\})\right]_{\langle\rangle\to[]}\,. \tag{3.80}$$

It's easy to convince yourself that an amplitude with only external fermions and no external bosons, always contains an even number of spinor products.[2] On the other hand, each vector boson adds one single spinor product (in the denominator of eqs. (3.73)), so a simple rule to account for the action of parity on an amplitude with bosons and fermions is to swap all square and angle brackets also *swapping the order of the momenta in each spinor product.* This is exactly the action of *complex conjugation* on the spinor products! So we could equally state that *parity acts by complex conjugating all spinor products in the amplitude.* In formulas

$$\begin{aligned} A(\{p_1,...,p_n\},\{p_{n+1},...,p_N\}) \ &\stackrel{P}{\longrightarrow}\ \left[A(\{p_1,...,p_n\},\{p_{n+1},...,p_N\})\right]_{\langle ij\rangle\leftrightarrow[ji]} && (3.81)\\ &= \left[A(\{p_1,...,p_n\},\{p_{n+1},...,p_N\})\right]^* && (3.82)\end{aligned}$$

where complex conjugation acts only on the spinors.

3.4 Spinor Helicity Formalism in QED

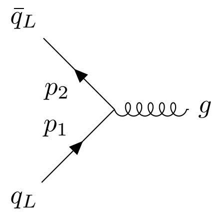


Figure 3: Feynman Diagram of the left-handed fermion line $\langle 2\gamma_\mu 1]$.

Let's see now how the spinor helicity formalism can be helpful to compute tree-level amplitudes of massless particles. We start with two examples in QED.

Example 1. $e^+e^- \to \mu^+\mu^-$

We consider the scattering of two massless electrons into two massless muons in QED as given in fig. 4. As always we assume all momenta incoming.

[2]We will see many examples of this below, but in general this is because external fermions always come in pairs, i.e. fermion lines always start with a fermion and end with an antifermion (in all incoming kinematics).

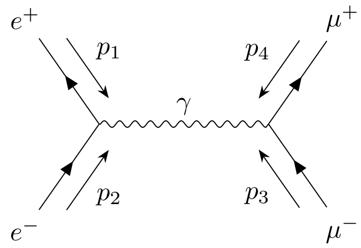


Figure 4: Feynman diagram of $e^+e^- \to \mu^+\mu^-$ at tree-level.

Using standard QED Feynman rules, the amplitude is

$$iM_{\lambda_1\lambda_2\lambda_3\lambda_4} = \frac{-ie^2}{q^2}\overline{u}_{\lambda_1}(p_1)\gamma^\mu u_{\lambda_2}(p_2)\overline{u}_{\lambda_4}(p_4)\gamma_\mu u_{\lambda_3}(p_3) \tag{3.83}$$

with $q = (p_1 + p_2)^2 = (p_3 + p_4)^2$. Electrons and muons can have two polarizations each, which we call L or R. Let us use spinor-helicity to compute the so-called *helicity amplitudes*. Helicity must be conserved along a massless fermion line, and in fact we see that there are 4 possible non-zero combinations, which we can easily compute directly from (3.83)

$$\begin{aligned}
M_{LL,LL} &= -\frac{e^2}{q^2}\langle 1\gamma^\mu 2]\langle 4\gamma_\mu 3] = -\frac{2e^2}{q^2}\langle 14\rangle\,[32] \\
M_{LL,RR} &= -\frac{e^2}{q^2}\langle 1\gamma^\mu 2][4\gamma_\mu 3\rangle = -\frac{e^2}{q^2}\langle 1\gamma^\mu 2]\langle 3\gamma_\mu 4] = -\frac{2e^2}{q^2}\langle 13\rangle\,[42] \\
M_{RR,LL} &= -\frac{e^2}{q^2}[1\gamma^\mu 2\rangle\langle 4\gamma_\mu 3] = -\frac{e^2}{q^2}[1\gamma^\mu 2\rangle[3\gamma_\mu 4\rangle = -\frac{2e^2}{q^2}[13]\,\langle 42\rangle \\
M_{RR,RR} &= -\frac{e^2}{q^2}[1\gamma^\mu 2\rangle[4\gamma_\mu 3\rangle = -\frac{2e^2}{q^2}[14]\,\langle 32\rangle\ .
\end{aligned}$$

All other combinations, which involve an helicity flip along the fermion line, are identically zero by direct calculation.

Clearly, not all combinations are independent. QED is invariant under CP transformations and using the transformation properties under parity derived in section 3.3, as expected

$$M_{LL,LL} = (M_{RR,RR})^*\ , \qquad M_{RR,LL} = (M_{LL,RR})^*\ . \tag{3.84}$$

It is also easy to see that all amplitudes scale as expected under the little group transformation associated to each external particle. For example, for $M_{LL,LL}$ we see that by doing a little group transformation on p_1 or p_4, the amplitude scales as $\mathcal{M} \to z\mathcal{M}$, while the same transformation on p_2 or p_3 generate the opposite scaling $\mathcal{M} \to 1/z\mathcal{M}$, as expected (remember that particles and antiparticles scale in the opposite way). Finally, one can see that the helicity amplitudes squared (and therefore the cross-sections) are individually invariant under little group transformations.

Introducing the usual Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$ and $u = (p_2 + p_3)^2$ it is easy to see that

$$|M_{LL,LL}|^2 = |M_{RR,RR}|^2 = \frac{4e^4}{q^4}\langle 14\rangle\,[32][41]\,\langle 23\rangle = \frac{4e^4}{q^4}(2p_1\cdot p_4)(2p_2\cdot p_3) = \frac{4e^4}{s^2}u^2\,.$$

Similarly, we get

$$|M_{LL,RR}|^2 = |M_{RR,LL}|^2 = \frac{4e^4}{q^4} \langle 13 \rangle [42][31] \langle 24 \rangle = \frac{4e^4}{q^4}(2p_1 \cdot p_3)(2p_2 \cdot p_4) = \frac{4e^4}{s^2} t^2 \, .$$

From these results, we can compute the spin averaged amplitude

$$\begin{aligned} \frac{1}{4} \sum_{\text{spins}} |M|^2 &= \frac{1}{4} 2(|M_{LL,LL}|^2 + |M_{LL,RR}|^2) \\ &= 2e^4 \left(\frac{t^2 + u^2}{s^2} \right) = e^4 \left(1 + \cos^2 \theta\right) , \end{aligned} \tag{3.85}$$

where we parametrized $t = -\frac{s}{2}(1 - \cos\theta)$ and $u = -\frac{s}{2}(1 + \cos\theta)$.

Example 2. $e^+e^- \to \gamma\gamma$

As a second example, we consider the scattering of an electron and a positron to produce two photons (which are spin-1 particles). Again, we assume all particles to be incoming. The two corresponding Feynman diagrams are depicted in fig.5.

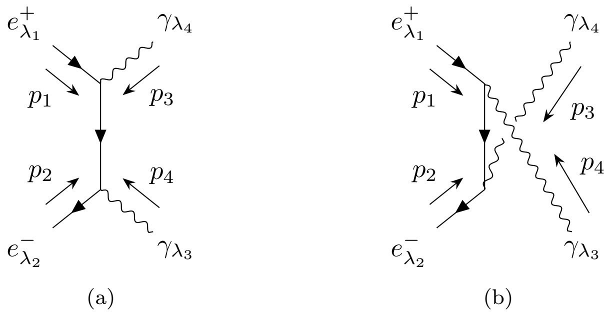


Figure 5: Feynman diagrams of $e^+e^- \to \gamma\gamma$ at tree-level

The Amplitude is given by

$$iM_{\lambda_1\lambda_2\lambda_3\lambda_4} = -ie^2 \overline{u}_{\lambda_2}(p_2) \Big[\Gamma_{\mu\nu} \Big] u_{\lambda_1}(p_1) \varepsilon^{\mu}_{\lambda_3} \varepsilon^{\nu}_{\lambda_4} \, , \tag{3.86}$$

where

$$\Gamma_{\mu\nu} = \frac{\gamma_\nu(\not{p}_2 + \not{p}_4)\gamma_\mu}{(p_2 + p_4)^2} + \frac{\gamma_\nu(\not{p}_2 + \not{p}_3)\gamma_\mu}{(p_2 + p_3)^2} = \frac{\gamma_\nu(\not{p}_2 + \not{p}_4)\gamma_\mu}{t} + \frac{\gamma_\nu(\not{p}_2 + \not{p}_3)\gamma_\mu}{u} \, . \tag{3.87}$$

Again, helicity along massless fermion lines is conserved, which implies $\lambda_1 = \lambda_2$ for the electron and the positron. This can be seen explicitly, as there are always three γ-matrices between the two spinors. Hence, we have 2 possible helicities for the fermions, times 2×2 for the photons, leaving us with a total of 8 helicity amplitudes. As for the previous example, invariance under CP transformations allows us to compute only half of these configurations. Let us start by fixing the fermion helicities in the two independent ways, the amplitudes then become

$$\begin{aligned} iM_{LL,\lambda_3\lambda_4} &= -ie^2 \langle 2| \, \Gamma_{\mu\nu} |1] \varepsilon^{\mu}_{\lambda_3} \varepsilon^{\nu}_{\lambda_4} \\ iM_{RR,\lambda_3\lambda_4} &= -ie^2 [2| \Gamma_{\mu\nu} \, |1\rangle \, \varepsilon^{\mu}_{\lambda_3} \varepsilon^{\nu}_{\lambda_4} \, . \end{aligned} \tag{3.88}$$

Consider now the case where the two photons have the same helicity. For two photons of helicity $+$, the polarization vectors are given by

$$\varepsilon_+^\mu(p_3, q_3) = -\frac{[q_3\gamma^\mu 3\rangle}{\sqrt{2}[q_3 3]}\,, \qquad \varepsilon_+^\nu(p_4, q_4) = -\frac{[q_4\gamma^\nu 4\rangle}{\sqrt{2}[q_4 4]}\,, \tag{3.89}$$

where we have left the gauge vectors q_i unspecified for now. In order to compute explicitly the amplitudes in Eq. (3.88), we start by obtaining an expression for the polarization vectors contracted with γ^μ. Using Little Group scaling we can write the following general Ansatz

$$\gamma^\mu \frac{[q_3\gamma_\mu 3\rangle}{\sqrt{2}[q_3 3]} = \frac{1}{\sqrt{2}[q_3 3]}\Big[A|q_3]\,\langle 3| + B\,|3\rangle\,[q_3|\Big]\,. \tag{3.90}$$

The coefficients A and B can be determined by taking Matrix elements with generic momenta

$$\begin{aligned}\langle k\left(\gamma^\mu \frac{[q_3\gamma_\mu 3\rangle}{\sqrt{2}[q_3 3]}\right) l] = \frac{\langle k\gamma^\mu l][q_3\gamma_\mu 3\rangle}{\sqrt{2}[q_3 3]} = \frac{2\langle k3\rangle[q_3 l]}{\sqrt{2}[q_3 3]} &\stackrel{!}{=} \frac{B\langle k3\rangle[q_3 l]}{\sqrt{2}[q_3 3]}\\ \Rightarrow B = 2\,,\end{aligned} \tag{3.91}$$

where we used the Fierz identity in the third step. Similarly, for A

$$\begin{aligned}[k|\left(\gamma^\mu \frac{[q_3\gamma_\mu 3\rangle}{\sqrt{2}[q_3 3]}\right)|l\rangle = \frac{[k\gamma^\mu l\rangle[q_3\gamma_\mu 3\rangle}{\sqrt{2}[q_3 3]} = \frac{2[kq_3]\,\langle 3l\rangle}{\sqrt{2}[q_3 3]} &\stackrel{!}{=} \frac{A[kq_3]\,\langle 3l\rangle}{\sqrt{2}[q_3 3]}\\ \Rightarrow A = 2\,,\end{aligned} \tag{3.92}$$

where we again used the Fierz identity. Using then Eq. (3.90), the contraction of the polarization vectors with the gamma matrices gives

$$\begin{aligned}\not\varepsilon_+(p_3, q_3) &= \gamma_\mu \varepsilon_+^\mu(p_3, q_3) = -\frac{\sqrt{2}}{[q_3 3]}\big(|q_3]\,\langle 3| + |3\rangle\,[q_3|\big)\,,\\ \not\varepsilon_+(p_4, q_4) &= \gamma_\nu \varepsilon_+^\nu(p_4, q_4) = -\frac{\sqrt{2}}{[q_4 4]}\big(|q_4]\,\langle 4| + |4\rangle\,[q_4|\big)\,.\end{aligned} \tag{3.93}$$

Using these equations, we see that in $M_{LL,++}$ only two pieces survive

$$\begin{aligned}\langle 2\,|\not\varepsilon_+(p_4,q_4)(\not p_2 + \not p_4)\not\varepsilon_+(p_3,q_3)\,|1] &\propto \langle 24\rangle\,[q_4(...)3\rangle[q_3 1]\,,\\ \langle 2\,|\not\varepsilon_+(p_3,q_3)(\not p_2 + \not p_3)\not\varepsilon_+(p_4,q_4)\,|1] &\propto \langle 23\rangle\,[q_3(...)4\rangle[q_4 1]\,,\end{aligned} \tag{3.94}$$

where the explicit expression in the brackets (...) is immaterial. It is then easy to see that by making the gauge choice $q_3 = p_1$ and $q_4 = p_1$, one finds $M_{LL,++} = 0$, i.e. the amplitude to produce two photons of equal positive helicity is zero. By using invariance under CP, this also implies $M_{RR,--} = 0$.

We can repeat the calculation for $M_{RR,++}$ and we find

$$\begin{aligned}[2\,|\not\varepsilon_+(p_4,q_4)(\not p_2 + \not p_4)\not\varepsilon_+(p_3,q_3)\,|1\rangle &\propto [2q_4]\langle 4(...)q_3]\langle 31\rangle\,,\\ [2\,|\not\varepsilon_+(p_4,q_4)(\not p_2 + \not p_4)\not\varepsilon_+(p_3,q_3)\,|1\rangle &\propto [2q_3]\langle 3(...)q_4]\langle 41\rangle\,,\end{aligned} \tag{3.95}$$

which again is identically zero if we choose $q_4 = p_2$ and $q_3 = p_2$. By CP invariance this also implies $M_{LL,--} = 0$. So we have shown, that *at tree level*, all helicity amplitudes with photons with equal helicity are all identically zero

$$M_{RR,++} = M_{LL,--} = M_{LL,++} = M_{RR,--} = 0\,. \tag{3.96}$$

We will see more of such cases later on in the course.

Let us proceed by considering the amplitudes where the photons have opposite helicities. Again, CP invariance implies that

$$M_{LL,+-} = M_{RR,-+}\,, \qquad M_{LL,-+} = M_{RR,+-} \tag{3.97}$$

and consequently there are only 2 independent non-zero amplitudes which we need to compute. Repeating the same exercise as before for photons of negative helicities we find

$$\begin{aligned} \not{\varepsilon}_{3-} &= \gamma_\mu \varepsilon^\mu_-(p_3, q_3) = \frac{\sqrt{2}}{\langle q_3 3\rangle}\big(|3]\,\langle q_3| + |q_3\rangle\,[3|\big) \\ \not{\varepsilon}_{4-} &= \gamma_\nu \varepsilon^\nu_-(p_4, q_4) = \frac{\sqrt{2}}{\langle q_4 4\rangle}\big(|4]\,\langle q_4| + |q_4\rangle\,[4|\big)\,. \end{aligned} \tag{3.98}$$

With this, we can now compute M_{LL-+}:

$$M_{LL-+} = e^2\,\langle 2|\left[\frac{\not{\varepsilon}_{4+}(\not{p}_2 + \not{p}_4)\not{\varepsilon}_{3-}}{t} + \frac{\not{\varepsilon}_{3-}(\not{p}_2 + \not{p}_3)\not{\varepsilon}_{4+}}{u}\right]|1]\,. \tag{3.99}$$

Again we exploit the freedom to choose the gauge of the external photons to simplify the computation as much as possible. We notice that by choosing in particular $q_3 = p_2$ and $q_4 = p_1$ only one term for the first diagram (t-channel) survives, while the entire diagram corresponding to the u-channel drops out. We can simplify this further as follows

$$\begin{aligned} M_{LL-+} =& e^2 \frac{2}{\langle 23\rangle\,[14]}\,\frac{\langle 24\rangle\,[1(\not{2} + \not{4})2\rangle[31]}{t} \\ =& e^2 \frac{2}{\langle 23\rangle\,[14]}\,\frac{\langle 24\rangle\,[1\not{4}2\rangle[31]}{t} \\ =& \frac{2e^2}{\langle 23\rangle\,[14]}\,\frac{\langle 24\rangle\,[14]\langle 42\rangle[31]}{t} = -2e^2\frac{\langle 24\rangle^2\,[31]}{\langle 23\rangle\,t}\,, \end{aligned} \tag{3.100}$$

and using $t = \langle 13\rangle\,[13]$, we finally find for the amplitude

$$M_{LL-+} = -2e^2 \frac{\langle 24\rangle^2}{\langle 13\rangle\,\langle 23\rangle}\,. \tag{3.101}$$

Similarly for the other choice of photon helicities, by fixing $q_3 = p_1$ and $q_4 = p_2$, we find

$$M_{LL+-} = -2e^2 \frac{\langle 23\rangle^2}{\langle 14\rangle\,\langle 24\rangle}\,, \tag{3.102}$$

which, as expected, is just M_{LL-+} with $3 \leftrightarrow 4$ exchanged.

We have now everything to compute the amplitude squared

$$\begin{aligned} |M|^2 &= 2\big(|M_{LL+-}|^2 + |M_{LL-+}|^2\big) \\ &= 8e^4\left(\frac{(\langle 23\rangle\,[32])^2}{\langle 24\rangle\,[42]\,\langle 14\rangle\,[41]} + \frac{(\langle 24\rangle\,[42])^2}{\langle 23\rangle\,[32]\,\langle 13\rangle\,[31]}\right) \\ &= 8e^4\left(\frac{u^2}{tu} + \frac{t^2}{ut}\right) = 8e^4\left(\frac{u}{t} + \frac{t}{u}\right)\,. \end{aligned} \tag{3.103}$$

4 Colour Ordering

4.1 A First Example of Colour Ordering: $q\overline{q} \to gg$

We have seen how spinor helicity-formalism simplifies the computation of scattering amplitudes with massless particles in QED, which is an abelian gauge theory based on the group $U(1)$. Let us now discuss what changes in a non-abelian gauge theory as QCD based on $SU(N)$. We will start with an explicit example. We consider the production of two gluons in quark anti-quark annihilation. Again, we take all particles to be incoming

$$q(p_1) + \overline{q}(p_2) + g(p_3) + g(p_4) \to 0\,. \tag{4.1}$$

There are three tree-level diagrams, see Figure 6.

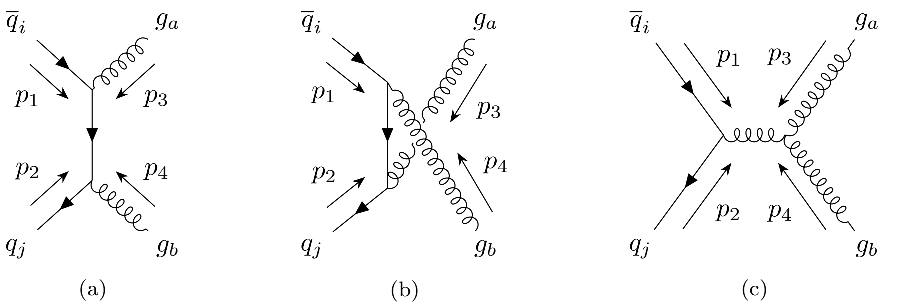


Figure 6: Feynman diagrams of $q\overline{q} \to gg$ at tree-level.

Recall, that the quark-gluon coupling is given by $-ig_s\gamma^\mu t^a_{ij}$, where g_s is the strong coupling, a in an index in the adjoint representation and i, j are in the fundamental representation. The t^a_{ij} are the generators of $SU(N)$ obeying

$$\mathrm{Tr}[t^a t^b] = \frac{\delta^{ab}}{2}\,, \qquad [t^a, t^b]_{ij} = if^{abc}t^c_{ij}\,. \tag{4.2}$$

The first two diagrams correspond to those we already know from QED – we refer to them as the Abelian part $M^{[A]}$. They can just be obtained from $e^+e^- \to \gamma\gamma$ by exchanging $e \leftrightarrow g_s$. Let us show this more explicitly:

$$\begin{aligned} iM^{[A]}_{\lambda_1 1\lambda_2\lambda_3\lambda_4} &= -ig_s^2\overline{u}_{\lambda_2}(p_2)\left[\Gamma_{\mu\nu}\right]u_{\lambda_1}(p_1)\varepsilon^\mu_{\lambda_3}\varepsilon^\nu_{\lambda_4} \\ \text{with}\quad \Gamma_{\mu\nu} &= \frac{\gamma_\nu(\not{p}_2 + \not{p}_4)\gamma_\mu}{t}t^b_{jk}t^a_{ki} + \frac{\gamma_\mu(\not{p}_2 + \not{p}_3)\gamma_\nu}{u}t^a_{jk}t^b_{ki} \end{aligned} \tag{4.3}$$

where the only difference to the QED amplitude is the appearance of the $SU(N)$ generators t^a_{ij}. It is easy to see that, by repeating the arguments in the previous section, that the Abelian part $M^{[A]}$ will only be non-vanishing for opposite gluon helicities.

Let us move on and consider the non-Abelian part corresponding to the third diagram. The 3-gluon (and 4-gluon) vertices are produced by the non-Abelian part of the field strength tensor

$$F_{\mu\nu} = \delta_\mu A_\nu - \delta_\nu A_\mu - \frac{ig}{\sqrt{2}}[A_\mu, A_\nu]\,, \tag{4.4}$$

where $A_\mu = A_\mu^a t_a$. The third diagram contribution to the scattering amplitude reads then

$$
\begin{aligned}
iM^{[NA]}_{\lambda_1 1\lambda_2\lambda_3\lambda_4} =& i g_s^2 \overline{u}_{\lambda_2}(p_2)\gamma^\sigma u_{\lambda_1}(p_1)\left[-i\frac{\delta_{cd}g_{\rho\sigma}}{(p_1+p_2)^2}\right] t^d_{ji} f^{abc} \\
&\times \left[g^{\mu\nu}(p_3^\rho - p_4^\rho) + g^{\nu\rho}(p_4^\mu - p_{12}^\mu) + g^{\mu\rho}(p_{12}^\nu - p_3^\nu)\right]\varepsilon_{3\mu}\varepsilon_{4\nu} \\
=& g_s^2 \overline{u}_{\lambda_2}\gamma_\rho u_{\lambda_1}\frac{f^{abd}}{s}t^d_{ji} \\
&\times \left[\varepsilon_3\cdot\varepsilon_4(p_3-p_4)^\rho + \varepsilon_4^\rho(\varepsilon_3\cdot p_4 - \varepsilon_3\cdot p_{12}) + \varepsilon_3^\rho(\varepsilon_4\cdot p_{12} - \varepsilon_4\cdot p_3)\right] \\
=& g_s^2 \overline{u}_{\lambda_2}\gamma_\rho u_{\lambda_1}\frac{f^{abd}}{s}t^d_{ji}\left[\varepsilon_3\cdot\varepsilon_4(p_3-p_4)^\rho + 2\varepsilon_4^\rho(\varepsilon_3\cdot p_4) + 2\varepsilon_3^\rho(\varepsilon_4\cdot p_{12})\right],
\end{aligned}
\tag{4.5}
$$

where in the last step, we used momentum conservation

$$p_{12} = p_1 + p_2 = -p_3 - p_4 \tag{4.6}$$

and transversality for the external physical gluons $\epsilon_i \cdot p_i = 0$, which allows us to write $\varepsilon_3 \cdot p_{12} = -\varepsilon_3 \cdot p_4$ and $\varepsilon_4 \cdot p_{12} = -\varepsilon_4 \cdot p_3$.

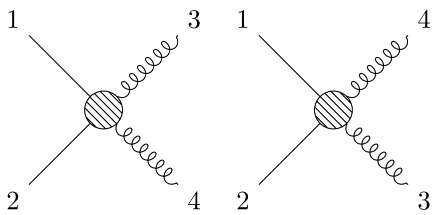
Let us now focus on the colour factors. Using (4.2) we find

$$f^{abd} t^d_{ji} = -i\left[t^a_{jk}t^b_{ki} - t^b_{jk}t^a_{ki}\right] = i t^b_{jk}t^a_{ki} - i t^a_{jk}t^b_{ki}\,. \tag{4.7}$$

These are the same colour factors of diagram 1 and diagram 2. We can therefore write the total amplitude including all three diagrams as

$$M = M_1(t^b t^a)_{ji} + M_2(t^a t^b)_{ji}\,. \tag{4.8}$$

We call M_1 and M_2 *colour ordered amplitudes*. Note, that the non-Abelian part contributes to both colour ordered amplitudes. As we will discuss more in general below, one of the reasons why it is useful to work with colour ordered amplitudes is that they are independently gauge invariant. This can be seen explicitly for the present case, realising that any gauge transformation cannot move terms from M_1 to M_2 or vice versa, since gauge transformations do not affect the colour factors and the two colour factors are independent. The two diagrams can be denoted as $M(1243)$ and $M(1234)$ as depicted in fig.7, where the numbers indicate the ordering of the coloured particles following them in the anti clock-wise direction.



(a) $M(1243)$ (b) $M(1234)$

Figure 7: Colour ordered amplitudes.

Clearly, since $M(1243) = M(1234)|_{3\leftrightarrow 4}$, it is sufficient to compute one of the two pieces to obtain the other by swapping the two gluons. So we only need one of the Abelian diagrams and one piece of the non-Abelian one. Let's do this explicitly, by computing first the second colour ordered amplitude:

$$\begin{aligned} M(1234) = g_s^2 \Big\{ & \overline{u}_{\lambda_2} \not\varepsilon_3^{\lambda_3} \frac{\not p_2 + \not p_3}{s_{23}} \not\varepsilon_4^{\lambda_4} u_{\lambda_1} \\ & + \frac{\overline{u}_{\lambda_2}\gamma^\mu u_{\lambda_1}}{s_{12}} \Big[\varepsilon_3^{\lambda_3} \cdot \varepsilon_4^{\lambda_4} (p_3 - p_4)_\mu + 2\varepsilon_3^{\lambda_3} \cdot p_4 \varepsilon_{4\mu}^{\lambda_4} - 2\varepsilon_4^{\lambda_4} \cdot p_3 \varepsilon_{3\mu}^{\lambda_3} \Big] \Big\} . \end{aligned} \tag{4.9}$$

We can now easily find the other colour ordered amplitude by swapping $3 \leftrightarrow 4$:

$$\begin{aligned} M(1243) = -g_s^2 \Big\{ & \overline{u}_{\lambda_2} \not\varepsilon_4^{\lambda_4} \frac{\not p_1 + \not p_3}{s_{13}} \not\varepsilon_3^{\lambda_3} u_{\lambda_1} \\ & + \frac{\overline{u}_{\lambda_2}\gamma^\mu u_{\lambda_1}}{s_{12}} \Big[\varepsilon_3^{\lambda_3} \cdot \varepsilon_4^{\lambda_4} (p_3 - p_4)_\mu + 2\varepsilon_3^{\lambda_3} \cdot p_4 \varepsilon_{4\mu}^{\lambda_4} - 2\varepsilon_4^{\lambda_4} \cdot p_3 \varepsilon_{3\mu}^{\lambda_3} \Big] \Big\} . \end{aligned} \tag{4.10}$$

Even with the addition of the non-abelian contribution, it is easy to show that for equal photon helicity, the amplitude vanishes, i.e.

$$M(1243)_{LL--} = M(1243)_{LL++} = M(1243)_{RR--} = M(1243)_{RR++} = 0 . \tag{4.11}$$

Therefore, as in QED, there are only two independent helicity configurations that should be computed $(LL-+, LL+-)$ since the other two $(RR+-, RR-+)$ are related by charge and parity conjugation.[3] Let us start by computing the following amplitude:

$$\begin{aligned} M(1234)_{LL-+} = & g_s^2 \Big[\langle 2| \not\varepsilon_3^- \left(\frac{\not p_2 + \not p_3}{s_{23}} \right) \not\varepsilon_4^+ |1] \\ & + \frac{\langle 2\gamma^\mu 1]}{s_1 2} \left(\varepsilon_3^- \cdot \varepsilon_4^+ (p_3 - p_4)^\mu + 2\varepsilon_3^- \cdot p_4 \varepsilon_{4\mu}^+ - 2\varepsilon_4^+ \cdot p_3 \varepsilon_{3\mu}^-) \right) \Big] . \end{aligned} \tag{4.12}$$

Taking $q_3 = p_4$ and $q_4 = p_3$ we have $\varepsilon_3 \cdot p_4 = \varepsilon_4 \cdot p_3 = 0$ and the second and third term in the round bracket vanish. For the first term in the bracket, we can realize that

$$\varepsilon_3^- \cdot \varepsilon_4^+ \sim \langle 4\gamma^\mu 3][3\gamma_\mu 4\rangle = \langle 4\gamma^\mu 3 \langle 4\gamma^\mu 3 = 0 , \tag{4.13}$$

to see that it also vanishes. Hence, with this gauge choice the non-Abelian part does not contribute at all (!), and the amplitude reads

$$M(1234)_{LL-+} = g_s^2 \langle 2| \not\varepsilon_3^- \left(\frac{\not p_2 + \not p_3}{s_{23}} \right) \not\varepsilon_4^+ |1] . \tag{4.14}$$

Using the expressions we found for the polarization vectors in (3.93) and (3.98) the amplitude becomes

$$\begin{aligned} M(1234)_{LL-+} &= -\frac{2g_s^2}{[34]\langle 43\rangle} \frac{1}{s_{23}} \langle 24\rangle [3(\not p_2 + \not p_3) 4\rangle [31] \\ &= -2g_s^2 \frac{\langle 24\rangle [32] \langle 24\rangle [31]}{\langle 43\rangle [34] s_{23}} \\ &= -2g_s^2 \frac{\langle 24\rangle^2 [32][31]}{\langle 43\rangle [34] \langle 23\rangle [32]} \\ &= +2g_s^2 \frac{\langle 24\rangle^2}{\langle 34\rangle \langle 23\rangle} \frac{[31]}{[34]} \end{aligned} . \tag{4.15}$$

[3] We are using the fact that QCD, as QED, is CP invariant. Charge conjugation is required to swap quark and antiquark.

We can manipulate this result a bit further to bring it into a more standard form

$$
\begin{aligned}
M(1234)_{LL-+} &= 2g_s^2 \frac{\langle 24\rangle^2}{\langle 34\rangle\langle 23\rangle}\frac{[31]}{[34]} \\
&= 2g_s^2 \frac{\langle 24\rangle^2}{\langle 34\rangle\langle 23\rangle}\frac{[31]}{[34]}\frac{\langle 12\rangle\langle 41\rangle\langle 24\rangle}{\langle 12\rangle\langle 41\rangle\langle 24\rangle} \\
&= 2g_s^2 \frac{\langle 24\rangle^3\langle 41\rangle}{\langle 12\rangle\langle 34\rangle\langle 23\rangle\langle 41\rangle}\frac{[31]\langle 12\rangle}{[34]\langle 24\rangle}\,.
\end{aligned}
\tag{4.16}
$$

Finally, using momentum conservation

$$
[34]\langle 24\rangle = -[24]\langle 42\rangle = [31]\langle 12\rangle\;, \tag{4.17}
$$

we find

$$
M(1234)_{LL-+} = 2g_s^2 \frac{\langle 24\rangle^3\langle 41\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,. \tag{4.18}
$$

From here, we can also easily check the scaling for the various particles

$$
\begin{aligned}
p_1 &\to z^{1-1-1} = 1/z \\
p_2 &\to z^{3-1-1} = 1/z \\
p_3 &\to z^{-2} \\
p_4 &\to z^{1+3-2} = z^2\,,
\end{aligned}
\tag{4.19}
$$

which is what could have been expected from the helicity of the individual particles in the game.

4.2 Colour Ordering for n-Gluon Amplitudes

Colour ordered amplitudes are simple, and from one amplitude we can obtain the others using crossing symmetries. Let's see how this works for n-gluon amplitudes. First of all, we follow the standard conventions and rescale the colour generators as follows:

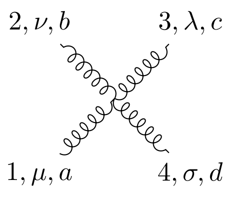
$$
t^a = \frac{T^a}{\sqrt{2}} \Rightarrow \mathrm{Tr}[T^aT^b] = \delta^{ab}\,, \tag{4.20}
$$

which has the practical effect of moving a factor of $\sqrt{2}$ from the colour algebra identities to the Feynman rules.

Let us then start off by considering the simplest gluon amplitude, 4-gluon scattering, taking as usual all momenta to be incoming:[4]

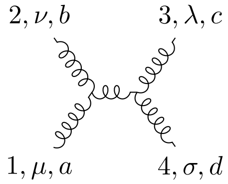
$$
g^a_\mu(p_1) + g^b_\nu(p_2) + g^c_\lambda(p_3) + g^d_\sigma(p_4) \to 0\,. \tag{4.21}
$$

There are four different tree-level diagrams, where we highlight in particular the colour structure

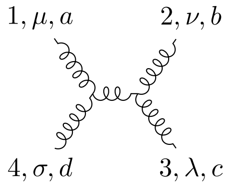


$$
\sim \quad -ig^2\Big[f^{abe}f^{cde}(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\lambda}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\sigma}g^{\nu\lambda}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma})\Big]\,, \tag{4.22}
$$

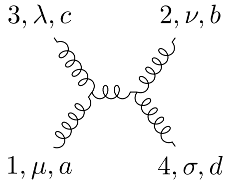
[4]Three gluon scattering for real momenta cannot happen due to momentum conservation. We'll say more about this later in these lectures.



$$\sim f^{abe} f^{cde} \,, \tag{4.23}$$



$$\sim f^{ade} f^{bce} \,, \tag{4.24}$$



$$\sim f^{ace} f^{bde} \,. \tag{4.25}$$

There are three different products of structure constants f^{abc}. In order to rewrite them in standard form, we can use $\mathrm{Tr}[T^a T^b] = \delta^{ab}$ and $[T^a, T^b] = i\sqrt{2} f^{abc} T^c$ to prove

$$\mathrm{Tr}\big([T^a, T^b] T^c\big) = i\sqrt{2} f^{abc} \,, \tag{4.26}$$

which implies for the structure constants f^{abc}

$$\begin{aligned} f^{abc} &= -\frac{i\sqrt{2}}{2} \mathrm{Tr}\big([T^a, T^b] T^c\big) \\ &= -\frac{i}{\sqrt{2}} \Big[\mathrm{Tr}\big(T^a T^b T^c\big) - \mathrm{Tr}\big(T^a T^c T^b\big) \Big] \,, \end{aligned} \tag{4.27}$$

where we used cyclicity of the trace in the last step. Using this relation, each product of the form $f^{abc} f^{cde}$ generates four terms:

$$f^{abe} f^{cde} \sim \Big[\mathrm{Tr}\big(T^a T^b T^e\big) - \mathrm{Tr}\big(T^a T^e T^b\big) \Big] \times \Big[\mathrm{Tr}\big(T^c T^d T^e\big) - \mathrm{Tr}\big(T^c T^e T^d\big) \Big] \,. \tag{4.28}$$

Let us consider them separately. We start with the first term:

$$\begin{aligned} \mathrm{Tr}\big(T^a T^b T^e\big) \mathrm{Tr}\big(T^c T^d T^e\big) &= T^a_{ij} T^b_{jk} T^e_{ki} T^c_{lm} T^d_{mn} T^e_{nl} \\ &= T^a_{ij} T^b_{jk} T^c_{km} T^d_{mi} - \frac{1}{N} T^a_{ij} T^b_{ji} T^c_{lm} T^d_{ml} \\ &= \mathrm{Tr}\big(T^a T^b T^c T^d\big) - \frac{1}{N} \mathrm{Tr}\big(T^a T^b\big) \mathrm{Tr}\big(T^c T^d\big) \,, \end{aligned} \tag{4.29}$$

where we used the Fierz identity

$$T^e_{ki} T^e_{lm} = \delta_{kl} \delta_{in} - \frac{1}{N} \delta_{ki} \delta_{nl} \tag{4.30}$$

in the second step. For the other terms, we find equivalently

$$\begin{aligned}
\mathrm{Tr}\big(T^aT^bT^e\big)\mathrm{Tr}\big(T^cT^eT^d\big) &= \mathrm{Tr}\big(T^aT^bT^dT^c\big) - \frac{1}{N}\mathrm{Tr}\big(T^aT^b\big)\mathrm{Tr}\big(T^cT^d\big) \\
\mathrm{Tr}\big(T^aT^eT^b\big)\mathrm{Tr}\big(T^cT^dT^e\big) &= \mathrm{Tr}\big(T^aT^cT^dT^b\big) - \frac{1}{N}\mathrm{Tr}\big(T^aT^b\big)\mathrm{Tr}\big(T^cT^d\big) \\
\mathrm{Tr}\big(T^aT^eT^b\big)\mathrm{Tr}\big(T^cT^eT^d\big) &= \mathrm{Tr}\big(T^aT^dT^cT^b\big) - \frac{1}{N}\mathrm{Tr}\big(T^aT^b\big)\mathrm{Tr}\big(T^cT^d\big)\,.
\end{aligned} \tag{4.31}$$

We note here in passing that all terms proportional to $1/N$ originate from the fact that we are working in $SU(N)$ instead of in $U(N)$. We will have more to say about this soon. Interestingly, summing the four terms, we find that the $1/N$ parts cancel and we obtain

$$\begin{aligned}
f^{abe}f^{cde} \sim\; & \mathrm{Tr}\big(T^aT^bT^cT^d\big) - \mathrm{Tr}\big(T^aT^bT^dT^c\big) \\
& - \mathrm{Tr}\big(T^aT^cT^dT^b\big) + \mathrm{Tr}\big(T^aT^dT^cT^b\big) \\
=\; & \mathrm{Tr}\big(1234\big) - \mathrm{Tr}\big(1243\big) - \mathrm{Tr}\big(1342\big) + \mathrm{Tr}\big(1432\big)\,.
\end{aligned} \tag{4.32}$$

For the other two combinations of colour factors we find similar expressions. Putting everything together, we get

$$\begin{aligned}
M_{gggg} &= M_1\mathrm{Tr}(1234) + \text{all permutations of (2,3,4)} \\
&= \sum_{\sigma\in P_3} M_{4g}[1\sigma(2,3,4)]\mathrm{Tr}(1\sigma(2,3,4))\,,
\end{aligned} \tag{4.33}$$

where $M_{4g}[ijkl]$ is a colour ordered 4 gluon amplitude amplitudes. The total number of terms corresponds to the number of permutations of $(2,3,4)$ which is $3! = 6$. The result and the procedure to obtain it deserve some discussion.

Remark 1

Let's first of all go back to our comment about $SU(N)$ versus $U(N)$. In order to expose the colour traces, we have used two identities valid for $SU(N)$:

$$\begin{aligned}
f^{abc} &= -\frac{i}{\sqrt{2}}\Big[\mathrm{Tr}\big(T^aT^bT^c\big) - \mathrm{Tr}\big(T^aT^cT^b\big)\Big] \\
T^e_{ki}T^e_{lm} &= \delta_{kl}\delta_{in} - \frac{1}{N}\delta_{ki}\delta_{nl} \quad \text{(Fierz)}\,.
\end{aligned} \tag{4.34}$$

The Fierz identity above, can be also interpreted as the statement that the generators of $SU(N)$ form a complete set of *traceless*, hermitian $N\times N$ matrices, where the tracelessness condition derives from the fact that matrices in $SU(N)$ have unit determinant.

If instead of $SU(N)$ we were to consider $U(N) = SU(N)\times U(1)$, we would get an additional generator

$$T^{a_{U(1)}}_{ij} = \frac{1}{\sqrt{N}}\delta_{ij}\,. \tag{4.35}$$

The extra generator modifies the completeness relation above which becomes

$$T^a_{ij}T^a_{kl} = \delta_{il}\delta_{jk} \quad \text{for} \quad U(N)\,. \tag{4.36}$$

The additional generator, being an identity matrix, commutes with all other generators, i.e. it does not *interact* with the gluons. For this reason, this extra generator is often referred to as a "photon". In general, this extra photon could couple to fermions via the usual QED vertex $\overline{\psi}\gamma^\mu\psi A_\mu^{U(1)}$, but it cannot couple to gluons. It is now easy to see that we could have guessed from the beginning that all terms proportional to $\frac{1}{N}\delta_{ij}\delta_{kl}$ would vanish in any tree-level n-gluon amplitude, because at this order fermions cannot appear in any Feynman diagram! For this reason, when computing tree-level n-gluon amplitudes in YM theory, we can effectively use the $U(N)$ colour algebra instead of the $SU(N)$ one.

Remark 2

The argument above leads to an important consequence. In fact, at tree level, the colour factors of all amplitudes involving n-gluons are products of

$$f^{abc} \sim \mathrm{Tr}([T^a, T^b]T^c)\,. \tag{4.37}$$

Since we can work in $U(N)$, it is easy to see that a repeated use of the Fierz identity

$$T^a_{ij}T^a_{kl} = \delta_{il}\delta_{jk} \tag{4.38}$$

can only produce single traces of the generators in the form $\mathrm{Tr}(T^1T^2...T^n)$ (and the $(n-1)!$ permutations thereof). Therefore, all n-gluon tree level amplitudes can be written in terms of colour ordered amplitudes as

$$M_{ng} = \sum_{\sigma\in S_{n-1}} M_n(1\sigma(2...n))\mathrm{Tr}(1\sigma(2...n))\,. \tag{4.39}$$

Clearly, this does not hold at loop level or if external fermions are involved, where the $SU(N)$ colour algebra must be used. For example, for 4-gluon scattering at one loop, we would get in addition also *double traces*

$$\mathrm{Tr}(1\sigma(234))\,, \quad \mathrm{Tr}(12)\mathrm{Tr}(34)\,, \quad \mathrm{Tr}(13)\mathrm{Tr}(24)\,, \quad \mathrm{Tr}(14)\mathrm{Tr}(23)\,. \tag{4.40}$$

Colour Ordered Feynman Rules

Which Feynman diagrams do contribute to each trace at tree level? We have seen that Tr(1234) receives contributions from the colour algebras of the diagram shown in figure 8a.

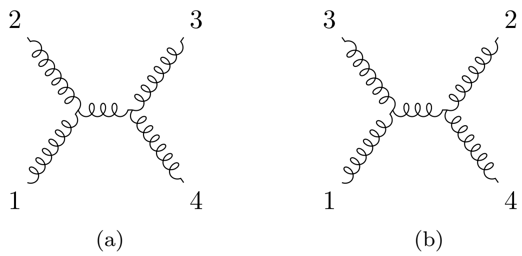


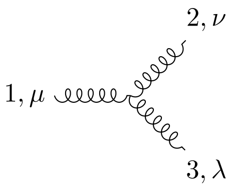
Figure 8: Four-point gluon amplitudes with different colour ordering.

In this diagram, gluons are arranged in the right order to contribute to the trace. If we consider a different ordering, such as that shown in figure 8b, we find that its colour factors are $f^{13l}f^{24l}$, which corresponds to what we found in eq. (4.32), with $2 \leftrightarrow 3$. The traces appearing in the amplitude are therefore

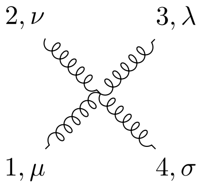
$$\mathrm{Tr}(1324),\ \mathrm{Tr}(1342),\ \mathrm{Tr}(1243),\ \mathrm{Tr}(1423)\,. \tag{4.41}$$

So we see that the gluons in this diagram are not properly ordered, and this diagram does not contribute to the color ordering 1234.

It is then convenient to introduce *colour ordered Feynman rules*:



$$\sim i\frac{g}{\sqrt{2}}\left[g^{\mu\nu}(p_1-p_2)^{\lambda}+g^{\nu\lambda}(p_2-p_3)^{\mu}+g^{\lambda\mu}(p_3-p_1)^{\nu})\right]\,, \tag{4.42}$$



$$\sim i\frac{g^2}{2}\left[2g^{\mu\lambda}g^{\nu\sigma}-g^{\mu\nu}g^{\lambda\sigma}-g^{\mu\sigma}g^{\nu\lambda}\right]\,. \tag{4.43}$$

All and only graphs which are properly ordered should be included when a process is computed with these Feynman rules.

Remark 3

The trace basis is over complete. For example, for four gluons, there are 6 traces, but we only started from 3 colour factors written as products of the structure constants. Moreover, due to the Jacobi identity

$$f^{abe}f^{ecd}+f^{ace}f^{edb}+f^{ade}f^{ebc}=0\,. \tag{4.44}$$

only two of the traces should be independent.

Remark 4

Why are then colour ordered amplitudes useful? It is true that only colour ordered diagrams contribute, so they are simpler to compute, but if they are not independent, one could argue that it is not guaranteed that they should be even be gauge invariant. As a matter of fact, they are. This is guaranteed by their so-called *partial orthogonality*

$$\sum \mathrm{Tr}(12...n)\left[\mathrm{Tr}(\sigma(12...n))\right]^{*}=N^{(n-2)}(N^2-1)\left(\delta_{\sigma\mathbb{1}}+\mathcal{O}(\frac{1}{N^2})\right)\,. \tag{4.45}$$

This is enough to guarantee gauge invariance of the partial amplitudes, because gauge invariance must hold order by order in $1/N$.

Their over-completeness manifests as further relations among these amplitudes. Important properties of n-gluon colour ordered amplitudes at tree-level are:

Cyclicity. $A(12...n) = A(2...n1)$. This is the reason why there are $(n-1)!$ colour ordered amplitudes.

Reflection. $A(12...n) = (-1)^N A(n...21)$. This can be proved using anti-symmetry of colour ordered Feynman rules. If holds for gluon-amplitudes at all loop orders.

Photon Decoupling. $A^{\text{tree}}_{ng}(123...n) + A^{\text{tree}}_{ng}(213...n) + ... + A^{\text{tree}}_{ng}(23...1n) = 0$. To see this, consider the tree-level decomposition

$$A = \sum_{\sigma(12...n)} S(1\sigma(12...n))\text{Tr}(1\sigma(12...n)) \,. \tag{4.46}$$

Remember that n-gluon amplitudes at tree level are equal if computed in $SU(N)$ or $U(N)$. However, in $U(N)$ if we pick for one gluon the $U(1)$ photon, the amplitude must vanish. The photon decoupling identity can be obtained choosing $T^1 = \mathbb{1}$, and enforcing that the resulting amplitude vanishes. Let us work this out explicitly for the 4- gluon case, for clarity. The complete amplitude can be written as

$$\begin{aligned} A =& A_1\text{Tr}(1234) + A_2\text{Tr}(1243) + A_3\text{Tr}(1324) \\ &+ A_4\text{Tr}(1342) + A_5\text{Tr}(1423) + A_6\text{Tr}(1432) \,. \end{aligned} \tag{4.47}$$

Setting $1 = \mathbb{1}$, this becomes

$$\begin{aligned} &A_1\text{Tr}(234) + A_2\text{Tr}(243) + A_3\text{Tr}(324) \\ &+ A_4\text{Tr}(342) + A_5\text{Tr}(423) + A_6\text{Tr}(432) = 0 \end{aligned} \tag{4.48}$$

Using cyclicity of the traces and collecting for the equal ones, this gives

$$0 = A(1234) + A(1342) + A(1423) = A(1234) + A(2134) + A(2314). \tag{4.49}$$

where we used cyclicity of the partial amplitudes to obtain the second equality. This is indeed the photon decoupling identity for four gluon scattering at tree level.

Photon decoupling is sometimes also called *dual ward identity.* It can be derived in string theory in a dual theory of some QFT through AdS/CFT correspondence.

There are even more identities, and one can prove that there are only $(n-3)!$ independent Colour ordered amplitudes. So in 4-gluon scattering there is only 1 independent amplitude. The missing identities are the **Kleiss-Kuijf relations** due to overcompleteness of the Colour traces (for four gluons this corresponds to the $U(1)$ decoupling) and the **BCJ relations** that we will see later on an example.

Colour Decomposition with Fermions

Recall, that external quark lines start and end with T^a_{ij}, where a fundamental index is open. For a quark line with m gluon vertices, we get $[T^{a_1}...T^{a_m}]_{ij}$. Consequently, the tree level amplitude of $q\bar{q}g...g$ with $n-2$ gluons has the following Colour decomposition:

$$A^{\text{tree}} = \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma}(3)}...T^{a_{\sigma}(n)})_{ij} A^{\text{tree}}(1\bar{q}, 2q, \sigma(3), ..., \sigma(n)) \,. \tag{4.50}$$

Work this out explicitly for $q\bar{q}Q\overline{Q}g$ as an exercise!

4.3 The MHV Amplitude

Let us once more go back to the tree level amplitude of 4-gluon scattering. In principle, we have $2^4 = 16$ helicity amplitudes and $3! = 6$ colour ordered amplitudes. However, many of them are related by Bose symmetry and crossings of the external legs, through the identities described in the previous sections. Let us then pick one color ordering $A(1234)$. For what concerns the helicities, 8 are independent, while the remaining 8 are trivially related by parity.

- $A(1^+2^+3^+4^+)$ There is one "all-plus" amplitude, related to $A(1^-2^-3^-4^-)$ by parity
- $A(1^-2^+3^+4^+)$ There are 4 "one-minus" amplitudes. Again, the 4 "one-plus" amplitudes are related by parity.
- $A(1^-2^-3^+4^+)$ There are in total 6 amplitudes having two particles of + helicity and two of - helicity. They are related by parity, leaving 3 of them independent.

The all + (or all -) amplitudes are always zero at tree level. To see this, remember that if we choose the same gauge vector for two gluons, we get

$$\varepsilon^+(p_i, q)^\mu \varepsilon^+_\mu(p_j, q) = 0\,. \tag{4.51}$$

Moreover, at tree level, all gluon amplitudes have the structure

$$\varepsilon_1^{\mu_1}...\varepsilon_1^{\mu_1} A_{\mu_1---\mu_n}\,, \tag{4.52}$$

where $A_{\mu_1...\mu_n}$ includes

$$\{p_{1,\mu_1}...p_{m,\mu_n}, g_{\mu_i\mu_j}, \text{etc}\}\,. \tag{4.53}$$

However, at tree-level only 3-gluon vertices can provide factors of p_μ. It's also easy to convince oneself that at tree level there are always fewer vertices than external lines, so each term must have at least one term proportional to a product of two polarisations $\varepsilon_i^\pm \varepsilon_j^\pm = 0$. This proves easily that at tree-level all n-gluon amplitudes with equal helicities are zero.

If one of the helicities is minus (pick for definiteness ε_1^-), we have

$$\varepsilon_1^- \cdot \varepsilon_j^+ \text{ and } \varepsilon_i^+ \cdot \varepsilon_j^+ \text{ where } i, j = 2, ..., n\,. \tag{4.54}$$

Let's choose for all plus polarisations the same reference vector p_1 $\varepsilon_j^+(p_j, p_1)$ (for $j \neq 1$) such that, as we saw before, $\varepsilon_i^+ \cdot \varepsilon_j^+ = 0$ $\forall i, j \neq 1$ since they have equal reference momenta. Consider now the one negative polarization ε_1^-, for which we pick a generic gauge momentum q, we then get

$$\varepsilon_1^-(p_1, q) \cdot \varepsilon_j^+(p_j, p_1) \sim \frac{[1\gamma_\mu q\rangle [1\gamma^\mu j\rangle}{\langle 1q\rangle\, [ji]} \sim [11] \sim 0. \tag{4.55}$$

Therefore,

$$A(1^-2^+ ... + n^+) = 0 \tag{4.56}$$

at tree-level. More generally, at tree level all n-gluon amplitudes with one helicity different from the rest are zero.

The first amplitudes that are non zero at tree-level are those with two helicities different from the rest. These amplitudes are called *Maximally Helicity Violating (MHV).* Thinking of the n-gluon scattering as $2 \to (n-2)$ scattering (2 incoming and $n-2$ outgoing), this peculiar naming comes from the fact that an all-equal helicity amplitude (in all-incoming or

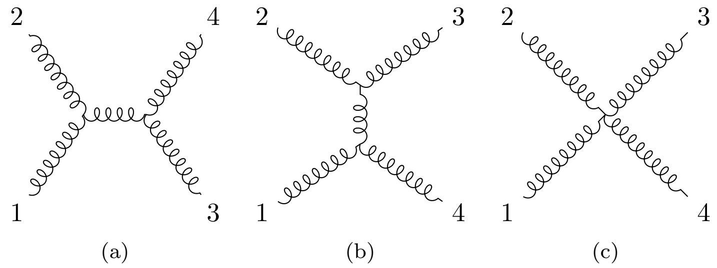


Figure 9: Colour ordered Feynman diagrams for the MHV amplitude.

all out-going kinematics) can be interpreted as an amplitude where gluons of equal helicity (say $+$) produce $n-2$ gluons of opposite helicity (say $-$). In this sense, an all plus or all minus amplitude, violates helicity maximally and turns out to be zero because of that. Similarly, a one-minus amplitude, can be interpreted as an amplitude where two $+$ gluons, produce 1 $+$ and $(n-3)$ $-$ gluons, which in this sense violates helicity "a little bit less" than the all plus. These amplitudes are nevertheless still zero, as we saw. Finally, the MHV amplitudes have two $+$ gluons in initial state, and then 2 more $+$ gluons in the final state, with $(n-4)$ additional $-$ gluons. These amplitudes are non-zero and are therefore the ones where helicity can be maximally violated, without trivializing the amplitude.

Let's consider now the specific case of 4-gluon scattering. We are left with one amplitude to compute $A(1^-2^-3^+4^+)$. All other amplitudes can then be obtained by parity, crossing of external legs and photon decoupling. To see this, recall

$$A(1234) + A(2134) + A(2314) = 0 \tag{4.57}$$

by fixing the helicities and using cyclicity this gives

$$A(1^-2^+3^-4^+) + A(1^-3^-4^+2^+) + A(1^-4^+2^+3^-) = 0\,. \tag{4.58}$$

Moreover the reflection identities state

$$\begin{aligned} A(1234) &= A(4321) \\ A(1243) &= A(3421) \\ A(1324) &= A(4231)\,. \end{aligned} \tag{4.59}$$

Again using cyclicity we finally find

$$A(1^-2^+3^-4^+) = -A(1^-3^-2^+4^+) - A(1^-3^-4^+2^+)\,. \tag{4.60}$$

Hence we can obtain $-+-+$ from $--++$ without having to perform any permutations. As you will see in one of the exercises, using the BCJ relations we are only left with one amplitude to compute.

The Parke-Taylor Formula

By direct calculation of the relevant colour-ordered Feynman diagrams shown in fig. 9, one can show that (taking all gluons incoming as usual)

$$A(1^+2^+3^-4^-) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,. \tag{4.61}$$

As a check let us have a look at the little group scaling. We have

$$\begin{aligned} 1 &\Rightarrow z^2 \\ 2 &\Rightarrow z^2 \\ 3 &\Rightarrow 1/z^2 \\ 4 &\Rightarrow 1/z^2 , \end{aligned} \tag{4.62}$$

which is as expected.

Notice that in many references in the literature, the "all-outgoing" convention is used. Taking all gluons outgoing instead of incoming reverses all the helicities

$$A(1^-2^-3^+4^+)_{\text{out}} = A(1^+2^+3^-4^-)_{\text{in}} , \tag{4.63}$$

such that the formula in (4.61) is often quoted for $A(1^-2^-3^+4^+)$ instead of $A(1^+2^+3^-4^-)$.

Let's finally discuss how we can obtain the non-adjacent MHV amplitude $-+-+$. We have (staying outgoing for now)

$$A(1^-2^+3^-4^+) = -A(1^-3^-2^+4^+) - A(1^-3^-4^+2^+) . \tag{4.64}$$

where

$$A(1^-3^-2^+4^+) = \frac{\langle 13\rangle^4}{\langle 13\rangle \langle 32\rangle \langle 24\rangle \langle 41\rangle} , \qquad A(1^-3^-4^+2^+) = \frac{\langle 13\rangle^4}{\langle 13\rangle \langle 34\rangle \langle 42\rangle \langle 21\rangle} . \tag{4.65}$$

which we obtained by exchanging $2 \leftrightarrow 3$ for the first and $2 \leftrightarrow 4$ for the second amplitude. Summing them we find

$$\begin{aligned} A(1^-2^+3^-4^+) &= -\frac{\langle 13\rangle^4}{\langle 13\rangle \langle 24\rangle}\left[\frac{1}{\langle 32\rangle \langle 41\rangle}\frac{1}{\langle 34\rangle \langle 12\rangle}\right] \\ &= \frac{\langle 13\rangle^4 [\langle 34\rangle \langle 12\rangle + \langle 32\rangle \langle 41\rangle]}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle \langle 13\rangle \langle 24\rangle} \\ &= \frac{\langle 13\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}\left[\frac{\langle 34\rangle \langle 12\rangle + \langle 32\rangle \langle 41\rangle}{\langle 13\rangle \langle 24\rangle}\right] \\ &= \frac{\langle 13\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}\left[\frac{[13] \langle 34\rangle [42] \langle 21\rangle + [42] \langle 23\rangle [31] \langle 14\rangle}{[13] \langle 13\rangle [24] \langle 24\rangle}\right] \\ &= -\frac{\langle 13\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}\left[\frac{[13] \langle 34\rangle [43] \langle 31\rangle + [41] \langle 13\rangle [31] \langle 14\rangle}{s_{13}s_{24}}\right] \\ &= -\frac{\langle 13\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}\left[\frac{s_{13}s_{24} + s_{13}s_{14}}{s_{13}s_{24}}\right] \\ &= -\frac{\langle 13\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}\left[\frac{s_{24} + s_{14}}{s_{24}}\right] \\ &= -\frac{\langle 13\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}\left[\frac{s_{12} + s_{23}}{s_{13}}\right] \\ &= -\frac{\langle 13\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}\left[-1\right] . \end{aligned} \tag{4.66}$$

Hence we find

$$A(1^-2^+3^-4^+) = \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,. \tag{4.67}$$

In general, one can show that for n-gluon scattering the MHV amplitude is

$$A(1^+..i^-...j^-...n^+) = \frac{\langle ij\rangle}{\langle 12\rangle\langle 23\rangle ... \langle n1\rangle}\,. \tag{4.68}$$

This formula is the celebrated Parke-Taylor formula for MHV amplitudes with arbitrary numbers of gluons. Note, that for $n = 5$ we would have to compute 10 diagrams and for $n = 7$ already 157 diagrams. There appears to be impressive structure, hidden behind hundreds or thousands of different terms which have to conspire together in a Feynman diagram calculation, to produce such a simple result. We will prove the Parke-Taylor formula shortly using on-shell recursion techniques, in particular the BCFW relations. However, before getting there, we need to say a bit more about how amplitudes factorise in special limits and kinematical configurations.

5 Soft and Collinear Factorization

In this section we will study universal properties of on-shell amplitudes in special kinematical regions, in particular when one external particle of momentum p_i becomes soft ($p_i \to 0$) or collinear to another particle of momentum p_j ($p_i || p_j$). This is the first example of factorization and it will furnish general contraints that can be used to determine amplitudes in a recursive way, i.e. one can relate an n-particle amplitude in special configurations to simpler $(n-1)$-particle amplitudes. We will have a closer look at this by considering the *colour ordered* n-gluon amplitude $A(1, 2, ..., n)$ in these limits.

5.1 Soft Limits

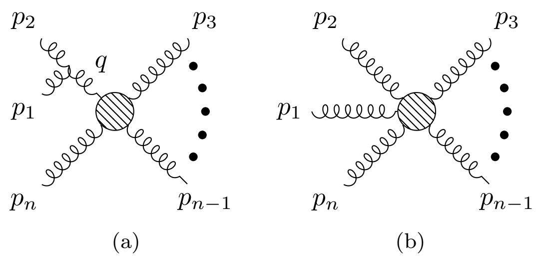


Figure 10: Amplitudes with a soft leg.

Let's consider the amplitude shown in figure 10a, assuming all external particles to be on-shell and massless $p_i^2 = 0$. The propagator of momentum q reads then

$$\frac{1}{q^2} = \frac{1}{(p_1 + p_2)^2} = \frac{1}{(p_1 + p_2)^2} = \frac{1}{2p_1 \cdot p_2} \to \infty \tag{5.1}$$

which blows up when $p_1 \to 0$ i.e. when p_1 becomes soft. This happens whenever the soft momentum is attached to another external leg (even if the external leg is a massive one!). If instead the soft momentum is attached to an off-shell virtual particle k^μ, as shown in figure 10b, the correspoding diagram does not diverge in the soft limit since

$$\frac{1}{q^2} = \frac{1}{k^2 + 2p_1 \cdot k} \, . \tag{5.2}$$

Moreover, note that if p_1 is attached through a 4-gluon vertex, again there can be no divergence.

We are interested in extracting the *leading* behaviour in the soft limit, i.e. we will only focus on the divergent contributions. Due to the colour ordering, there are only two configurations that can produce soft divergences. They are shown in figure 11. In the limit $p_1 \to 0$, we can write in particular

$$\begin{aligned}\lim_{p_1 \to 0} A(1, 2, ..., n) = & - i \frac{V_{3g}(1, 2, q_{12})^\rho}{2p_1 \cdot p_2} M_\rho(q_{12}, 3, ..., n) \\ & - i \frac{V_{3g}(n, 1, q_{1n})^\rho}{2p_1 \cdot p_n} M\rho(2, 3, ..., q_{1n}) \, ,\end{aligned} \tag{5.3}$$

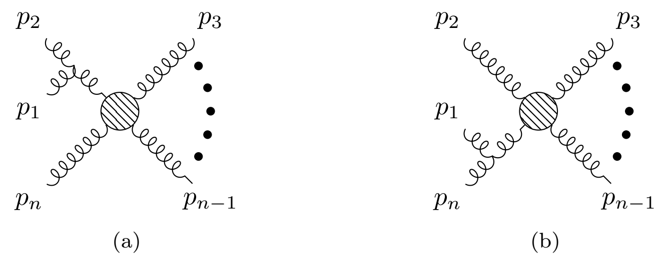


Figure 11: n-gluon amplitudes with a soft divergence.

where

$$V^{\rho}_{3g}(1,2,q_{12}) = \frac{ig}{\sqrt{2}}\left[g^{\mu\nu}(p_1-p_2)^{\rho} + g^{\nu\rho}(p_2+q_{12})^{\mu} + g^{\rho\mu}(-q_{12}-p_1)^{\nu}\right]\varepsilon_{1\mu}\varepsilon_{2\nu}\,, \tag{5.4}$$

and $M\rho(q_{12},3,...,n)$, $M\rho(2,3,...,q_{1n})$ are the amplitudes stripped of the divergent contributions. Here we defined $q_{ij} = p_i + p_j$. Note that q_{ij} has to be taken incoming in the definition of the three-gluon vertex, hence the minus signs. Using $p_2 - q_{12} = p_1 + 2p_2$ and $-q_{12} - p_1 = -2p_1 - p_2$ and recalling that $\varepsilon_1 \cdot p_1 = \varepsilon_2 \cdot p_2 = 0$, we can write

$$V^{\rho}_{3g}(1,2,q_{12}) = \frac{ig}{\sqrt{2}}\left[\varepsilon_1\cdot\varepsilon_2(p_1-p_2)^{\rho} + 2\varepsilon_2^{\rho}\varepsilon_1\cdot p_2 - 2\varepsilon_1^{\rho}\varepsilon_2\cdot p_1\right]. \tag{5.5}$$

Now taking the soft limit $p_1 \to 0$ and using the Ward identity

$$p_2^{\rho}M_{\rho}(2,3,...,n) = 0\,, \tag{5.6}$$

we find

$$V^{\rho}_{3g}(1,2,q_{12}) \sim \frac{ig}{\sqrt{2}}\left[2\varepsilon_1\cdot p_2\right]\varepsilon_2^{\rho}\,. \tag{5.7}$$

Repeating the same manipulations for the three-gluon vertex in the second diagram we have

$$V^{\rho}_{3g}(n,1,q_{1n}) \sim \frac{ig}{\sqrt{2}}\left[\varepsilon_n\cdot\varepsilon_1 p_n^{\rho} - 2\varepsilon_n^{\rho}\varepsilon\cdot p_n\right] \sim -\frac{ig}{\sqrt{2}}\left[2\varepsilon_1\cdot p_n\right]\varepsilon_2^{\rho}\,. \tag{5.8}$$

So the amplitude becomes

$$\lim_{p_1\to 0} A(1,2,...,n) \sim \frac{g}{\sqrt{2}}\left(\frac{\varepsilon_1\cdot p_2}{p_1\cdot p_2} - \frac{\varepsilon_1\cdot p_n}{p_1\cdot p_n}\right)\varepsilon_2^{\rho}M_{\rho}(2,...,n)\,. \tag{5.9}$$

Note that $\varepsilon_2^{\rho}M_{\rho}(2,...,n) = A(2,...,n)$ is just the color ordered amplitude for the scattering of $n-1$ gluons. We call the factor

$$\left(\frac{\varepsilon_1\cdot p_2}{p_1\cdot p_2} - \frac{\varepsilon_1\cdot p_n}{p_1\cdot p_n}\right) \tag{5.10}$$

the soft-factor or *Eikonal* factor. It is universal and in fact it is the same if the soft gluon is instead emitted by quark lines. We will show this later.

Let us now write the Eikonal in spinor helicity formalism. We chose p_n as gauge momentum, such that one of the terms is identically zero.

$$\varepsilon_1(p_1,p_n)\cdot p_n = 0\,. \tag{5.11}$$

The Eikonal factor is gauge invariant and therefore the result will not depend on the gauge choice we make. Recall that in the spinor helicity formalism

$$\varepsilon_1^{+,\mu} = -\frac{[n\gamma^\mu 1\rangle}{\sqrt{2}[n1]}; \qquad \varepsilon_1^{-,\mu} = -\frac{\langle n\gamma^\mu 1]}{\sqrt{2}\langle n1\rangle} . \tag{5.12}$$

So we find

$$\frac{\varepsilon_1^+ \cdot p_2}{p_1 \cdot p_2} = \frac{[n2]\langle 21\rangle}{\sqrt{2}[n1]} \frac{2}{\langle 12\rangle [21]} = -\frac{\sqrt{2}[n2]}{[n1][12]} \tag{5.13}$$

and equivalently

$$\frac{\varepsilon_1^- \cdot p_2}{p_1 \cdot p_2} = \frac{\langle n2\rangle [21]}{\sqrt{2}\langle n1\rangle} \frac{2}{\langle 12\rangle [21]} = \frac{\sqrt{2}\langle n2\rangle}{\langle n1\rangle \langle 12\rangle} . \tag{5.14}$$

Hence we find

$$\begin{aligned} \lim_{p_1\to 0} A(1^+, 2, ..., n) &\sim -g\frac{[n2]}{[n1][12]} A(2, ..., n) \\ \lim_{p_1\to 0} A(1^-, 2, ..., n) &\sim g\frac{\langle n2\rangle}{\langle n1\rangle \langle 12\rangle} A(2, ..., n) . \end{aligned} \tag{5.15}$$

More generally, due to the universality of the Eikonal factor, we then write

$$\lim_{p_j\to 0} A(1, 2, ..., j^\lambda, ...n) = S(j+1, j^\lambda, j-1) A(1, 2, ..., j-1, j+1, ..., n) , \tag{5.16}$$

where the soft factor reads

$$S(a, s^\lambda, b) = \begin{cases} -g\frac{[ab]}{[as][sb]} & \lambda = + \\ +g\frac{\langle ab\rangle}{\langle as\rangle\langle sb\rangle} & \lambda = - \end{cases} . \tag{5.17}$$

As an example of this universality, let's consider a soft gluon being emitted from a quark line, as shown in figure 12. The colour ordered vertex depends on q or $\overline{q}$ and is

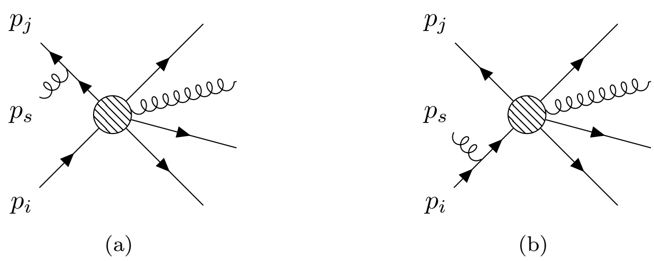


Figure 12: Amplitudes with a soft gluon being emitted from a quark line.

$$V^\mu_{q\bar{q}q} = \pm \frac{ig}{\sqrt{2}\gamma^\mu} . \tag{5.18}$$

This leads to the amplitude

$$A(1, ...i, s, j, ..., n) = \frac{ig}{\sqrt{2}} \overline{u}_j(p_j) \left\{ [M]_{ji} \frac{\not{p}_s + \not{p}_i}{(p_s + p_i)^2} \gamma^\mu - \gamma^\mu \frac{\not{p}_s + \not{p}_j}{(p_s + p_j)^2} [M]_{ji} \right\} u_i(p_i) \varepsilon^s_\mu , \tag{5.19}$$

where as before $[M]_{ji}$ is the amplitude stripped of the external quark-gluon structure, with ji the color indices, not to be confused with the labels of the momenta. In the limit $p_s \to 0$ this becomes

$$\begin{aligned}\lim_{p_s \to 0} A(1, ...i, s, j, ..., n) \sim & \frac{ig}{\sqrt{2}} \overline{u}_j(p_j) \left\{ [M]_{ji} \frac{\not{p}_i \not{\varepsilon}_s^+}{2p_i \cdot p_s} - \frac{\not{\varepsilon}_s^+ \not{p}_j}{2p_s \cdot p_j} [M]_{ji} \right\} u_i(p_i) \\ = & \frac{ig}{\sqrt{2}} \overline{u}_j(p_j) \left\{ [M]_{ji} \frac{2\varepsilon_s^+ \cdot p_i}{2p_i \cdot p_s} - \frac{2\varepsilon_s^+ \cdot p_j}{2p_s \cdot p_j} [M]_{ji} \right\} u_i(p_i) \, ,\end{aligned} \tag{5.20}$$

where we used $\not{p}_i \not{\varepsilon}_s^+ = 2p_i \cdot \varepsilon_s^+ - \not{\varepsilon}_s^+ \not{p}_i$ and $\not{p}_i u_i(p_i) = 0$. In this way, the amplitude becomes

$$\begin{aligned}\lim_{p_s \to 0} A(1, ..., i, s, j, ..., n) \to & \frac{ig}{\sqrt{2}} \left\{ \frac{\varepsilon_s^+ \cdot p_i}{p_i \cdot p_s} - \frac{\varepsilon_s^+ \cdot p_j}{p_s \cdot p_j} \right\} \overline{u}_j(p_j) [M]_{ji} u_i(p_i) \\ = & \frac{ig}{\sqrt{2}} \left\{ \frac{\varepsilon_s^+ \cdot p_i}{p_i \cdot p_s} - \frac{\varepsilon_s^+ \cdot p_j}{p_s \cdot p_j} \right\} A(1, ..., i, j, ..., n).\end{aligned} \tag{5.21}$$

Again one could choose the gauge $\varepsilon_s^+ \cdot p_j = 0$, and find

$$\frac{\varepsilon_s^+ \cdot p_i}{p_i \cdot p_s} = -\sqrt{2} \frac{[ji]}{[js][si]} \, , \qquad \frac{\varepsilon_s^- \cdot p_i}{p_i \cdot p_s} = +\sqrt{2} \frac{\langle ij \rangle}{\langle js \rangle \langle si \rangle} \, . \tag{5.22}$$

In conclusion, this demonstrates that the soft factor is universal, i.e. it is the same for a soft gluon that is emitted from a gluon or a quark line.

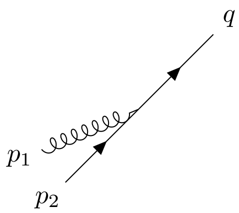


Figure 13: Gluon collinear to a quark line.

5.2 Collinear Limits

Another important configuration where a form of factorization happens, is the so-called collinear limit. To study it more in detail, let us consider an amplitude with quarks and gluons, as before, and study, what happens when a gluon becomes collinear to a quark as shown in figure 13 The propagator of momentum q^μ is the one that enters in the "hard process". In the collinear limit it becomes

$$\frac{i\not{q}}{q^2 + i\varepsilon} = \frac{i(\not{p}_1 + \not{p}_2)}{2p_1 \cdot p_2 + i\varepsilon}\,, \tag{5.23}$$

where

$$2p_1 \cdot p_2 = 2E_1E_2\left(1 - \frac{2|\vec{p_1}||\vec{p_2}|}{2E_1E_2}\cos\theta_{12}\right) = 2E_1E_2\left(1 - \cos\theta_{12}\right)\,. \tag{5.24}$$

In the last step, we assumed that p_1 and p_2 are massless. Now in the collinear limit $\theta_{12} \to 0$ and therefore $2p_1 \cdot p_2 \sim \theta_{12}^2$. This is the naive manifestation of a collinear singularity. We will see that *in gauge theories* the real divergence is milder, i.e. it is only logarithmic.

One can immediately make an important observation: other than soft singularities, collinear singularities are associated to one external parton only. Let us consider than an amplitude with n such partons, two of which,, of momenta p_1 and p_2 become collinear. No assumption of color ordering is required here and we can write in general

$$M(1, ..., n) = gT^a_{ij}\overline{u}(p_2)\gamma^\mu \frac{\not{p}_1 + \not{p}_2}{(p_1 + p_2)^2} M_j(p_{12}, p_3, ..., p_n) + \dots\,, \tag{5.25}$$

where by ... we denote other Feynman diagrams that are not singular in collinear limit. To study the collinear limit, we introduce a so-called Sudakov Decomposition. Given a light-like direction p^μ, ($p^2 = 0$), we parametrize the two "collinear" momenta as

$$\begin{aligned} p_1 &= x_1 p + y_1 \overline{p} + p_\perp^\mu \\ p_2 &= x_2 p + y_2 \overline{p} - p_\perp^\mu \end{aligned} \tag{5.26}$$

where we choose the reference frame such that $p^\mu_{1\perp} = -p^\mu_{2\perp}$. We introduced a complementary light-like momentum $\overline{p}$, such that $p \cdot \overline{p} \neq 0$. Moreover, the orthogonal momentum $p_\perp$ fulfils

$$p \cdot p_\perp = \overline{p} \cdot p_\perp = 0\,. \tag{5.27}$$

Note that the momentum in the orthogonal plane is a Euclidean vector, so it is convenient to define

$$p^\mu_\perp p_{\perp,\mu} = -\mathbf{p}^2_\perp\,. \tag{5.28}$$

Explicitly, a possible choice could be to have the z-axis to be the collinear direction and choose

$$p^\mu = (E, 0, 0, E), \quad \overline{p}^\mu = (E, 0, 0, -E) \quad \text{and} \quad p_\perp = (0, p_x, p_y, 0)\,. \tag{5.29}$$

For both momenta we then find

$$p_i^\mu p_{i,\mu} = -\mathbf{p}_\perp^2 + 2x_i y_i p \cdot \overline{p}\,. \tag{5.30}$$

Since they are both light-like we find

$$y_i = \frac{\mathbf{p}_\perp^2}{2x_i p \cdot \overline{p}}\,. \tag{5.31}$$

Moreover,

$$\begin{aligned}(p_1+p_2)^2 &= \left[(x_1+x_2)p + (y_1+y_2)\overline{p}\right]^2 \\ &= 2(x_1+x_2)(y_1+y_2)p\cdot\overline{p} \\ &= (x_1+x_2)\left(\frac{1}{x_1}+\frac{1}{x_2}\right)\frac{\mathbf{p}_\perp^2}{p\cdot\overline{p}}p\cdot\overline{p} = \frac{(x_1+x_2)^2}{x_1x_2}\mathbf{p}_\perp^2\,,\end{aligned} \tag{5.32}$$

which also corresponds to the invariant mass of the two particles. So now the collinear limit clearly corresponds to $\mathbf{p}_\perp \to 0$. In that limit clearly

$$y_i \sim \mathcal{O}(p_\perp^2)\,, \qquad p_{12}^2 \sim \mathcal{O}(p_\perp^2) \quad \to \quad \mathbf{p}_\perp^2 \sim \theta_{12}^2\,.$$

Then in the strict collinear limit $p_1 \sim x_1 p$ and $p_2 \sim x_2 p$ and therefore

$$p_1 + p_2 = (x_1 + x_2)p\,. \tag{5.33}$$

We choose the decomposition such that in the strict collinear limit $x_1 + x_2 = 1$ and hence $p_1 + p_2 \to p$. Note that this is equivalent to choosing as collinear direction the direction of one of the two massless partons.

Let us now use the Sudakov Decomposition to examine the amplitude in the collinear limit. As in the soft case, we will only keep leading divergent terms.

$$\begin{aligned}M(1,...,n) &\sim gT^a_{ij}\overline{u}(p_2)\gamma^\mu\frac{\not{p}_1+\not{p}_2}{(p_1+p_2)^2}M_j(p_{12},p_3,...,p_n) \\ &\sim gT^a_{ij}\overline{u}(p_2)\gamma^\mu\frac{(x_1+x_2)\not{p}}{p_{12}^2}M_j(px_{12},p_3,...,p_n)\,.\end{aligned} \tag{5.34}$$

Let us now fix the helicities. Setting the quark to be left-handed for definiteness, and using spinor helicity, we can then write

$$M_L^\lambda \sim gT^a_{ij}\frac{\langle 2|\not{\epsilon}_1^\lambda p](x_1+x_2)}{p_{12}^2}\langle p|M_j(px_{12},...,p_n)\,. \tag{5.35}$$

The gluon can then have two possible helicities, $\lambda = \pm$. Using spinor-helicity for the gluon polarization we then find

$$\langle 2\not{\epsilon}_1^+ p] = -\frac{\langle 2\gamma^\mu p][r_1\gamma_\mu 1\rangle}{\sqrt{2}[r_1 1]} = -\frac{2\,\langle 21\rangle\,[r_1 p]}{\sqrt{2}[r_1 1]} \tag{5.36}$$

$$\langle 2\not{\epsilon}_1^- p] = +\frac{\langle 2\gamma^\mu p]\langle r_1\gamma_\mu 1]}{\sqrt{2}\,\langle r_1 1\rangle} = -\frac{2\,\langle 2r_1\rangle\,[1p]}{\sqrt{2}\,\langle r_1 1\rangle}\,. \tag{5.37}$$

(5.38)

For the amplitude with gluon polarization +, we thus find

$$M_L^+ \sim -gT_{ij}^a \frac{2\,\langle 21\rangle\,[r_1 p](x_1+x_2)}{\sqrt{2}[r_1 1]\,\langle 12\rangle\,[21]}\langle p|M_j(px_{12},...,p_n)\,. \tag{5.39}$$

The collinear singularity manifests as

$$\langle 12\rangle\,[21] = p_{12}^2 \sim \theta_{12}^2 \to 0\,. \tag{5.40}$$

However, notice that $\langle 12\rangle$ cancels between numerator and denominator, which softens the collinear divergence. We have

$$M_L^+ \sim -gT_{ij}^a \frac{2[r_1 p](x_1+x_2)}{\sqrt{2}[r_1 1][12]}\langle p(x_1+x_2)|M_j(px_{12},...,p_n) + \text{non divergent terms}\,. \tag{5.41}$$

The divergent term scales only like

$$\frac{1}{[12]} \sim \frac{1}{\sqrt{2p_1\cdot p_2}} \sim \frac{1}{\theta_{12}}\,. \tag{5.42}$$

At leading order, we can take the limit everywhere and compute the residue at $\theta_{12}=0$, using $x_1+x_2=1$. We find

$$\frac{[r_1 p]}{[r_1 1]} \cong \frac{\sqrt{2r_1\cdot p}}{\sqrt{2r_1\cdot p}}\frac{1}{\sqrt{x_1}} = \frac{1}{\sqrt{x_1}}\,, \tag{5.43}$$

where we used $p_1 \to x_1 p$. So finally the amplitude for a left handed quark and a gluon with plus helicity factorizes as follows in the collinear limit

$$M_L^+ \sim -gT_{ij}^a \frac{\sqrt{2}}{\sqrt{x_1}}\frac{1}{[12]}M_j^L(p,...p_n) \sim -gT_{ij}^a \frac{\sqrt{2}}{\sqrt{1-z}}\frac{1}{[12]}M_j^L(p,...p_n)\,, \tag{5.44}$$

where we introduced the standard parametrization for the momentum fractions $x_1 = 1-z$ and $x_2 = z$, which guarantees $x_1+x_2=1$.

Let's have a look also at the amplitude for a gluon with negative helicity. As before we write

$$M_L^- \sim +gT_{ij}^a \frac{2\,\langle 2r_1\rangle\,[1p]}{\sqrt{2}\,\langle r_1 1\rangle\,\langle 12\rangle\,[21]}\langle p_{12} M_j(px_{12},...,p_n)\,. \tag{5.45}$$

Here we should also see a cancellation of the $\frac{1}{\theta_{12}^2}$ divergence to leave $\frac{1}{\theta_{12}^2}$. In this case the cancellation is slightly more subtle, in fact if $1 \parallel p$ we have $[1p] \to 0$ and we can write

$$\frac{[1p]}{[21]} = \frac{[1p]\,\langle 12\rangle}{[21]\,\langle 12\rangle} = -\frac{[1p]\,\langle 21\rangle}{2p_1\cdot p_2} = -\frac{\langle 2\,1\,p]}{2p_1\cdot p_2}\,. \tag{5.46}$$

Using $p_1+p_2 = (x_1+x_2)p + (y_1+y_2)\overline{p}$, we find

$$\frac{[1p]}{[21]} = -\frac{\langle 2((x_1+x_2)p+(y_1+y_2)\overline{p}-p_2)p]}{2p_1\cdot p_2} = -\frac{(y_1+y_2)}{2p_1\cdot p_2}\,\langle 2\overline{p}\rangle\,[\overline{p}p]\,. \tag{5.47}$$

Inserting $p_1 \cdot p_2 = (x_1+x_2)(y_1+y_2)p\cdot\overline{p}$ then gives

$$\begin{aligned}\frac{[1p]}{[21]} &= -\frac{(y_1+y_2)}{(x_1+x_2)(y_1+y_2)2p\cdot\overline{p}}\langle 2\overline{p}\rangle\,[\overline{p}p] \\ &= -\frac{1}{(x_1+x_2)\,\langle p\overline{p}\rangle\,[\overline{p}p]}\langle 2\overline{p}\rangle\,[\overline{p}p] = -\frac{\langle 2\overline{p}\rangle}{(x_1+x_2)\,\langle p\overline{p}\rangle}\,,\end{aligned} \tag{5.48}$$

such that in the collinear limit we can write

$$\frac{[1p]}{[21]} = -\frac{\langle 2\overline{p}\rangle}{\langle p\overline{p}\rangle}\frac{1}{(x_1+x_2)} \to -\frac{\langle 2\overline{p}\rangle}{\langle p\overline{p}\rangle}\,. \tag{5.49}$$

The divergent spinor [21] has cancelled and the result is explicitly is convergent. So the amplitude now becomes

$$M_L^- \sim -gT^a_{ij}\frac{\sqrt{2}\,\langle 2r_1\rangle\,\langle 2\overline{p}\rangle}{\langle r_1 1\rangle\,\langle 12\rangle\,\langle p\overline{p}\rangle}\langle pM_j(px_{12},...,p_n)\,. \tag{5.50}$$

Finally taking the collinear limit in the remaining spinors, $p_i \to x_i\,p$, we find

$$\frac{\langle 2\overline{p}\rangle}{\langle p\overline{p}\rangle} \to \sqrt{x_2}\,, \qquad \frac{\langle 2r_1\rangle}{\langle r_1 1\rangle} \to -\sqrt{\frac{x_2}{x_1}} \tag{5.51}$$

and finally the amplitude becomes

$$\begin{aligned}M_L^- &\sim gT^a_{ij}\sqrt{2}\sqrt{\frac{x_2}{x_1}}\sqrt{x_2}\frac{1}{\langle 12\rangle}M_j^L(p,...,p_n) \\ &\sim gT^a_{ij}\frac{\sqrt{2}z}{\sqrt{1-z}}\frac{1}{\langle 12\rangle}M_j^L(p,...,p_n)\,.\end{aligned} \tag{5.52}$$

Now we can compute the sum over the gluon polarizations

$$\begin{aligned}\sum_{\text{pol},\lambda_q}|M|^2 &= |M_L^-|^2 + |M_L^+|^2 \\ &= g^2(T^a_{ik}T^a_{kj})\left[\frac{2z^2}{1-z}\frac{1}{p_{12}^2} + \frac{2}{1-z}\frac{1}{p_{12}^2}\right]\sum_{\lambda_q}|M|^2 \\ &= g^2(T^a_{ik}T^a_{kj})\left[\frac{2z^2}{1-z}\frac{1}{p_{12}^2} + \frac{2}{1-z}\frac{1}{p_{12}^2}\right]\sum_{\lambda_q}|M|^2 \\ &= 2g^2\frac{\delta_{ij}C_F}{p_{12}^2}\left[\frac{1+z^2}{1-z}\right]\sum_{\lambda_q}|M|^2\,.\end{aligned} \tag{5.53}$$

It is common to define the splitting function

$$P_{qq}(z) = \frac{1+z^2}{1-z} \tag{5.54}$$

which represents the probability that a (parent) quark splits into a gluon and collinear quark (daughter partons), and that the daughter quark carries the fraction z of the parent momentum. In the same way one can define $P_{qg}(z)$ as the probability of a quark to emit a gluon of momentum z. The two splitting functions are connected by

$$P_{qq}(1-z) = \frac{1+(1-z)^2}{z} = P_{qg}(z)\,. \tag{5.55}$$

The last splitting we have not considered explicitly is a gluon splitting to two gluons, which gives rise to the splitting functions $P_{gg}(z)$. Note in particular that while in the case of P_{qq} the quark helicity is conserved along the quark line, this is not true in general. Generally, the collinear splitting can mix the helicities of the parent and daughter partons in a non trivial way, as you will see computing explicitly $P_{gg}(z)$. For this reason, the general factorization in the collinear limit can be given as

$$A_n(1...a^{\lambda_a}, b^{\lambda_b}, ..., n) \underset{a\|b}{\longrightarrow} \sum_{\lambda_p=\pm} \text{Split}_{-\lambda p}(a^{\lambda_a} b^{\lambda_b}; z) A_{n-1}(...p^{\lambda_p})\,, \tag{5.56}$$

where a sum over the helicities of the daughter parton is included. In the special case of the quark-gluon splitting analysed before, the only non-trivial splitting kernels are

$$\begin{aligned} \text{Split}_{\text{R}}(q^L, g^-, z) &= -\frac{z}{\sqrt{1-z}}\frac{1}{\langle qg\rangle} \\ \text{Split}_{\text{R}}(q^L, g^+, z) &= +\frac{1}{\sqrt{1-z}}\frac{1}{[qg]}\,. \end{aligned} \tag{5.57}$$

Note that we have $R = -L$ for incoming/outgoing particles.

6 Complex Momenta and Uniqueness of Yang-Mills

One of the motivation for all the techniques we have introduced till this point, was to discuss to which level it is possible to compute on-shell scattering amplitudes without resorting to an expansion in off-shell Feynman diagrams. Until now, nevertheless, we always used Feynman diagrams to construct amplitudes and only resorted to spinor helicity and color ordering to simplify the explicit calculation of so-called helicity amplitudes, from their Feynman diagram representation. In this section and next sections, we finally start introducing methods that allow us to skip Feynman diagrams alltogether.

The first important example of this, is the by now famous proof of uniqueness of Yang Mills theories. The idea is that, by only imposing the correct Little group scaling plus general factorization requirements on on-shell amplitudes, one can prove that massless spin one particles must interact through a theory based on a Lie group. This proof will also rely on analytic continuation of momenta to complex values. The same analytic continuation will be at the core of the on-shell recursion techniques that we will discuss later on.

To begin, let us recall some general properties of scattering amplitudes that we will use in the following. Together with global, gauge and space-time symmetries, at the core of QFT are the concepts of unitarity and locality. In particular, locality of interactions implies that poles of the S-matrix are always associated to on-shell intermediate states. Let us consider a general momentum-space Green function

$$G_n(p_1, ...p_n) = \int d^4x_1 e^{ip_i \cdot x_1} ... \int d^4x_n e^{ip_n \cdot x_n} = \langle \Omega | T\phi(x_1)...\phi(x_n)|\Omega\rangle \,, \tag{6.1}$$

where $|\Omega\rangle$ is the vacuum. Suppose now that there exists a one-particle state $|\psi\rangle$ with mass m_Ψ, and a subset of external momenta

$$p^\mu = p_1^\mu + ... + p_r^\mu = p_{r+1}^\mu + ... + p_n^\mu \tag{6.2}$$

with $p^\mu p_\mu = m_\psi^2$, such that

$$\langle \psi | \phi(x_1)...\phi(x_r)|\Omega\rangle \neq 0 \,. \tag{6.3}$$

What this means is that the one-particle state can be created from the vacuum by the operators $\phi(x_1), ..., \phi(x_r)$.

If this is the case, then one can prove that the Green function G will have a pole at $p^2 = m_\psi^2$ and it will factorize close to that pole as

$$G(p_1...p_n) \sim M_\psi^{1,r} \frac{1}{p^2 - m_\psi^2 + i\varepsilon} \left(M_\psi^{r+1,n} \right)^\dagger + ...(\text{non-div. terms}) \,, \tag{6.4}$$

where $M_\psi^{1,r}$ is the matrix element for $\phi_1, ..., \phi_r \to \Psi$ and $M_\psi^{r+1,n}$ for $\psi \to \phi_{r+1}, ..., \phi_n$. Note that $|\psi\rangle$ does not have to be an elementary particle state in general, but could be anything, for example a bound state (you can think of positronium).

We can easily convince ourselves that this is true in the context of tree-level amplitudes, where we clearly get a pole from each internal propagator corresponding to a one-particle state. If the amplitude is colour ordered, propagators will also only contain consecutive external legs. In conclusion, locality at tree-level means that any poles with non-vanishing residue must correspond to the propagator of a physical particle going on shell. Spurious poles must have zero residue!

How can we use this? To answer that question, let use take one step back. So far we have considered four-point amplitudes, but there exist simpler things. Two-point "amplitudes" are just propagators, i.e. by construction off-shell. Three-point amplitudes are instead a bit more interesting. Clearly, we cannot have any non-trivial three-point on-shell amplitude (with massless external states) if momenta are real. In fact, consider for example figure 14. If the three external states are massless and on shell, this also implies $2p_1 \cdot p_2 = 0$, which implies the whole amplitude is zero. Let's see how this works in spinor helicity. The amplitude must depend on external polarizations ε_j^μ and momenta p_j^μ, which means it will be some function of all possible spinors $|1\rangle, |2\rangle, |3\rangle, |1], |2], |3]$.

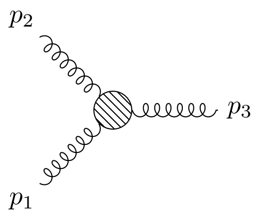


Figure 14: Three-point gluon amplitude.

Momentum conservation ($p_1 + p_2 + p_3 = 0$) written in spinor helicity implies separately

$$\begin{aligned} |1\rangle [1| + |2\rangle [2| + |3\rangle [3| = 0 \\ |1] \langle 1| + |2] \langle 2| + |3] \langle 3| = 0\,. \end{aligned} \tag{6.5}$$

Note that the two equations are true separately because $|i\rangle [i|$ and $|i] \langle i|$ do act on orthogonal representations of the Lorentz group. In fact, if we contract them from the left by angle spinors $\langle 1|$ and $\langle 2|$. The second equation does not contribute anything non-trivial, while the first gives

$$\begin{aligned} \langle 1| \left(|1\rangle [1| + |2\rangle [2| + |3\rangle [3| \right) = \langle 12\rangle [2| + \langle 13\rangle [3| = 0 \\ \langle 2| \left(|1\rangle [1| + |2\rangle [2| + |3\rangle [3| \right) = \langle 21\rangle [1| + \langle 23\rangle [3| = 0\,, \end{aligned} \tag{6.6}$$

leading to

$$\langle 12\rangle [2| = -\langle 13\rangle [3| \qquad \langle 12\rangle [1| = +\langle 23\rangle [3|\,. \tag{6.7}$$

Hence there are two possibilites. Either $\langle 12\rangle = \langle 13\rangle = \langle 23\rangle = 0$ or $\langle ij\rangle \neq 0$ and then

$$[2| = -\frac{\langle 13\rangle}{\langle 12\rangle}[3|\,, \qquad [1| = +\frac{\langle 23\rangle}{\langle 12\rangle}[3|\,, \tag{6.8}$$

which implies that the square spinors are all proportional to each other, which gives

$$[12] = [13] = [23] = 0\,. \tag{6.9}$$

So either $\langle ij\rangle = 0$ or $[ij] = 0$. Now, if the momenta are real, we have seen that

$$\langle ij\rangle^* = [ji]\,, \tag{6.10}$$

which finally implies that all spinor products are zero. As there is nothing else that the tree-level amplitude could depend on, the tree-point tree-level amplitude with real momenta is zero.

Let us then relax this assumptions and consider amplitudes as function of complex momenta. This is ok, since scattering amplitudes are analytic functions modulo poles and branch cuts and can be uniquely analytically continued to the whole complex plane. With this

$$\langle ij\rangle^* \neq [ji]\,, \tag{6.11}$$

and we have two possibilities for the three point amplitude in fig. 14:

$$A_3(1^{\lambda_1}2^{\lambda_2}3^{\lambda_3}) = \begin{cases} C^{abc}\,\langle 12\rangle^A\,\langle 23\rangle^B\,\langle 31\rangle^C \\ C^{abc}[12]^A[23]^B[31]^C \end{cases}, \tag{6.12}$$

where C^{abc} is for now a constant as far as kinematics goes, and it carries the information about any unspecified additional global degree of freedom associated to the external gluons (for example, color). Little group scaling can already help us constrain this general form. Recall that under little group

$$\begin{aligned} |p\rangle &\to z\,|p\rangle \qquad && \varepsilon_+^\mu \to z^2\varepsilon_+^\mu \\ |p] &\to \frac{1}{z}|p] \qquad && \varepsilon_-^\mu \to \frac{1}{z^2}\varepsilon_-^\mu\,. \end{aligned} \tag{6.13}$$

Focusing on the first case, little group scaling then implies

$$\begin{cases} A+C = 2\lambda_1 \\ A+B = 2\lambda_2 \\ B+C = 2\lambda_2 \end{cases} \Rightarrow \quad \begin{aligned} -A &= \lambda_3 - \lambda_1 - \lambda_2 \\ -B &= \lambda_1 - \lambda_2 - \lambda_3 \\ -C &= \lambda_2 - \lambda_1 - \lambda_3\,. \end{aligned} \tag{6.14}$$

Repeating the analogous analysis for the second case we find finally

$$A_3(1^{\lambda_1}2^{\lambda_2}3^{\lambda_3}) = \begin{cases} C^{abc}\,\langle 12\rangle^{\lambda_1+\lambda_2-\lambda_3}\,\langle 23\rangle^{\lambda_2+\lambda_3-\lambda_1}\,\langle 31\rangle^{\lambda_3+\lambda_1-\lambda_2} \\ C^{abc}[12]^{\lambda_3-\lambda_1-\lambda_2}[23]^{\lambda_1-\lambda_2-\lambda_3}[31]^{\lambda_2-\lambda_3-\lambda_1} \end{cases}. \tag{6.15}$$

Little group then fixes the structure of the amplitude up to C^{abc}.

To make more progress, let us focus on the first case and specify the helicities to the all-plus case. We find

$$A_3(1^+2^+3^+) = \begin{cases} C^{abc}\,\langle 12\rangle\,\langle 23\rangle\,\langle 31\rangle \\ C^{abc}\frac{1}{[12][23][31]}\,. \end{cases} \tag{6.16}$$

We see immediately that, since for real momenta all spinor products must go to zero, the second possibility blows up and does not correctly reproduce the correct result. This means it cannot be the correct analytic continuation of the three point amplitude for complex momenta and only the first possibility remains. It is then easy to see by dimensional analysis that since the cross-section must have $[\sigma]_E = -2$, a scattering amplitude with n external legs must have dimension $[A_n]_E = 4-n$ in $D=4$ space-time dimensions.

Since $[\langle ij\rangle]_E \sim [\sqrt{2p_i\cdot p_j}]_E = 1$, it's easy to see that we wouldd need $[C^{abc}] = -2$. However, C^{abc} can only depend on the coupling constant, and a dimensionful coupling implies a non-renormalizable theory. We can then conclude that for renormalizable theories, the only solution is $C^{abc} = 0$. Hence, we found that the scattering amplitude for equal helicities vanishes, even for complex momenta!

Let us next consider the amplitude with one minus helicity. Following the same reasoning, little group scaling imposes

$$A_3(1^+2^+3^-) = \begin{cases} C^{abc}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \\ \\ C^{abc}\frac{[23][31]}{[12]^3}\,, \end{cases} \tag{6.17}$$

Once more, the second possibility must be expluded since it would diverge for real momenta and it therefore cannot be the correct analytic continuation of the one-minus three point amplitude. We are then left with the three one-minus amplitudes

$$\begin{aligned} M(1^+2^+3^-) &= C^{abc}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \\ M(1^+2^-3^+) &= C^{abc}\frac{\langle 31\rangle^3}{\langle 12\rangle\langle 23\rangle} \\ M(1^+2^+3^-) &= C^{abc}\frac{\langle 23\rangle^3}{\langle 12\rangle\langle 31\rangle}\,. \end{aligned} \tag{6.18}$$

Since we are dealing with identical bosonic particles, we expect the three formulas to be symmetric under a swap of the two bosons with plus helicity, namely $1 \leftrightarrow 2$, $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$ respectively. It is then easy to see that the spinor product parts are all antisymmetric for these exchanges, which in turn implies that C^{abc} must be totally antisymmetric under exchanges of all pairs! So little group scaling in addition to Bose symmetry, fixes C^{abc} to be completely antisymmetric.

We can constrain C^{abc} further, by considering 4-particle scattering. We consider in particular the MHV helicity configuration for the color-ordered scattering of four gluons of momenta p_1,...,p_4. We focus on the standard ordering $A_4(1,2,3,4,)$. Based on little group scaling, we can parametrize these amplitudes as

$$A_4(1^+2^+3^-4^-) = \langle 12\rangle^2\,[34]^2 F^{abcd}(s,t,u)\,. \tag{6.19}$$

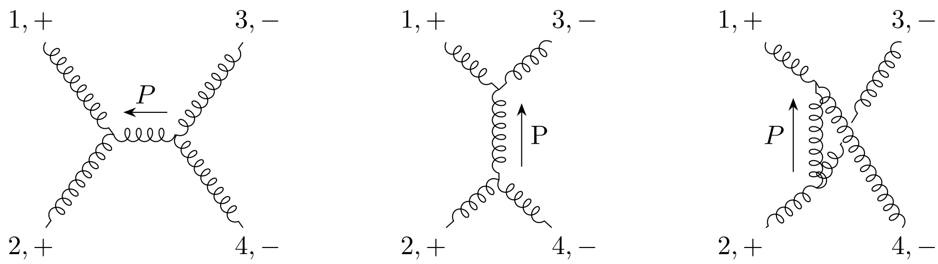
Note, that other than for the 3-point amplitude, this is not unique. In fact, we know that at tree level these amplitudes are usually written as Parke-Taylor factors (4.61). The parametrization in eq. (6.19) provides the same overall spinor scaling, and in fact its ratio to the Parke-Taylor factor is just a combination of Mandelstam invariants or, as one usually says in jargon, "helicity free":

$$\begin{aligned} \langle 12\rangle^2\,[34]^2 \Big/ \left[\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right] &= \frac{[34]^2\,\langle 23\rangle\langle 34\rangle\langle 41\rangle}{\langle 12\rangle} \\ &= \frac{[21]\,\langle 14\rangle\,[43]\,\langle 32\rangle\,\overbrace{[43]\,\langle 34\rangle}^{s_{12}}}{\underbrace{[21]\,\langle 12\rangle}_{s_{12}}} \\ &= s_{12}s_{23} \;=\; s\,u\,. \end{aligned}$$

We will use for convenience the spinor factor $\langle 12\rangle^2\,[34]^2$, remembering that $F^{abcd}(s,t,u)$ is now in general a function of the kinematics. To start constraining it, let's do some dimensional analysis. Since in this case the amplitude must be dimensionless $[A]_E = 0$, we have

$$[F^{abcd}]_E \sim -4\,. \tag{6.20}$$

Consequently, F^{abcd} must have some poles. Unitarity implies that poles in the S-matrix must correspond to physical particles going on-shell. For a generic four-point scattering amplitude, we expect three possible factorizations channels, s, t and u, corresponding respectively to the three sub-processes $12 \to 34$, $13 \to 24$ and $23 \to 41$, shown in fig. 15.



(a) s-channel (b) t-channel (c) u-channel

Figure 15: Factorization channels of the 4-gluon amplitude.

Let us start considering factorization in the s-channel. The general expected factorization pattern implies that when $s \to 0$ the amplitude can be written in terms of two 3-point amplitudes $12 \to P$ and $P \to 34$, where $P^\mu = -(p_1+p_2)^\mu = (p_3+p_4)^\mu$ is the momentum of the on-shell intermediate state that goes on shell. In principle, we must sum over all possible states of momentum P. Assuming that the only particle content of our theory are the spin-one gluons, the state of momentum P can only be a gluon of helicity $\pm$ and we must sum over both possibilities. Following fig 15, we assign the helicity to the gluon P assuming the momentum goes from right to left, i.e. it is incoming in the left amplitude and outgoing for the right amplitude. This also implies that if P has positive helicity as seen from the left amplitude, it then has negative helicity for the right one.

As we have seen, even for complex momenta, three-point gluon amplitudes with all equal helicities are zero, which implies that when the gluon of momentum P has positive helicity the left amplitude is zero and therefore only the negative helicity state contributes. Considering only P^- we find then

$$\begin{aligned}\lim_{s\to 0} A_4(1^+2^+3^-4^-) &= \frac{\delta^{ef}}{p^2} A_4(1^+2^+P^-)M(3^-4^-(-P)^+)\\ &= +\frac{C^{abe}C^{cde}}{s}\frac{\langle 12\rangle^3}{\langle 2P\rangle\langle P1\rangle}\frac{[34]^3}{[3(-P)][(-P)4]}\\ &= -\frac{C^{abe}C^{cde}}{s}\frac{\langle 12\rangle^3\,[34]^3}{\langle 2P\rangle\,[P4][3P]\,\langle P1\rangle}\,,\end{aligned} \tag{6.21}$$

where in the second step we used

$$A_3(1^+2^+3^-) = C^{abc}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,, \tag{6.22}$$

and in the third we used our convention for analytic continuation of the spinors

$$|(-P)\rangle = i|P\rangle\,, \qquad |(-P)] = i|P]\,.$$

Recalling that $P = -p_1 - p_2 = p_3 + p_4$, we can write

$$\langle 2P\rangle\,[P4] = -\,\langle 21\rangle\,[14] = \langle 12\rangle\,[14] \tag{6.23}$$

$$[3P]\,\langle P1\rangle = [34]\,\langle 41\rangle\ , \tag{6.24}$$

which gives for the limit of the 4-point amplitude:

$$\lim_{s\to 0} A_4(1^+2^+3^-4^-) = -\frac{C^{abe}C^{cde}}{s}\frac{\langle 12\rangle^2\,[34]^2}{u}\,. \tag{6.25}$$

Comparing this to eq. (6.19), we can see that in the limit $s \to 0$

$$\lim_{s\to 0}\left[suF^{abcd}(s,t,u)\right] = -C^{abe}C^{cde}\,. \tag{6.26}$$

Let's now have a look at the other factorization channels. Differently from the previous case, in the u-channel, corresponding to $u = s_{23} \to 0$, the on-shell gluon can have both helicities, as it is easy to see from fig. 15. We find

$$\begin{aligned}\lim_{u\to 0} M(1^+2^+3^-4^-) &= \frac{C^{ade}C^{bce}}{u}\left[\frac{\langle 1P\rangle^3\,[3P]^3}{\langle 14\rangle\,\langle 4P\rangle\,[32][2P]} + \frac{[4P]^3\,\langle 2P\rangle^3}{[41][1P]\,\langle 23\rangle\,\langle 3P\rangle}\right]\\ &= \frac{C^{ade}C^{bce}}{u}\left[\frac{\langle 41\rangle\,[34]^3}{[32][21]} + \frac{[14]\,\langle 21\rangle^3}{\langle 23\rangle\,\langle 34\rangle}\right]\,,\end{aligned} \tag{6.27}$$

where we used $P = p_2 + p_3$. Recall, that the u-pole corresponds to $\langle 14\rangle\,[41] \to 0$. Since we are using complex momenta, remember that $\langle ij\rangle^* \neq [ji]$ and therefore only one of the two factors must go to zero - and we can choose which one. Let's choose $[14] = 0$, such that the second term drops. Then we find

$$\begin{aligned}\lim_{u\to 0} M(1^+2^+3^-4^-) &= \frac{C^{ade}C^{bce}}{u}\frac{\langle 41\rangle\,[34]^3}{[32][21]} = \frac{C^{ade}C^{bce}}{u}\frac{\langle 41\rangle\,[34]^3\,\langle 12\rangle\,\langle 23\rangle}{su}\\ &= \frac{C^{ade}C^{bce}}{u}\frac{[34]\,\langle 41\rangle\,[34]^2\,\langle 12\rangle\,\langle 23\rangle}{su} \quad \Big(\text{use }\ [34]\langle 41\rangle = -[32]\langle 21\rangle\Big)\\ &= -\frac{C^{ade}C^{bce}}{u}\frac{[32]\,\langle 23\rangle\,\langle 21\rangle\,\langle 12\rangle\,[34]^2}{su}\\ &= +\frac{C^{ade}C^{bce}}{u}\frac{\langle 12\rangle^2\,[34]^2}{s}\,.\end{aligned} \tag{6.28}$$

So we found

$$\lim_{u\to 0} M(1^+2^+3^-4^-) = \frac{C^{ade}C^{bce}}{su}\,\langle 12\rangle^2\,[34]^2\,, \tag{6.29}$$

which implies

$$\lim_{u\to 0}\left[usF^{abcd}(s,t,u)\right] = C^{ade}C^{bce}\,. \tag{6.30}$$

Carrying out the same steps for the t-channel cut leads to the corresponding limit. Putting everything together we find

$$\begin{cases} \lim_{s\to 0} F^{abcd}(s,t,u) = -\frac{C^{abe}C^{cde}}{su} \\ \\ \lim_{u\to 0} F^{abcd}(s,t,u) = +\frac{C^{ade}C^{bce}}{su} \\ \\ \lim_{t\to 0} F^{abcd}(s,t,u) = +\frac{C^{ace}C^{bde}}{st} \end{cases} . \tag{6.31}$$

Now, amplitudes are homogeneous functions (by dimensional analysis, they scale trivially with one single dimensional variable, while the non-trivial remaining dependence is only on dimensionless ratios). Since for four massless particles, momentum conservation implies $s+t+u=0$, we can parametrize the function $F^{abcd}(s,t,u)$ in general as

$$\begin{aligned} F^{abcd}(s,t,u) &= \frac{1}{su} f_1^{abcd}\Big(\frac{s}{u}\Big) + \frac{1}{tu} f_2^{abcd}\Big(\frac{t}{u}\Big) \\ &= \frac{1}{su}\sum_{n=0}^{\infty} a_n^{abcd}\left(\frac{s}{u}\right)^n + \frac{1}{tu}\sum_{n=0}^{\infty} b_n^{abcd}\left(\frac{t}{u}\right)^n , \end{aligned} \tag{6.32}$$

where the first term has poles in s and t and the second term in t and u. Note, that in the second line we inserted a Taylor expansion for both terms. As things stands, both terms have only single poles in s and t, as required by locality, but could have higher poles in u.

Let's study the different limits more closely, keeping only the leading divergent contributions. The $s \to 0$ limit is straightforward, and we find

$$\lim_{s\to 0} F^{abcd} = \frac{1}{su} a_0^{abcd} + \mathcal{O}(1) . \tag{6.33}$$

Comparing this with eq. (6.31) we can then infer

$$a_0^{abcd} = -C^{abe}C^{cde} . \tag{6.34}$$

Similarly, for $t \to 0$ we find

$$\lim_{t\to 0} F^{abcd} = \frac{1}{tu} b_0^{abcd} + \mathcal{O}(1) , \tag{6.35}$$

which using $u \to -s$ implies

$$b_0^{abcd} = -C^{ace}C^{bde} . \tag{6.36}$$

Finally, let us consider the $u \to 0$ limit. Due to the parametrization we have chosen, which allows for higher poles in u, this limit is more delicate. Equating both sides and using $t = -s$ in the $u \to 0$ limit, we find

$$\begin{aligned} \lim_{u\to 0} F^{abcd} = \frac{C^{ade}C^{bce}}{su} &= \frac{1}{su}\sum_{n=0}^{\infty} a_n^{abcd}\left(\frac{s}{u}\right)^n + \frac{1}{tu}\sum_{n=0}^{\infty} b_n^{abcd}\left(\frac{t}{u}\right)^n \\ \Rightarrow \quad C^{ade}C^{bce} &= \sum_{n=0}^{\infty} a_n^{abcd}\left(\frac{s}{u}\right)^n - \sum_{n=0}^{\infty} b_n^{abcd}\left(-\frac{s}{u}\right)^n \\ \Rightarrow \quad C^{ade}C^{bce} &= \sum_{n=0}^{\infty}\left[a_n^{abcd} - (-1)^n b_n^{abcd}\right]\left(\frac{s}{u}\right)^n . \end{aligned} \tag{6.37}$$

Remember that we started assuming C^{ade} is independent of the kinematics and definitely should not diverge when $u \to 0$. This requires all coefficients multiplying poles in u to go to zero, i.e.

$$a_n^{abcd} = (-1)^n b_n^{abcd} \quad \forall\, n > 0\,, \tag{6.38}$$

which leaves us with

$$C^{ade}C^{bce} = a_0^{abcd} - b_0^{abcd} = -C^{abe}C^{cde} + C^{ace}C^{bde}\,. \tag{6.39}$$

Note that in the second equality we used the results of eqs. (6.34) and (6.36). Using the antisymmetry of C^{abc}, we can rewrite this identity as

$$C^{abe}C^{cde} + C^{cae}C^{bde} + C^{ade}C^{bce} = 0\,, \tag{6.40}$$

which is nothing by the *Jacobi Identity* for the (structure) constants C^{abc}. This is a very powerful result! It demonstrates that Gauge theories based on a Lie Algebra are the unique solution for massless spin-one particles! This result was obtained just using the requirements of little group scaling, renormalizability and locality, namely that scattering amplitudes factorize properly on one particle states.

7 Recursion Relations

In the proof of the uniqueness of Yang-Mills theory, we have seen a first example of how general requirements as those of little group covariance and locality are enough to fix most of the structure of three- and four-point scattering amplitudes. We will now see that this can be generalized for all tree-level scattering amplitudes, such that they can be computed only resorting to on-shell, gauge invariant quantities, without having to use Feynman diagrams. The crucial step that will allow us to extract so much information from on-shell data only, is analytic continuation to complex kinematics.

Let's consider a tree-level on-shell amplitude for the scattering of n particles A_n. We assume again that all particles are massless

$$p_i^2 = 0 \tag{7.1}$$

and obey momentum conservation

$$\sum_i p_i = 0\,. \tag{7.2}$$

The amplitude A_n is a function of the momenta p_i and the external polarizations $\varepsilon_j, u, \overline{u}$. As we have seen, the helicity amplitudes will then all be function of spinor products $|i\rangle\,, |i]$, $\langle i|\,, [i|$ only. Now since we are assuming that the momenta are complex, let us imagine performing the following complex shift of the loop momenta:

$$p_i^\mu \to \hat{p}_i^\mu = p_i^\mu + z\, r_i^\mu\,, \tag{7.3}$$

where $z \in \mathbb{C}$.

We would like to guarantee that the shifted momenta $\hat{p}_i$ still obey momentum conservation

$$\sum_i \hat{p}_i = 0 \tag{7.4}$$

and remain massless

$$\hat{p}_i^2 = p_i^2 + 2z p_i \cdot r_i + z^2 r_i^2 = 0\,. \tag{7.5}$$

To do that, as one can easily prove, we can choose the new momenta r_i^μ such that

$$\sum_i r_i^\mu = 0\,, \qquad r_i \cdot r_j = 0\,, \qquad p_i \cdot r_i = 0\,. \tag{7.6}$$

Let us now take a subset of I momenta with $2 \leq \#I \leq n-2$ and define

$$P_I^\mu = \sum_{i \in I} p_i^\mu\,. \tag{7.7}$$

This corresponds to the momentum flow in the tree-level diagram with the momenta contained in P_I on the left hand side as shown in figure 16.

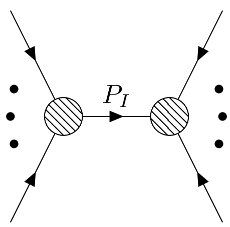


Figure 16: Tree-level amplitude with momentum P_I in the propagator.

After the shift defined above, the total momentum of any subset of particles becomes

$$\hat{P}_I^\mu = \sum_{i\in I} p_i^\mu + z\sum_{i\in I} r_i^\mu = P_I^\mu + zR_I^\mu\,, \tag{7.8}$$

where we defined

$$R_I^\mu = \sum_{i\in I} r_i^\mu\,. \tag{7.9}$$

Importantly, with these definitions $\hat{P}_I^2$ depends linearly on z

$$\hat{P}_I^2 = P_I^2 + 2zP_I\cdot R_I = -\frac{P_I^2}{z_I}(z-z_I)\,, \tag{7.10}$$

where we introduced

$$z_I = -\frac{P_I^2}{2P_I\cdot R_I}\,. \tag{7.11}$$

Notice that, if we consider $\hat{P}_I$ as a function of z, we have $\hat{P}_I^2 = 0$ for $z = z_I$.

Let us now study the amplitude after this shift as a function of z. At tree-level, we have seen that the scattering amplitude can only have poles, no branch cuts (logs, roots, etc). Therefore, $A_n(z)$ must be a rational function of z. Moreover,

$$A_n(z=0) = A_n \tag{7.12}$$

is the original unshifted amplitude. $A_n(z)$ can in general have *single* poles at different values of $z = z_I$, i.e. $A_n(z\to z_I) \sim 1/(z-z_I)$. This is obvious since, as we have discussed, all poles come from the intermediate states going on shell, i.e. they are all propagators of the general form $1/\hat{P}_I^2$, for some arbitrary subset of momenta. At tree-level, there can never be two equal propagators, as long as all external momenta area generic. The poles are

$$\frac{1}{\hat{P}_I^2} = -\frac{z_I}{P_I^2}\frac{1}{(z-z_I)}\,. \tag{7.13}$$

Since $z_I \neq 0$ the poles are all away from the origin of the z-$\mathbb{C}$ plane. This is again a consequence of locality.

Consider now the quantity $\frac{A_n(z)}{z}$. Since $A_n(z)$ does not have a pole at $z=0$, $\frac{A_n(z)}{z}$ has a single pole at the origin. We can then use Cauchy's theorem to write the original, unshifted amplitude, as

$$\oint_{\mathcal{C}} \frac{A_n(z)}{z}dz = A_n(0) = A_n\,. \tag{7.14}$$

According to the global residue theorem, we can deform the contour to infinity and write

$$\operatorname*{Res}_{z=0}\left(\frac{A_n(z)}{z}\right) + \sum_{z_I}\operatorname*{Res}_{z=z_I}\left(\frac{A_n(z)}{z}\right) = \operatorname*{Res}_{z=\infty}\left(\frac{A_n(z)}{z}\right)\,. \tag{7.15}$$

Using then the value of the residue at zero

$$\operatorname*{Res}_{z=0}\left(\frac{A_n(z)}{z}\right) = A_n\,, \tag{7.16}$$

we can finally write

$$A_n = -\sum_{z_I}\operatorname*{Res}_{z=z_I}\left(\frac{A_n(z)}{z}\right) + B_n\,, \tag{7.17}$$

where B_n denotes the boundary term at infinity

$$B_n = \operatorname*{Res}_{z=\infty} \left(\frac{A_n(z)}{z} \right) . \tag{7.18}$$

We can inspect the boundary term by substituting $z \to 1/y$ such that

$$\oint dz \frac{A_n(z)}{z} = \oint \frac{dy}{y} A_n(y) \tag{7.19}$$

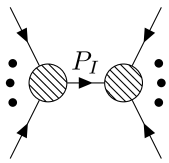
which brings the residue from infinity to zero. With this we can then write

$$B_n = A_n(y \to 0) = A_n(z \to \infty) , \tag{7.20}$$

which corresponds to the first coefficient of the Laurent expansion at infinity.

Now let us consider the poles at $z = z_I$. As we discussed proving YM uniqueness, locality implies that $\hat{P}_I^2 = 0$ and the amplitude must factorize appropriately onto products of lower point amplitudes, summing over all intermediate (on-shell) particle states that can be exchanged:

$$A_n(z \sim z_I) \longrightarrow$$



$$\begin{aligned} &= \hat{A}_L(z_I) \frac{1}{\hat{P}_I^2} \hat{A}_R(z_I) \\ &= - \frac{z_I}{z - z_I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) . \end{aligned} \tag{7.21}$$

where $\hat{A}_{L,R}$ are the left-hand and right-hand subamplitudes and in the last line the intermediate momentum $\hat{P}_I^2$ is not shifted. Hence, we find

$$\operatorname*{Res}_{z=z_I} \left(\frac{A_n(z)}{z} \right) = - \lim_{z \to z_I} \left[\frac{(z - z_I)}{z} \frac{z_I}{z - z_I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) \right] = - \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) . \tag{7.22}$$

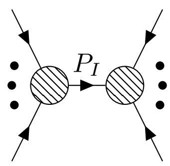
Note, that $A_{L,R}$ involve fewer legs than the original amplitude. So if $B_n = 0$ (no residue at infinity), we can get A_n from on-shell amplitudes with fewer particle. This can clearly be seen as the basis of a recursion relation.

A sufficient but not necessary condition for the absence of a boundary term is that $A_n(z) \to 0$ as $z \to \infty$. In jargon, if $A_n(z) \to 0$ for $z \to \infty$, we say that we started from a *valid shift*. Then:

$$A_n = \sum_I \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) . \tag{7.23}$$

In other words, we sum over all possible factorization channels and over all helicity states, etc. explicitly:

$$A_n = \sum_I$$



(7.24)

This is the general form of an on-shell recursion relation, since it allows one to build higher point scattering amplitudes from lower point, gauge invariant, on-shell building blocks.

7.1 BCFW Recursion Relation

The most famous on-shell recursion relation is the so called BCFW relation (from Britto, Cachazo, Feng, Witten, see [2]). It uses a special type of shift, where we shift only two momenta, say p_i, p_j. We also choose the extra momenta r_i, r_j such that the spinor products are shifted as follows

$$\begin{aligned} |\hat{i}\rangle &= |i\rangle\,; \qquad |\hat{i}] \to |i] + z|j] \\ |\hat{j}] &= |j\rangle\,; \qquad |\hat{j}\rangle \to |j\rangle - z\,|i\rangle\;. \end{aligned} \tag{7.25}$$

We call this a $[i, j\rangle$ shift. This implies that the momentum p_i is transformed as follows

$$\begin{aligned} \not{p}_i = |i\rangle\,[i| + |i]\,\langle i| \;\to |i\rangle\,[i| + z\,|i\rangle\,[j| + |i]\,\langle i| + z|j]\,\langle i| \\ = \not{p}_i + z(|i\rangle\,[j| + |j]\,\langle i|) \end{aligned} \tag{7.26}$$

and equivalently

$$\not{p}_j \to \not{p}_j - z(|i\rangle\,[j| + |j]\,\langle i|)\,. \tag{7.27}$$

Let us use this to write explicitly the corresponding shift momenta r_i and r_j. First of all, clearly from the formulas above $r_i^\mu = -r_j^\mu = q^\mu$ and $\not{q} = |i\rangle\,[j| + |j]\,\langle i|$ must hold.

The form of the $\not{q}$ is very reminiscent of what we would obtain by contracting a gluon polarizations vector of positive helicity with γ^μ and in fact, the shift momentum q^μ is

$$q^\mu = \frac{[j\,\gamma^\mu\,i\rangle}{2}\,. \tag{7.28}$$

One can easily verify that q^μ fulfils all properties of a "proper" shift momentum, as given in eq. (7.6)

$$\hat{p}_i + \hat{p}_j = p_i + p_j\,, \qquad q\cdot p_i = q\cdot p_j = 0\,, \qquad q\cdot q = 0\,,$$

where the second identity is a consequence of Dirac equation and the last can be proved using

$$[j\gamma^\mu i\rangle[j\gamma_\mu i\rangle \sim \langle ii\rangle\,[jj] = 0\,. \tag{7.29}$$

Note that q as we just defined it, does not exist as a real momentum. This becomes obvious using an explicit parametrization for the two massless momenta

$$p_i = (E, 0, 0, E); \quad P_j = (E, 0, 0, -E)\,. \tag{7.30}$$

Then the requirement $q\cdot p_i = q\cdot p_j = 0$ implies $q = (0, q_1, q_2, 0)$, and $q\cdot q = 0$ further requires $q_1^2 + q_2^2 = 0$, which for real momenta means $q^\mu = 0$. Thus, for a non trivial solution $q^\mu \neq 0$, q has to be complex. In fact,

$$q^\mu = \frac{[j\gamma^\mu i\rangle}{2}\,, \qquad (q^\mu)^* = \frac{\langle j\gamma^\mu i]}{2} \neq q^\mu\,. \tag{7.31}$$

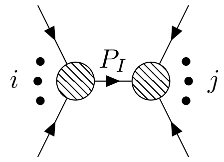
So to recapitulate, the BCFW shift $[i, j\rangle$ is

$$p_i \to p_i + z\frac{1}{2}[j\gamma^\mu i\rangle \qquad p_j \to p_j - z\frac{1}{2}[j\gamma^\mu i\rangle\,. \tag{7.32}$$

The general recursion formula simplifies. In fact, if the two shifted momenta are on the same side of the cut then the dependence on z cancels in the intermediate momentum

$\hat{P}_I^2 = P_I^2$. Consequently, there can be no pole in z (and no residue) and that particular configuration does not contribute to the recursion relation. We can therefore write

$$A_n = \sum_{I \in \Sigma_{ij}} \tag{7.33}$$



where I runs *only* over all set of indices Σ_{ij} where the momenta p_i and p_j are on the two opposite sides of the cut.

Finally, one could wonder what happens to the polarization vectors under this shift. Could they produce extra poles that generate unphysical contributions to the recursion relations? Under the BCFW shift we find

$$\varepsilon_{1+}^{\mu} \to -\frac{[r_i \gamma^{\mu} i\rangle}{\sqrt{2}([r_i i] + z[r_i j])}\,, \tag{7.34}$$

where r_i is the gauge momentum. The denominator might look like a new spurious pole. Nevertheless, it is easy to see that if we choose $r_i = p_j$ the extra term drops and $\hat{\varepsilon}_{1+}^{\mu} = \varepsilon_{1+}^{\mu}$. In conclusion, there always exists a gauge choice such that the polarization vectors are non shifted.

7.2 The Parke-Taylor Formula for N-Gluon Scattering

We will now use the BCFW recursion to prove the Parke-Taylor Formula

$$A(1^+2^+3^-...n^-) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle} \tag{7.35}$$

for the MHV n-gluon amplitudes. More generally, this is true for any two adjacent or non-adjacent + helicities. We will consider here the adjacent case and study the non-adjacent one in the exercises.

We will prove Parke-Taylor formula inductively, starting from $n = 3$ For three particles, we derived all amplitudes and found that little group scaling alone imposes (6.18)

$$M(1^+2^+3^-) = C^{abc}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,. \tag{7.36}$$

Which is exactly formula (7.35) for $n = 3$.

To use BCFW recursion, we first of all need to find a valid shift, i.e. a shift that does not generate any boundary terms at infinity in the recursion relation. We will cheat a bit here, and use the known form of the result (7.35) to study the behaviour at infinity of the amplitude under a specific shift. Let us consider in particular $[1,2\rangle$ shift, defined as

$$\begin{aligned} &|\hat{1}] = |1] + z|2]; \qquad &&|\hat{2}] = |2] \\ &|\hat{1}\rangle = |1\rangle\,; &&|\hat{2}\rangle = |2\rangle - z\,|1\rangle\,. \end{aligned} \tag{7.37}$$

The spinor products get shifted as

$$\begin{aligned} \langle 12\rangle &\to \langle 12\rangle - z\,\langle 11\rangle = \langle 12\rangle \\ \langle 23\rangle &\to \langle 23\rangle - z\,\langle 13\rangle\,, \end{aligned} \tag{7.38}$$

which shows that the amplitude $A(1^+2^+3^-...n^-)$ in Eq.(7.35) goes as $\frac{1}{z}$ for $z \to \infty$, which is enough to guarantee that there is no boundary term.

Clearly, this is not very satisfactory. Indeed, one can prove from general arguments that if we perform the shift $[i,j\rangle$ for any two adjacent spinors of different helicities i, j, the amplitude goes always as

$[i,j\rangle$:	$[++\rangle$	$[+-\rangle$	$[-+\rangle$	$[-+\rangle$
$A_n(z)$:	$\frac{1}{z}$	$\frac{1}{z}$	$\frac{1}{z}$	z^3

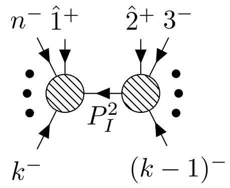
Table 2: Behaviour of the amplitude under different adjacent shifts.

The full proof can be found in [1]. With this, we see that the only shift that is not allowed is $[-+\rangle$. Let us exemplify this by shifting $[n1\rangle$ in $A(1^+2^+3^-...n^-)$. Then the spinor products go like

$$\begin{aligned} \langle 12\rangle &\to \langle 12\rangle - z\,\langle n2\rangle \\ \langle n1\rangle &\to \langle n1\rangle - z\,\langle nn\rangle = \langle n1\rangle \ . \end{aligned} \tag{7.39}$$

Consequently, the amplitude will go like $A_n(z) \sim \langle 12\rangle^3 \to z^3$, i.e. it would be divergent.

Let us then consider the shift $[1,2\rangle$. It is easy to convince ourselves that we can write (taking all momenta to be incoming)

$$A_n(1^+2^+3^-...n^-) = \sum_{k=4}^{n}$$ 

, (7.40)

where we sum over all possibilities such that

$$\hat{P}_I = \hat{p}_2 + p_3 + ... + p_{k-1} \tag{7.41}$$

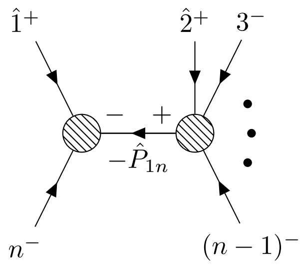
and all amplitudes are colour ordered. Note, that we need $\hat{1}$ and $\hat{2}$ to be on opposite sides, such that P_I depends on z. Moreover, we need at least a 3-point function on each side. Consequently, we have

$$\begin{aligned} A_n(1^+2^+3^-...n^-) = \sum_{k=4}^{n}\sum_{n_I=\pm} \Big[&\hat{A}_{n-k+3}(\hat{1}^+\hat{P}_I^{h_I}k^-...n^-) \\ &\times \frac{1}{P_I^2}\hat{A}_{k-1}(-\hat{P}_I^{-h_I}2^+3^-...(k-1)^-)\Big] , \end{aligned} \tag{7.42}$$

where we are summing over all possible helicities exchanged in the intermediate states.

Now recall that, even for complex values of the momenta, all amplitudes with only one plus helicity state $A(1^-...j^+...k^-)$ vanish, except for the three-point amplitudes, see discussion in Section 4.3. Therefore, only two diagrams can contribute to the recursion relation and the amplitude becomes

$$A_n(1^+2^+3^-...n^-) =$$



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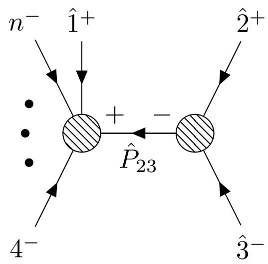


$$, \qquad (7.43)$$

where the helicity of the intermediate particle is fixed by the requirement that the corresponding amplitudes do not vanishing.

Let us focus first on the amplitude corresponding to the first diagram, remembering that $\hat{P}_I = \hat{P}_{1n}$ is evaluated at the residue $\hat{P}_{1n}^2 = 0$. The three-point amplitude reads

$$\hat{A}_3(\hat{1}^+, -\hat{P}_{1n}^-, n^-) = -\frac{[\hat{P}_{1n}n]^3}{[n\hat{1}][\hat{1}\hat{P}_{1n}]}\,, \qquad (7.44)$$

where we used our usual conventions for the analytic continuation $|-\hat{P}_{1n}] = i|\hat{P}_{1n}]$, etc. Since $\hat{P}_{1n}$ must be on-shell, we have

$$\hat{P}_{1n}^2 = 2\hat{p}_1 \cdot p_n = \langle 1n\rangle\, [n\hat{1}] = 0\,. \qquad (7.45)$$

Only $[n\hat{1}]$ depends on z, which means that the pole corresponds to the value of z for which $[n\hat{1}] = 0$. Now let us see what this implies for the numerator of the three point amplitude. Consider

$$|\hat{P}_{1n}\rangle\, [\hat{P}_{1n}n] = \hat{\not{P}}_{1n}|n] = (\hat{\not{p}}_1 + \not{p}_n)|n] = \hat{\not{p}}_1|n] = |\hat{1}\rangle\, [\hat{1}n]\,, \qquad (7.46)$$

This means that $[\hat{P}_{1n}n] \to 0$ as $[\hat{1}n] \to 0$. Similarly, one can show that also $[\hat{1}\hat{P}_{1n}] \to 0$ as $[\hat{1}n] \to 0$. Putting everything together this means that in the on-shell limit

$$\hat{A}_3(\hat{1}^+, -\hat{P}_{1n}^-, n^-) \sim [\hat{1}n] \to 0$$

So only the second diagram in Eq. (7.43) survives. Let us compute it.

Its amplitude reads

$$A_n(1^+2^+3^-...n^-) = \hat{A}_{n-1}(\hat{1}^+\hat{P}_{23}^+4^-...n^-)\frac{1}{P_{23}^2}A_3(-\hat{P}_{23}^-, 2^+, 3^-)\,. \qquad (7.47)$$

The three-point amplitude evaluates explicitly to

$$A_3(-\hat{P}_{23}^-, 2^+, 3^-) = -\frac{[3\hat{P}_{23}]^3}{[\hat{P}_{23}\hat{2}][\hat{2}3]}\,. \qquad (7.48)$$

Here $|\hat{2}] = |2]$, so none of the spinor products change in the on-shell limit. For the $(n-1)$-point amplitude, we use instead the induction approach and write

$$\hat{A}_{n-1}(\hat{1}^+\hat{P}_{23}^+4^-...n^-) = \frac{\langle\hat{1}\hat{P}_{23}\rangle^4}{\langle\hat{1}\hat{P}_{23}\rangle\,\langle\hat{P}_{23}4\rangle\,\langle 45\rangle\,...\,\langle n\hat{1}\rangle}\,. \qquad (7.49)$$

Sewing the two amplitudes together and writing the intermediate propagator in the on-shell limit in terms of spinor products $\hat{P}_{23}^2 \to 2p_2 \cdot p_3 = \langle 23\rangle [32]$, the n-point amplitude becomes

$$\begin{aligned} A_n(1^+2^+3^-...n^-) &= \frac{\langle \hat{1}\hat{P}_{23}\rangle^4}{\langle \hat{1}\hat{P}_{23}\rangle \langle \hat{P}_{23}4\rangle \langle 45\rangle ... \langle n\hat{1}\rangle} \frac{1}{\langle 23\rangle [32]} \frac{[3\hat{P}_{23}]^3}{[\hat{P}_{23}\hat{2}][3\hat{2}]} \\ &= \frac{\langle \hat{1}\hat{P}_{23}\rangle^3 [3\hat{P}_{23}]^3}{\langle 23\rangle \langle 45\rangle ... \langle n\hat{1}\rangle [32][3\hat{2}] \langle \hat{P}_{23}4\rangle [\hat{P}_{23}\hat{2}]} . \end{aligned} \tag{7.50}$$

Again the on-shell limit, we can manipulate the spinors as follows

$$\langle \hat{1}\hat{P}_{23}\rangle [3\hat{P}_{23}] = -\langle 1(\hat{\not{p}}_2 + \not{p}_3)3] = -\langle 1\hat{2}\rangle [\hat{2}3] = -\langle 12\rangle [23] , \tag{7.51}$$

where we used in the last step $|\hat{2}\rangle = |2\rangle - z\,|1\rangle$. Similarly, we also have

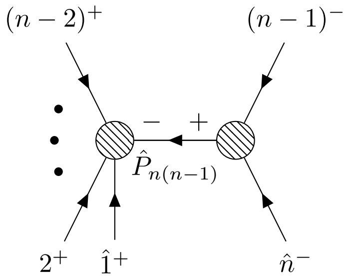
$$\langle \hat{P}_{23}4\rangle [\hat{P}_{23}\hat{2}] = -[\hat{2}\hat{P}_{23}] \langle \hat{P}_{23}4\rangle = -[\hat{2}(\hat{\not{p}}_2 + \not{p}_3)4\rangle = -[23] \langle 34\rangle . \tag{7.52}$$

Combining everything together, the amplitude becomes

$$\begin{aligned} A_n(1^+2^+3^-...n^-) &= \frac{\langle 12\rangle^3 [32]^3}{\langle 23\rangle \langle 45\rangle ... \langle n\hat{1}\rangle [32][3\hat{2}][32] \langle 34\rangle} \\ &= \frac{\langle 12\rangle^3}{\langle 23\rangle \langle 34\rangle \langle 45\rangle ... \langle n1\rangle} , \end{aligned} \tag{7.53}$$

where again we used that $|\hat{2}]$ and $|1\rangle$ are not shifted. This concludes the induction and proves the Parke-Taylor formula for n-gluon scattering and also a powerful application of the BCFW recursion relation.

One might wonder if $[1, 2\rangle$ is the only possible shift which one can use to prove the Parke-Taylor formula. The answer is obviously no! One could just as well have considered another shift, say $[1, n\rangle$, see table 2. In this case, it is easy to convince oneself that the only non-vanishing amplitude would be

$A_n(1^+2^+...n^-) =$  . (7.54)

So the expression for the n-gluon amplitude becomes

$$\begin{aligned} A_n(1^+2^+...n^-) =& \frac{\langle 12\rangle^4}{\langle 12\rangle \langle 23\rangle ... \langle n-2P_{nn-1}\rangle \langle P_{nn-1}1\rangle} \\ &\times \frac{1}{\langle nn-1\rangle [n-1n]} \frac{[n-1\hat{n}]^3}{[-\hat{P}_{nn-1}n-1][\hat{n}(-\hat{P}_{nn-1}]} . \end{aligned} \tag{7.55}$$

Similarly to the previous computation, we use the identities

$$\langle n-2\hat{P}_{nn-1}\rangle\,[\hat{n}\hat{P}_{nn-1}] = -\langle n-2(\hat{\not{p}}_n + \not{p}_{n-1})\hat{n}] = -\,\langle n-2n-1\rangle\,[n-1\hat{n}] \tag{7.56}$$

and

$$\langle\hat{P}_{nn-1}1\rangle\,[\hat{P}_{nn-1}n-1] = -\langle 1(\hat{\not{p}}_n + n \not{-}\, 1)n-1] = \langle n1\rangle\,[nn-1]\,. \tag{7.57}$$

Inserting this into the amplitude, we find after cancellations

$$A_n(1^+2^+...n^-) = \frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,...\,\langle n-2n-1\rangle\,\langle n-1n\rangle\,\langle n1\rangle}\,. \tag{7.58}$$

So using the $[1,n\rangle$ shift works just as well, and completely equivalently, to the $[1,2\rangle$ shift.

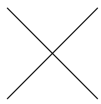
Let us make some comments on the validity of the BCFW recursion. We have been able to derive the amplitude for n-gluon scattering using only the formula for a 3-gluon on-shell amplitude. However, in Yang-Mills theory, there are two types of interactions: $A^2\delta A$, corresponding to a 3-gluon vertex, and A^4, corresponding to a 4-gluon vertex, where the A^4 term is required by off-shell gauge invariance on the Lagrangian. Why does this not matter for the amplitudes? The point is, that the 3-point amplitude we start from is on-shell and gauge invariant already! In this sense, together with our on-shell recursion relation, it already carries the (redundant) information contained in the 4-gluon vertex,which is why we never even have to consider it.

This feature becomes impressively powerful in gravity. Gravitational interactions are described by the following Lagrangian

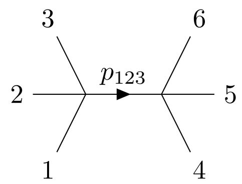
$$\mathcal{L} = \frac{1}{2k^2}\int d^4x\sqrt{-g}R\,, \tag{7.59}$$

where $g_{\mu\nu} = \eta_{\mu\nu} + kh_{\mu\nu}$. Expanding in k, we need infinitely many vertices to compute all tree-level amplitudes. On the other hand, one can prove that BCFW is valid for gravity at tree-level and that all infinite vertices are "redundant". Consequently, all tree-level on-shell amplitudes can be derived from 3-graviton scattering!

And what about $\lambda\phi^4$, the "simplest QFT"? The simplest amplitude in $\lambda\phi^4$-theory is the 4-point tree-level amplitude

$A_4 = \lambda =$  . (7.60)

The next-to simplest amplitude at tree-level is the 6-point amplitude

$A_6 =$ 

$+$ permutations. (7.61)

Hence, the 6-point amplitude is

$$\begin{aligned}A_6 &= \lambda^2 \left[\frac{1}{s_{123}} + ...\right] \\ &= \lambda^2 \left[\frac{1}{(p_1+p_2+p_3)^2} + ...\right] \\ &= \lambda^2 \left[\frac{1}{\langle 12\rangle [21] + \langle 13\rangle [31] + \langle 23\rangle [32]} + ...\right] .\end{aligned} \tag{7.62}$$

As it is easy to see, no matter which shift we do, there will always be terms not containing either of the shifted momenta. Consequently, $A_6(z) \to \mathcal{O}(z^0)$ when $z \to \infty$, i.e. there will always be boundary terms. Thus, the BCFW Recursion does not work for $\lambda\phi^4$-theory. In this sense, gauge theories are simpler than $\lambda\phi^4$-theory.

Part II

Introduction to 1-Loop Scattering Amplitudes

8 Introduction

In this section we will move from tree-level to one-loop amplitudes. Before delving into the specifics, we will recap some standard concepts about Feynman integrals, which will allow us also to establish the notation used throughout this section.

8.1 Tadpole Integral and Wick Rotation

We define the Feynman propagator in momentum space for a (scalar) particle with momentum q^μ and mass m as

$$\text{Feynman propagator } \sim \frac{1}{q^2 - m^2 + i0} = -\frac{1}{-q^2 + m^2 - i0}\,, \tag{8.1}$$

where $i0$ is is the usual Feynman prescription. We regulate ultraviolet (UV) and infrared (IR) divergences using *dimensional regularization* and set the spacetime dimension to $D = 4-2\epsilon$, with

(i) $\epsilon > 0$ in UV,

(ii) $\epsilon < 0$ in IR.

Moreover, we will often set the dimensional regularization scale to $\mu^2 = 1$.

The next step involves establishing conventions for four-momenta. Consider a general 1-loop diagram with N external lines, as depicted in Figure 17. Each line carries a momentum p_i^μ and a mass m_i. We assume that all external four-momenta are *incoming*. Additionally, denoting l^μ as the *1-loop four-momentum* and q_i^μ as the $N-1$ distinct *region momenta*, we summarize our conventions as follows:

$$\sum_{i=1}^{N} p_i^\mu = 0\,, \qquad p_i^\mu = q_i^\mu - q_{i-1}^\mu\,, \qquad q_i^\mu = \sum_{k=1}^{i} p_k^\mu. \tag{8.2}$$

Let us now consider the simplest example of a 1-loop integral, namely the *tadpole*, represented by Figure 17 ($N = 1$). We generalize the problem slightly and consider the tadpole for generic powers of the propagator

$$I_a = \int \frac{\mathrm{d}^D l}{i\pi^{D/2}} \frac{1}{(l^2 - m^2 + i0)^a} = \int \frac{\mathrm{d}^{D-1}\boldsymbol{l}}{i\pi^{D/2}} \int_{-\infty}^{\infty} \mathrm{d}l_0 \frac{1}{(l_0^2 - \boldsymbol{l}^2 - m^2 + i0)^a}\,, \tag{8.3}$$

where for now, we consider $a \in \mathbb{Z}$. The integral has a potential singularity at $l^2 = m^2$ and Feynman's prescription specifies along which contour we are supposed to integrate to avoid it. Let us consider first the integral in l_0. The integrand exhibits two poles,

$$l_0^\pm = \pm\sqrt{\boldsymbol{l}^2 + m^2} \mp i0\,, \tag{8.4}$$

as depicted in Figure 17 on the r.h.s.. But looking at the position of the poles, it is clear that in this case it is possible to redefine the integral by performing a so-called Wick rotation. Using Cauchy theorem, we can rotate the integration contour from the real axis onto the imaginary axis, avoiding the poles

$$l_0 \overset{\text{def}}{=} i \cdot l_{0,\mathrm{E}}\,, \qquad l_{0,\mathrm{E}} \in [-\infty, +\infty]\,, \tag{8.5}$$

where the subscript E signifies Euclidean.

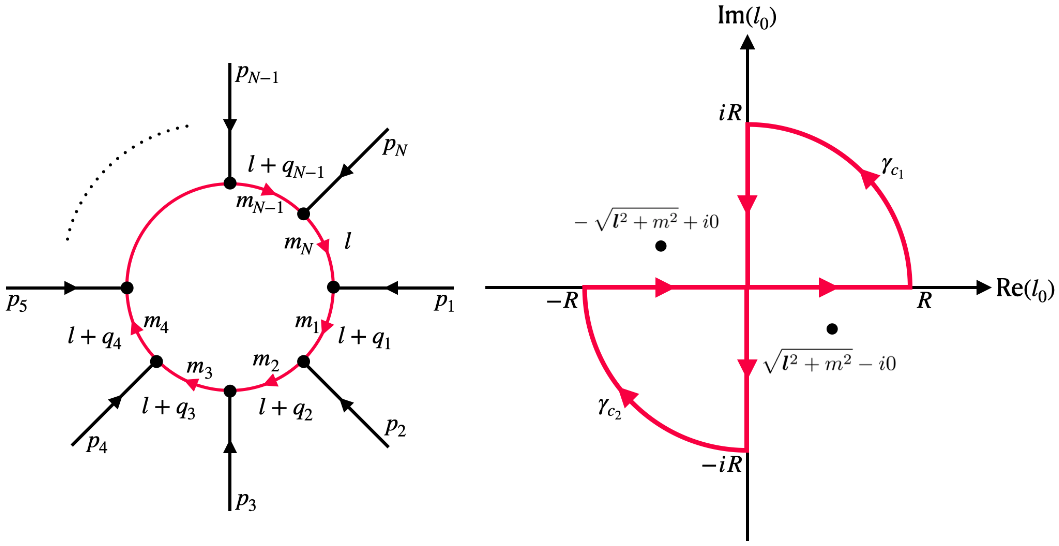


Figure 17: *On the left*: 1-loop diagram with N external lines. The arrows define the direction of the momenta.
On the right: The two balck dots represent the two poles of the tadpole propagator of Eq. (8.3), while the red path is the one that must be chosen in order to prove the Wick rotation.

We can now introduce a Euclidean momentum as

$$l_{\rm E}^{\mu} \stackrel{\rm def}{=} (l_{0,{\rm E}}, \boldsymbol{l}) , \tag{8.6}$$

such that the tadpole integral I_a becomes

$$I_a = (-1)^a \int \frac{{\rm d}^D l_{\rm E}}{\pi^{D/2}} \frac{1}{(l_{\rm E}^2 + m^2)^a} . \tag{8.7}$$

With this setup, evaluating I_a becomes straightforward. Going into spherical coordinates in D dimensions and using the fact that the integrand does not depend on any angles we can write

$$\int {\rm d}^D l_{\rm E} = \Omega(D) \int_0^\infty d|l_{\rm E}||l_{\rm E}|^{D-1} \tag{8.8}$$

such that

$$\begin{aligned} I_a &= (-1)^a \frac{\Omega(D)}{\pi^{D/2}} \int_0^\infty d|l_{\rm E}| \frac{|l_{\rm E}|^{D-1}}{(l_{\rm E}^2 + m^2)^a} \\ &= (-1)^a (m^2)^{(D-2a)/2} \frac{\Omega(D)}{\pi^{D/2}} \int_0^\infty dx \frac{x^{D-1}}{(x^2+1)^a} \\ &= (-1)^a \frac{(m^2)^{(D-2a)/2}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty dt \frac{t^{D/2-1}}{(t+1)^a} \\ &= (-1)^a \frac{\Gamma(a - D/2)}{\Gamma(a)} (m^2)^{(D-2a)/2} . \end{aligned} \tag{8.9}$$

It is important to note that the integral over $\mathrm{d}|l_\mathrm{E}|$ is well-defined in the ultraviolet (UV) kinematical region only when $2a > D$. In a $D = 4$ dimensional space, the tadpole integral diverges if $a \leq 2$. This divergence is not coincidental but is part of a broader convergence condition.

As manifestation of the fact that integrals in dimensional regularizations should be interpreted with care, consider the case $a = 1$ and $D = 3$. The integrals is clearly divergent in the UV. Nevertheless, plugging $D = 3$ and $a = 1$ in eq. (8.9) we get

$$I_1^{D=3} = -\int \frac{\mathrm{d}^3 l_\mathrm{E}}{\pi^{3/2}} \frac{1}{(l_\mathrm{E}^2 + m^2)} = -\Gamma(-1/2)\sqrt{m^2} = 2\sqrt{\pi}\sqrt{m^2}\,. \tag{8.10}$$

If we read this formula bringing the minus sign on the right hand side, i.e.

$$\int \frac{\mathrm{d}^3 l_\mathrm{E}}{\pi^{3/2}} \frac{1}{(l_\mathrm{E}^2 + m^2)} = -2\sqrt{\pi}\sqrt{m^2} \tag{8.11}$$

we see how the *divergent* integral of a *positive definite* integrand, generates in dimensional regularization a *negative* result! This is a consequence of the fact that integrals in dim reg should be interpreted with extreme care. Through analytic continuation, the integral becomes a meromorphic function of the complex variable D. The integral represents the function only when it converges, so for $D = 3$ the left-hand-side of eq. (8.11) has nothing to say about the right-hand-side. This is equivalent, in Feynman integral calculus, to the famous formula

$$\zeta(-1) = 1 + 2 + 3 + \ldots + \infty = -\frac{1}{12} \tag{8.12}$$

where the Riemann ζ-function is defined as

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n^s}\,.$$

8.2 Definition of 1-loop diagrams, UV and IR divergences

After this little warm up, let us move to consider more general integrals. Utilizing the parametrization from Eq. (8.2), we define the most general expression for a 1-loop integral in a D-dimensional Minkowski space, as depicted in Figure 17, i.e.

$$\begin{aligned} I_{N,\mathcal{N}(l^\mu)} &\stackrel{\text{def}}{=} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \\ &\times \frac{\mathcal{N}(l^\mu)}{\left[(l+q_1)^2 - m_1^2 + i0\right] \ldots \left[(l+q_{N-1})^2 - m_{N-1}^2 + i0\right]\left[l^2 - m_N^2 + i0\right]}\,. \end{aligned} \tag{8.13}$$

While we will be mainly interested in external massless particles, here we assume every external particle to be massive, for generality, which allows us to include also the case of an external leg splitting into two massless ones. The numerator $\mathcal{N}(l^\mu)$ is a polynomial in the loop momentum l^μ and external momenta. For example, the fermionic propagator corresponds to

$$\text{Fermionic propagator} \sim \frac{1}{\not{l} - m} = \frac{\not{l} + m}{l^2 - m^2}\,, \tag{8.14}$$

so $\mathcal{N}(l^\mu)$ includes its numerator $(\not{l} + m)$. In a renormalizable theory, if we consider a diagram with N external lines, the maximum polynomial degree is N, occurring in two configurations:

(i) all N propagators inside the loop are fermions, contributing $(\not{l} + m)$ each;

(ii) all fields within the loop and external lines are gluons due to the linearity of the triple-gluon vertex in l^μ.

Let $\mathcal{N}(l^\mu)$ be a polynomial of degree $r \leq N$, and let u_i, where $i = 1, \ldots, r$, be four-vectors dependent on the external momenta and polarizations. We rewrite I_N as

$$I_N^{(r)} \stackrel{\text{def}}{=} \prod_{i=1}^{r} u_i^{\mu_i} \overbrace{\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l_{\mu_1} \ldots l_{\mu_r}}{D_1 D_2 \ldots D_N}}^{\text{tensor integral of rank } r} = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\prod_{i=1}^{r}(u_i \cdot l)}{D_1 D_2 \ldots D_N}\,, \tag{8.15}$$

where

$$D_i = (l + q_i)^2 - m_i^2 + i0\,, \qquad q_N = 0\,. \tag{8.16}$$

Integrals with $r = 0$, denoted as $I_N^{(0)}$, are usually referred to as *scalar integrals*. Additionally, note that the scalar product between loop momentum and region momentum simplifies to a linear combination of propagators,

$$\begin{aligned} l \cdot q_i &= \frac{1}{2}\Big[(l + q_i)^2 - m_i^2 - (l^2 - m_N^2) - (q_i^2 - m_i^2) - m_N^2\Big] \\ &= \frac{1}{2}\Big[D_i - D_N - q_i^2 + m_i^2 - m_N^2\Big]\,. \end{aligned} \tag{8.17}$$

In general terms, $I_N^{(r)}$ exhibits two distinct types of divergences: ultraviolet (UV) and infrared (IR) divergences. Without delving into specifics at this point, let us briefly outline the primary characteristics of both.

UV divergences

Let us consider the behaviour of $I_N^{(r)}$ as given in Eq. (8.15) in the UV, i.e. when l^μ becomes large. As we are only interested in the UV behaviour, for simplicity we introduce an infrared (IR) cut-off Λ and write

$$I_N^{(r)} = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\prod_{i=1}^{r}(u_i \cdot l)}{D_1 D_2 \cdots D_N} \sim \int \frac{d^D l}{(2\pi)^D} \frac{l^{r+D-1}}{l^{2N}} \sim \int_\Lambda^\infty dl\; l^{D-1+r-2N} \tag{8.18}$$

Therefore, $I_N^{(r)}$ is UV-divergent when $2N - r - D + 1 \leq 1$, specifically $r \geq 2N - D$. Now recalling that in a renormalizable theory $r \leq N$, the general divergence condition becomes $N \leq D$. Assuming $D = 4$, any 1-loop diagram with $N \geq 5$ external lines must therefore be UV-convergent. Hence, only integrals up to four points (boxes) can yield UV divergences in renormalizable quantum field theories like QCD or $\mathcal{N} = 4$ SYM.

If we restrict to scalar integrals $I_N^{(0)}$, the divergence condition becomes $N \leq D/2$, i.e. $N \leq 2$ if $D = 4$. Consequently, only 1- and 2-point scalar integrals are UV divergent.

IR divergences

Here we explore the opposite scenario compared to the previous one, specifically when $I_N^{(r)}$ is divergent close to the lower extreme of integration, i.e. in the IR (soft) region. As illustration, consider the 3-point integral $I_3^{(r)}$, in a special configuration where $p_i^2 = m_i^2$ with

Table 3: Conditions of UV-divergence for both the tensor integral $I_N^{(r)}$ and the scalar integral $I_N^{(0)}$.

IF	Tensor Integral	Scalar Integral
$\boldsymbol{r \leq N}$, **any** $\boldsymbol{D}$	$r \geq 2N - D$	$N \leq D/2$
$\boldsymbol{r = N}$, **any** $\boldsymbol{D}$	$N \leq D$	-
$\boldsymbol{r \leq N}$, $\boldsymbol{D = 4}$	$r \geq 2N - 4$	$N \leq 2$
$\boldsymbol{r = N}$, $\boldsymbol{D = 4}$	$N \leq 4$	-

$i = 1, 2$ and $p_3^2 = 0$. A similar integral, for $m_1 = m_2$, arises when computing the 1-loop electron vertex within QED. Using Eq. (8.2), we write

$$\begin{aligned} I_3^{(0)} &\stackrel{\text{def}}{=} \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{l^2\left[(l-p_1)^2 - m_1^2\right]\left[(l+p_2)^2 - m_2^2\right]} \\ &= \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{l^2(l^2 - 2l\cdot p_1)(l^2 + 2l\cdot p_2)}\,. \end{aligned} \tag{8.19}$$

This time we focus on the IR regime and introduce an high-energy cut-off Λ. In the soft limit, $l \to 0$, we find

$$I_3^{(0)} \sim \int_0^\Lambda \mathrm{d}l\, \frac{l^{D-1}}{l^4}\,. \tag{8.20}$$

which converges at $l = 0$ if $4 - D + 1 < 1$, i.e. $D > 4$. Such a divergence is referred to as *soft*. In four dimensions, to regulate this divergence, we can set $D = 4 - 2\epsilon$, assuming $\epsilon < 0$. This regularization produces a pole of the form

$$I_3^{(0)} \sim \int_0^\Lambda \frac{\mathrm{d}l}{l^{1+2\epsilon}} \sim \frac{1}{\epsilon}\,. \tag{8.21}$$

Furthermore, there exists another divergence type from Eq. (8.19). Here, let l^μ be parallel (*collinear*) to one of the external momenta, say $l^\mu = c \cdot p_1^\mu$, and assume all the external fields to be massless. In this case

$$(l + p_1)^2 = (c \cdot p_1 + p_1)^2 = (1 + c)^2 p_1^2 = 0\,. \tag{8.22}$$

This shows that in the collinear configuration the integral develops a pole, referred to as a collinear divergence

$$I_3^{(0)} \sim \int \frac{\mathrm{d}^D l}{i\pi^{D/2}} \frac{1}{l^4(l^2 - 2l\cdot p_2)}\,. \tag{8.23}$$

As the loop momentum can still approach zero ($l^\mu \to 0$), there is an extra *overlapping* soft divergences, such that

$$I_3^{(0)} \sim \int \frac{\mathrm{d}^D l}{i\pi^{D/2}} \frac{1}{l^4(l^2 - 2l\cdot p_2)} \sim \frac{1}{\epsilon^2}\,. \tag{8.24}$$

8.3 Generalities on 1-loop amplitudes

Loop integrals are the backbone of loop scattering amplitudes. They are "hidden" inside Feynman diagrams at loop level, which brings a new type of complexity. Before proceeding to the next section elucidating the comprehensive theoretical framework for computing N-point functions, let us look at an explicit example to see how scalar integrals can be extracted in general from Feynman graphs.

Consider the Feynman diagram in Figure 18, corresponding to the one-loop QED corrections to the photon propagator. The amplitude corresponding to this diagram is given by ($p_{12}^\mu = p_1^\mu + p_2^\mu$)

$$i\mathcal{M} = \bar{u}_2\gamma^\mu u_1 \frac{i}{p_{12}^2} \int \frac{\mathrm{d}^D l}{i\pi^{D/2}} \frac{\mathrm{Tr}[\gamma_\mu \not{l} \gamma_\nu (\not{l} - \not{p}_{12})]}{(l^2 - m^2 + i\epsilon)\big[(l - p_{12})^2 - m^2 + i\epsilon\big]} \frac{i}{p_{12}^2} \bar{u}_3\gamma^\nu u_4 \,. \tag{8.25}$$

Let us introduce the usiual polarization tensor

$$\Pi_{\mu\nu}(p_{12}^2) \stackrel{\text{def}}{=} \int \frac{\mathrm{d}^D l}{i\pi^{D/2}} \frac{\mathrm{Tr}[\gamma_\mu \not{l} \gamma_\nu (\not{l} - \not{p}_{12})]}{(l^2 - m^2 + i\epsilon)\big[(l - p_{12})^2 - m^2 + i\epsilon\big]} \,, \tag{8.26}$$

such that

$$i\mathcal{M} = \bar{u}_2\gamma^\mu u_1 \frac{i}{p_{12}^2} \Pi_{\mu\nu}(p_{12}^2) \frac{i}{p_{12}^2} \bar{u}_3\gamma^\nu u_4 \,. \tag{8.27}$$

Note that $\Pi_{\mu\nu}(p_{12}^2)$ represents a proper *tensor integral*, which comprises a combination of the following integrals:

$$\int \frac{\mathrm{d}^D l}{i\pi^{D/2}} \frac{\{l^\mu l^\nu;\ l^\mu p_{12}^\nu;\ l^\nu p_{12}^\mu;\ p_{12}^\mu p_{12}^\nu\}}{(l^2 - m^2 + i\epsilon)\big[(l - p_{12})^2 - m^2 + i\epsilon\big]} \,. \tag{8.28}$$

To compute this, a convenient starting point is so-called *tensor decomposition*. We argue as follows

(i) If $q^\mu = p_{12}^\mu$, then $\Pi_{\mu\nu}$ is a function of q^2, that is $\Pi_{\mu\nu} = \Pi_{\mu\nu}(q^2)$.

(ii) Using *Lorentz covariance* we can say that

$$\Pi_{\mu\nu}(q^2) = F_1\, q_\mu q_\nu + F_2\, g_{\mu\nu} \,, \tag{8.29}$$

where $F_{1,2}$ are two *scalar form factors*.

(iii) $\Pi_{\mu\nu}(q^2)$ must be *gauge invariant*. This means that the *Ward Identity*

$$q^\mu \Pi_{\mu\nu}(q^2) = (F_1 q^2 + F_2) q_\nu \equiv 0 \tag{8.30}$$

must hold, which implies

$$F_1 = -\frac{F_2}{q^2} \,. \tag{8.31}$$

Relabelling $F_2 \mapsto F$, we find

$$\Pi_{\mu\nu}(q^2) = F \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \,. \tag{8.32}$$

The expression in brackets indeed represents the numerator of the transverse photon propagator. There is therefore one single form factor, denoted as F.

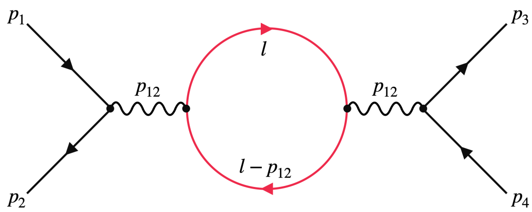


Figure 18: 1-loop Feynman diagram for the process $p_1 + p_2 \to p_3 + p_4$, where both the incoming and outgoing particles are fermions.

To compute F, it is convenient to define a projector $P^{\mu\nu}$ that fulfils the condition $P^{\mu\nu}\Pi_{\mu\nu}(q^2) = F$. We can use for $P^{\mu\nu}$ the Ansatz

$$P^{\mu\nu} \stackrel{\text{def}}{=} c\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right) , \tag{8.33}$$

and determine the prefactor c imposing

$$P^{\mu\nu}\Pi_{\mu\nu}(q^2) = c\,F\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) = c\,F(-1 + D) \equiv F\,, \tag{8.34}$$

which gives

$$c = \frac{1}{D-1} \qquad \Longrightarrow \qquad P^{\mu\nu} = \frac{1}{D-1}\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right) . \tag{8.35}$$

Without performing the explicit calculation it should be clear that upon applying $P^{\mu\nu}$ onto Eq. (8.28), we will be left with a linear combination of integrals which involve numerators built out of scalar products of the loop momentum and the external momentum q

$$\begin{aligned} P^{\mu\nu} &\int \frac{\mathrm{d}^D l}{i\pi^{D/2}} \frac{\{l_\mu l_\nu;\ l_\mu q_\nu;\ l_\nu q_\mu;\ q_\mu q_\nu\}}{(l^2 - m^2 + i\epsilon)\big[(l-q)^2 - m^2 + i\epsilon\big]} \\ &= \int \frac{\mathrm{d}^D l}{i\pi^{D/2}} \frac{\{l \cdot l;\ l \cdot q;\ q \cdot q\}^n}{(l^2 - m^2 + i\epsilon)\big[(l-q)^2 - m^2 + i\epsilon\big]} \,, \end{aligned} \tag{8.36}$$

where the power n can vary as an integer.

In what follows, we will elaborate on a general method to “reduce” this type of integrals to a unique basis of so-called master integrals, in terms of which any one-loop scattering amplitude can be computed.

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