

How to calculate observables at hadron colliders

- Collider physics uses scattering experiments to learn about fundamental particles and forces
 - observables at collider experiments are usually cross sections and related concepts
- Quantum mechanics tells us: calculate squared (transition) amplitudes (from some initial to some final state) and sum/integrate over unobserved degrees of freedom (average for initial state)
- At hadron colliders: we collide composite particles (usually protons), but our fundamental theories are formulated in terms of elementary particles (quarks, gluons, ...)
 - we need to rely on factorisation of physics at very different scales
 - non-perturbative, low-energy dynamics of bound states (protons)
 - perturbative, high-energy scattering of fundamental particles
 - describe the part of the initial state that's relevant for the high-energy scattering using parton distribution functions (PDFs)

- Fundamental formula for most predictions at collider experiments:

$$d\sigma \sim \sum_{a,b,X} \frac{1}{N_i} \int_0^1 dx_1 \int_0^1 dx_2 \int d\Phi \underbrace{f_{a/p}(x_1) f_{b/p}(x_2)}_{\text{PDFs}} \underbrace{|M_{ab \rightarrow X}(x_1, x_2, \Phi)|^2}_{\text{squared matrix elements}} \underbrace{F(\Phi)}_{\text{measurement function/observable}}$$

phase space integration

- A lot of things need to come together in order to make useful predictions
- Each of the elements in the formula above points to a different rabbit hole / set of expertise that we need to explore

→ **squared matrix elements** → often calculated in perturbation theory from Feynman diagrams & Feynman rules
→ involves integrals over virtual loop momenta → Feynman integrals
→ we'll spend the first week on how to calculate scattering amplitudes & Feynman integrals (me, Renato & Tom)

→ **PDFs**: describe structure of initial state protons in terms of quarks & gluons

- Have to be fitted from experiment, but are universal (valid for any proton scattering process)
- They are scale-dependent quantities („protons look different if you look closer”), but we can predict their scale evolution
- Rhorry will talk about them in the second week

→ **phase space integration**

- integrate over all valid final state momentum configurations
- subject to constraints (momentum conservation, on-shell condition, ...)
- need to find suitable parametrisations
- integration is often done numerically → needs Monte Carlo techniques
- Rhorry covers this in his lectures in the second week

→ observables:

- when we integrate over "everything" ($F(\{\Phi\})=1$), we obtain the total cross section: σ
- we can also restrict the integration region (e.g., $F(\{\Phi\})=\Theta(\hat{p}_{T,Z}>30\text{GeV})$) $\Rightarrow$ fiducial cross sections: $\sigma(p_{T,Z}>30\text{GeV})$
- or we can ask for an observable to have a precise value (e.g., $F(\{\Phi\})=\delta(\hat{y}_H - y_H)$) $\Rightarrow$ differential distributions: $\frac{d\sigma}{dy_H}$
- practical implementation: collection of "fiducial cross sections" ($F(\{\Phi\}) \in \{\Theta(0 < \hat{y}_H < 0.5), \Theta(0.5 < \hat{y}_H < 1.0), 1\}$) $\Rightarrow$ binned differential distribution/histograms
- important question: what are observables we can use in the measurement function?
 - key property of observables: infrared and collinear safety $\rightarrow$ Rhorry and Marius will touch on this in week 2
 - interesting observables can be built from "jets" and their "substructure" $\rightarrow$ Rhorry and in particular Andrea talk about this (week 2)

→ sum over relevant final states:

- (colour-)charged particles radiate more particles
- realistic descriptions should take into account fairly high-multiplicity final states
- parton showers allow to describe dominant contributions from such final states that would not be accessible using fixed order perturbation theory
- Marius will explain this in more detail in his lecture in the second week

Quick recap of some basics about QCD

Lagrangian of QCD:

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \sum_f \bar{\Psi}_f (i \not{D} - m_f) \Psi_f$$

colour index in adjoint rep. $\rightarrow$ number of quark flavours $\rightarrow$ mass of quark flavour m_f
 $\rightarrow$ $SU(3)_c$ structure constants

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c$ is the gluon field strength tensor

and $D_\mu = \partial_\mu - ig_s \vec{T}^a A_\mu^a$ is the covariant derivative.
 $\rightarrow$ $SU(3)_c$ generators in fundamental rep.

To quantize the theory, we have to fix a gauge and go through the Faddeev-Popov procedure that leaves us with ghost fields.

From this we can derive Feynman rules of QCD:

$\rightarrow$ propagators

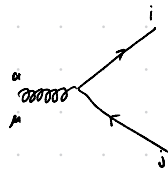
quarks $j \xrightarrow{p} i \rightarrow \frac{i(\not{p} + m)\delta_{ij}}{p^2 - m^2 + i0^+}$

gluons $a \xrightarrow{p} b \rightarrow \frac{i[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}]\delta^{ab}}{p^2 - m^2 + i0^+}$ (here in general covariant gauge; useful choice: Feynman gauge ($\xi=1$))

ghosts $a \cdots b \rightarrow \frac{i\delta^{ab}}{p^2 + i0^+}$

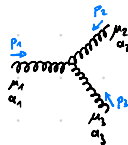
$\rightarrow$ vertices

quark gluon coupling



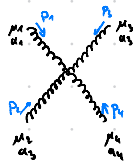
$$ig_s \gamma^\mu T_{ij}^a$$

3-gluon vertex



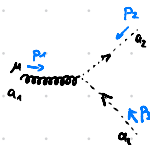
$$g_s f^{a_1 a_2 a_3} [g_{\mu_1 \mu_2} (p_1 - p_2)_{\mu_3} + g_{\mu_2 \mu_3} (p_2 - p_3)_{\mu_1} + g_{\mu_3 \mu_1} (p_3 - p_1)_{\mu_2}]$$

4-gluon vertex



$$-ig_s^2 \left[f^{a_1 a_2 b} f^{a_3 a_4 b} (g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} - g_{\mu_1 \mu_3} g_{\mu_2 \mu_4}) \right. \\ \left. + f^{a_1 a_2 b} f^{a_3 b \mu} (g_{\mu_1 \mu_2} g_{\mu_3 \mu} - g_{\mu_1 \mu} g_{\mu_2 \mu_3}) \right. \\ \left. + f^{a_1 a_3 b} f^{a_2 b \mu} (g_{\mu_1 \mu} g_{\mu_2 \mu_3} - g_{\mu_1 \mu_3} g_{\mu_2 \mu}) \right]$$

ghost vertex



$$-g_s f^{abc} p_{2\mu}$$

$\rightarrow$ external legs:

incoming quark: $\xrightarrow{p} u(p, \lambda)$

outgoing quark: $\xrightarrow{p} \bar{u}(p, \lambda)$

incoming antiquark: $\xleftarrow{p} \bar{v}(p, \lambda)$

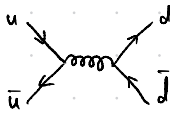
outgoing antiquark: $\xleftarrow{p} v(p, \lambda)$

incoming gluon: $\xrightarrow{p} \varepsilon_\mu^a(p, \lambda)$

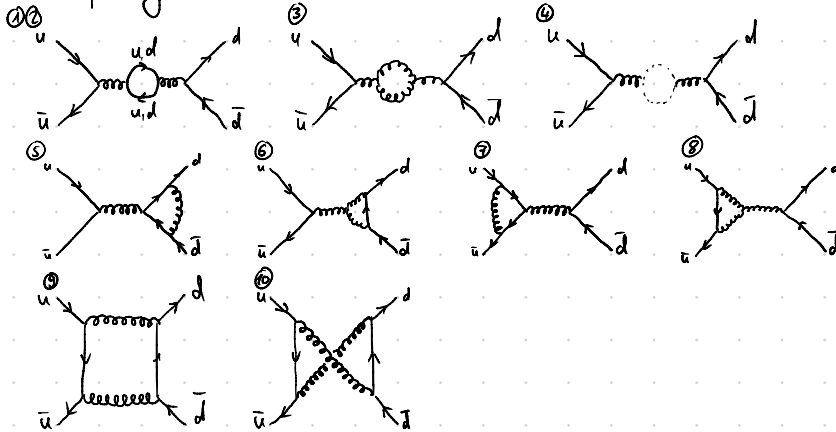
outgoing gluon: $\xrightarrow{p} \varepsilon_\mu^{a*}(p, \lambda)$

Example: One-loop corrections to $u\bar{u} \rightarrow d\bar{d}$

• Tree-level diagrams



• One-loop diagrams



• Use this as our running example, but focus on only one tree-level and one one-loop diagram to keep the workload manageable.

$\rightarrow i\mathcal{M}^{(0)} = \bar{v}(p_2, \lambda_2) i g_s T^a \gamma^\mu u(p_1, \lambda_1) \frac{i[-g_{\mu\nu} + (1-\xi) \frac{p_{2,\mu} p_{2,\nu}}{p_2^2}]}{p_2^2 + i0} \bar{u}(q_1, \lambda_1) i g_s T^b \gamma^\nu v(q_2, \lambda_2) \quad (p_{12} = p_1 + p_2 = q_1 + q_2)$

• Gauge invariance requires amplitude (sum of all diagrams, but here it's only one diagram) to be independent of gauge parameter ξ . Can we see that? Focus on term $\propto (1-\xi)$ and ignore colour, couplings and propagators for now.

$\bar{v}(p_2, \lambda_2) \not{p}_1 + \not{p}_2 u(p_1, \lambda_1) = 0$ using Dirac eq: $\not{p}_1 u(p_1) = m u(p_1) = 0$; $\bar{v}(p_2) \not{p}_2 = -m \bar{v}(p_2) = 0$

• Complex conjugation of amplitude:

$(\mathcal{M}^{(0)})^* = -i g_s^2 \bar{u}(p_1, \lambda_1) T^b \gamma^\nu v(p_2, \lambda_2) \frac{1}{p_{12}^2} v(q_2, \lambda_2) T^a \gamma^\mu u(q_1, \lambda_1)$

• Unpolarised squared matrix element:

$\sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} |\mathcal{M}^{(0)}|^2 = g_s^4 \frac{1}{(p_{12}^2)^2} \sum_{\lambda_1, \lambda_2} \bar{u}(p_1, \lambda_1) T^b \gamma^\nu v(p_2, \lambda_2) \bar{v}(p_2, \lambda_2) T^a \gamma^\mu u(p_1, \lambda_1) \sum_{\lambda_3, \lambda_4} \bar{v}(q_2, \lambda_2) T^b \gamma_\nu u(q_1, \lambda_1) \bar{u}(q_1, \lambda_1) T^a \gamma_\mu v(q_2, \lambda_2)$

[Use spin sums $\sum_\lambda u(p, \lambda) \bar{u}(p, \lambda) = \not{p} + m$ and $\sum_\lambda v(p, \lambda) \bar{v}(p, \lambda) = \not{p} - m$ and $p_{12}^2 = s$

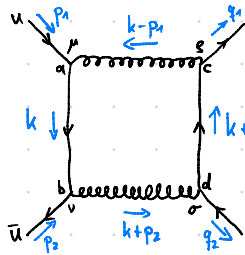
$= \frac{g_s^4}{s^2} \text{Tr}[\not{p}_1 \gamma^\nu \not{p}_2 \gamma^\mu] \text{Tr}[\not{q}_2 \gamma_\nu \not{q}_1 \gamma_\mu] \text{Tr}[T^b T^a] \text{Tr}[T^a T^b]$
 $= 4(\not{p}_1 \not{p}_2 + \not{p}_2 \not{p}_1 - \not{p}_1 \not{p}_2) = 4(\not{q}_1 \not{q}_2 + \not{q}_2 \not{q}_1 - \not{q}_1 \not{q}_2)$

$= \frac{g_s^4}{s^2} 16[(p_1 \cdot p_2)(q_1 \cdot q_2)(d-4) + 2(p_1 \cdot q_1)(p_2 \cdot q_2) + 2(p_1 \cdot q_2)(p_2 \cdot q_1)] \text{Tr}[T^a T^a]$
 $= \frac{g_s^4}{s^2} 16[(p_1 \cdot p_2)(q_1 \cdot q_2)(d-4) + 2(p_1 \cdot q_1)(p_2 \cdot q_2) + 2(p_1 \cdot q_2)(p_2 \cdot q_1)] (N_c^2 - 1)$

[Use $s = (p_1 + p_2)^2 = 2p_1 \cdot p_2 = (q_1 + q_2)^2 = 2q_1 \cdot q_2$; $t = (p_1 - q_1)^2 = -2p_1 \cdot q_1 = (p_2 - q_2)^2 = -2p_2 \cdot q_2$; $u = (p_1 - q_2)^2 = -2p_1 \cdot q_2 = (p_2 - q_1)^2 = -2p_2 \cdot q_1 = -s - t$

$= \frac{g_s^4}{s^2} 4 \text{Tr}(N_c^2 - 1) [4t^2 + 4st + (d-2)s^2]$

- One-loop box diagram (9) [work immediately in Feynman gauge ($\xi=1$)]



$$\rightarrow i\mathcal{M}_{\textcircled{9}} = \int \frac{d^d k}{(2\pi)^d} \bar{v}(p_2, \lambda_{\bar{u}}) i g_s T^b \gamma^\nu \frac{i \not{k}}{k^2 + i0} i g_s T^a \gamma^\mu u(p_1, \lambda_u) \frac{-ig_{\mu\sigma} \delta^{ac}}{(k-p_1)^2} \frac{-ig_{\sigma\tau} \delta^{bd}}{(k+p_2)^2} \\ \bar{u}(q_1, \lambda_d) i g_s T^c \gamma^s \frac{i(\not{k} + \not{q}_1 + \not{p}_1)}{(k+q_1+p_1)^2 + i0} i g_s T^d \gamma^\tau v(q_2, \lambda_{\bar{d}})$$

- Squared matrix element:

$$|\mathcal{M}|^2 = ((\mathcal{M}^{(0)})^* + (\mathcal{M}^{(1)})^* + \dots)(\mathcal{M}^{(0)} + \mathcal{M}^{(1)} + \dots)$$

$$= (\mathcal{M}^{(0)})^* \mathcal{M}^{(0)} + (\mathcal{M}^{(1)})^* \mathcal{M}^{(0)} + (\mathcal{M}^{(0)})^* \mathcal{M}^{(1)} + \dots$$

$$= |\mathcal{M}^{(0)}|^2 + 2\text{Re}[(\mathcal{M}^{(0)})^* \mathcal{M}^{(1)}] + \dots$$

- Contribution of diagram (9) to unpolarised squared matrix element:

$$\sum_{\substack{\lambda_u, \lambda_{\bar{u}}, \\ \lambda_d, \lambda_{\bar{d}}}} 2\text{Re}[(\mathcal{M}^{(0)})^* \mathcal{M}_{\textcircled{9}}] = 2\text{Re} \left[\frac{ig_s^6}{s} \int \frac{d^d k}{(2\pi)^d} \text{Tr}[\not{p}_1 \gamma^\tau \not{p}_2 \gamma^\nu \not{k} \gamma^\mu] \text{Tr}[\not{q}_2 \gamma_\tau \not{q}_1 \gamma_\mu (\not{k} + \not{q}_1 + \not{p}_1) \gamma_\nu] \right. \\ \left. \text{Tr}[T^e T^b T^a] \text{Tr}[T^c T^a T^b] [k^2 (k-p_1)^2 (k+p_2)^2 (k+q_1+p_1)^2]^{-1} \right]$$

- Dirac algebra: $\text{Tr}[\not{p}_1 \gamma^\tau \dots] \text{Tr}[\dots] = 4 \{ k^2 [8t^2 + 4st + s^2(3d-8)] + (p_1 k)^2 [8(s+t)(4-d)] + \dots \} \rightarrow \text{see exercises}$

- Colour algebra: Use Fierz identity for $SU(3)$ generators: $T_{ij}^a T_{kl}^b = T_F (\delta_{ik} \delta_{jl} - \frac{1}{N_c} \delta_{ij} \delta_{kl})$

$$\Rightarrow \text{Tr}[T^e T^b T^a] \text{Tr}[T^c T^a T^b] = T_F^3 \frac{(N_c^2 - 2)(N_c^2 - 1)}{N_c} = \frac{8}{3}$$

$\rightarrow \text{see exercises}$

Tensor decomposition of amplitudes

- Before we move on and deal with the loop integrals, let's discuss an alternative way of dealing with the amplitudes themselves (without squaring them)
- Why?
 - ↪ Maybe we're not just interested in unpolarised squared matrix element...
 - ↪ Maybe there are many diagrams we need to multiply $\Rightarrow$ quadratic explosion of terms

- What's the problem? Suppose we want to calculate $M_0^{(n)}$ directly. There will be terms like:

$$\int d^4k \frac{(\bar{u} \gamma^\mu \not{k} \gamma^\nu u)(\bar{u} \gamma_\mu \not{k} \gamma_\nu v)}{k^2 (k-p_1)^2 (k+p_2)^2 (k+p_1+p_2)^2}$$

This mixes the Lorentz and Dirac structure of the external particles (spinors & polarisation vectors) with the loop integration.

- Why wasn't this a problem above? Spin/polarisation sums eliminated spinors & polarisation vectors and turned open fermion chains into traces $\Rightarrow$ we were only left with scalar products.
- Goal: simplify the numerator & separate Dirac & Lorentz algebra as much as possible from loop integrals
- Two obvious directions:

$\rightarrow$ Use properties of tensor Feynman integrals to turn $\int d^4k \frac{k^\mu k^\nu}{[\dots]}$

into $g^{\mu\nu} \int d^4k \frac{1}{[\dots]} + p_1^\mu p_1^\nu \int d^4k \frac{1}{[\dots]} + \dots$

↑ ↑ ↑ ↑
 Tensor structures depending only on external kinematics Scalar integrals

↪ Tensor integral reduction; at 1-loop: Passarino-Veltman reduction

$\rightarrow$ Use properties of the amplitude to reduce it to tensor structures

↪ Tensor decomposition of amplitudes via projectors $\leftarrow$ our focus here following [Peraro, Taubert '19 & '20]

- Starting points:
 - amplitudes are (multi-)linear in external fields and therefore in their polarisation vectors and spinors
 - amplitudes transform as scalars under Lorentz transformations (but nontrivial phase factors under Poincaré transformations)

↪ Write amplitude as

$$M = \varepsilon^{\mu_1}(p_1) \varepsilon^{\mu_2}(p_2) \dots \bar{u}_{i_3}(p_3) u_{i_4}(p_4) \dots M_{\mu_1 \mu_2 \dots i_3 i_4 \dots}$$

- The stripped amplitude $M_{\mu_1 \mu_2 \dots i_3 i_4 \dots}$ must transform as a tensor under Lorentz transformations

↪ We can pick a basis for these tensors and decompose the stripped amplitude in this basis

Quick linear algebra aside / reminder:

- Assume we have a vector space with a scalar product
- We can pick a set of linearly independent vectors $\{\vec{b}_i\}$ as a basis of this space
- Obviously we can express a vector as a linear combination of these basis vectors: $\vec{v} = \sum_i v_i \vec{b}_i$
- How do we find the coordinates v_i ?
- For an orthonormal basis ($\vec{b}_i \cdot \vec{b}_j = \delta_{ij}$) it's enough to just calculate the scalar product with the corresponding basis vector:

$$\vec{b}_i \cdot \vec{v} = \sum_j v_j \vec{b}_i \cdot \vec{b}_j = v_i$$

- For a non-orthonormal basis, each scalar product is a linear combination of the coordinates: $\vec{b}_i \cdot \vec{v} = \sum_j \vec{b}_i \cdot \vec{b}_j v_j$
- Taking the set of all scalar products $\{\vec{b}_i \cdot \vec{v}\}$ as a vector we get a linear system:

$$\Rightarrow \begin{pmatrix} \vec{b}_1 \cdot \vec{v} \\ \vdots \\ \vec{b}_n \cdot \vec{v} \end{pmatrix} = \begin{pmatrix} \vec{b}_1 \cdot \vec{b}_1 & \dots & \vec{b}_1 \cdot \vec{b}_n \\ \vdots & \ddots & \vdots \\ \vec{b}_n \cdot \vec{b}_1 & \dots & \vec{b}_n \cdot \vec{b}_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

↑ That's called the Gram matrix G ($G_{ij} = \vec{b}_i \cdot \vec{b}_j$)

- If the $\{\vec{b}_i\}$ are linearly independent, G has full rank, is invertible and the system can be solved:

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = G^{-1} \begin{pmatrix} \vec{b}_1 \cdot \vec{v} \\ \vdots \\ \vec{b}_n \cdot \vec{v} \end{pmatrix} \quad \text{or} \quad v_i = \sum_j (G^{-1})_{ij} \vec{b}_j \cdot \vec{v}$$

- The stripped amplitude $M_{\mu_1 \mu_2 \dots i_3 i_4 \dots}$ "lives" on a (sub)space of the vector space spanned by the tensor structures.
- We can write it as a linear combination of these tensor structures with coefficients that are often called "form factors"

$$M_{\mu_1 \mu_2 \dots i_3 i_4 \dots} = \sum_k c^{(k)} T_{\mu_1 \mu_2 \dots i_3 i_4 \dots}^{(k)}$$

- The form factors are scalar (in the sense that they are scalars under Lorentz transformations) functions of the external kinematics.
- We can compute the form factors $c^{(k)}$ as explained in the aside above – if:
 - we can build a basis of tensor structures, i.e., ensure that the structures are indeed linearly independent.
 - we find a scalar product on this vector space.

① Building the basis

• Impose all relevant constraints from transformation properties of the amplitudes, symmetries, etc.

• In the end, we're interested in the amplitudes, including the spinors and polarisation vectors.
 → It makes sense to consider the contracted tensor structures when building the basis.
 → For example, for massless, on-shell spin-1 bosons transversality holds $k_\mu \epsilon^\mu(k) = 0$

• Example: $gg \rightarrow H$ (suppress colour indices for clarity)

$$\rightarrow \mathcal{M} = \epsilon_\mu(p_1) \epsilon_\nu(p_2) \mathcal{M}^{\mu\nu} \quad (\text{abbreviate } \epsilon_\mu(p_1) \equiv \epsilon_{1\mu}, \epsilon_\nu(p_2) \equiv \epsilon_{2\nu})$$

→ $\mathcal{M}^{\mu\nu}$ transforms as rank-2 Lorentz tensors

→ QCD is CP invariant $\Rightarrow$ no e.g. $\epsilon^{\mu\nu\sigma\rho} p_{1\sigma} p_{2\rho}$ can arise

$\left. \begin{array}{l} \Rightarrow \text{build rank-2 tensors from} \\ \{g^{\mu\nu}, p_1^\mu, p_2^\mu\} \end{array} \right\}$

$$\rightarrow \text{Ansatz: } \mathcal{M}^{\mu\nu} = c_{00} g^{\mu\nu} + c_{11} p_1^\mu p_1^\nu + c_{12} p_1^\mu p_2^\nu + c_{21} p_2^\mu p_1^\nu + c_{22} p_2^\mu p_2^\nu$$

→ Use transversality: $p_1 \cdot \epsilon_1 = p_2 \cdot \epsilon_2 = 0$

$\Rightarrow c_{11}, c_{12}$ & c_{22} will never contribute to the contracted amplitude

$\Rightarrow$ we might as well remove them from our basis

$\Rightarrow$ Split the stripped amplitude as:

$$\mathcal{M}^{\mu\nu} = \tilde{\mathcal{M}}^{\mu\nu} + \mathcal{M}_{\text{non-phys}}^{\mu\nu}, \quad \text{where } \mathcal{M}_{\text{non-phys}}^{\mu\nu} \epsilon_{1\mu} \epsilon_{2\nu} = 0$$

$\Rightarrow$ We split the vector space on which $\mathcal{M}^{\mu\nu}$ lives into a physical subspace on which $\tilde{\mathcal{M}}^{\mu\nu}$ lives and a non-physical subspace to which, e.g., $\{p_1^\mu p_1^\nu, p_1^\mu p_2^\nu, p_2^\mu p_2^\nu\}$ belong.

$\Rightarrow$ However, in order to ignore $\{p_1^\mu p_1^\nu, p_1^\mu p_2^\nu, p_2^\mu p_2^\nu\}$ completely, we have to choose the scalar product in a "compatible" way that decouples the two subspaces
 → see below

→ Use Ward identities: $p_{1\mu} \mathcal{M}^{\mu\nu} \epsilon_{2\nu} = 0$ and $\epsilon_{1\mu} \mathcal{M}^{\mu\nu} p_{2\nu} = 0$

$$\Rightarrow p_{1\mu} \mathcal{M}^{\mu\nu} \epsilon_{2\nu} = c_{00}(p_1 \cdot \epsilon_2) + c_{21}(p_1 \cdot p_2)(p_1 \cdot \epsilon_2) \stackrel{!}{=} 0 \Rightarrow c_{00} + c_{21}(p_1 \cdot p_2) = 0 \Rightarrow c_{00} = -c_{21}(p_1 \cdot p_2)$$

$$\Rightarrow \mathcal{M}^{\mu\nu} = c_{21} (-g^{\mu\nu}(p_1 \cdot p_2) + p_2^\mu p_1^\nu)$$

② Defining the scalar product

• We're looking for a map $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$, which is (for $\vec{x}, \vec{y}, \vec{z} \in V$ and $\lambda \in \mathbb{C}$)

- linear in the second argument $\langle \vec{x}, \vec{y} + \lambda \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \lambda \langle \vec{x}, \vec{z} \rangle$
- semilinear in the first argument $\langle \vec{x} + \lambda \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \lambda^* \langle \vec{y}, \vec{z} \rangle$
- hermitian $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle^*$
- positive definite: $\langle \vec{x}, \vec{x} \rangle \geq 0$ & $\langle \vec{x}, \vec{x} \rangle = 0 \Leftrightarrow \vec{x} = 0$

- In addition, since we're usually dealing only with a subspace, we would like the scalar product to be "compatible" with the subspace: if we multiply a vector from the subspace by a vector that has both a component in this subspace, and a component outside it, changing the component outside of the subspace shouldn't change the scalar product.

Let's look at a simple example where this goes wrong:

linear span of the unit vector in x-direction

→ We take $V = \mathbb{R}^2$ and as the subspace we're interested in we take $\langle \hat{e}_x \rangle$, i.e., the x-axis.

→ Suppose we choose as the scalar product the form: $\langle \vec{a}, \vec{b} \rangle = \vec{a}^T A \vec{b}$ where $A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$.
(You can easily check that this is a scalar product on both $\mathbb{R}^2$ and $\langle \hat{e}_x \rangle$.)

→ Take the vectors $\vec{v} = \begin{pmatrix} v_x \\ 0 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_x \\ w_y \end{pmatrix}$

→ Their scalar product $\langle \vec{v}, \vec{w} \rangle = v_x w_x + \frac{1}{2} v_x w_y$ changes if we change the y component of w.

→ So when we want to use it to project a vector in $\mathbb{R}^2$ onto $\langle \hat{e}_x \rangle$ we still have to build the basis for all of $\mathbb{R}^2$.

→ If instead we use $A = \mathbb{1}$, the scalar product "decouples" the subspace and its orthogonal complement.

→ In fact for our purposes it is enough for the form we choose to be a scalar product on the subspace and to project onto this subspace.

→ If we use $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y}$ is still a scalar product on the subspace $\langle \hat{e}_x \rangle$ but any component of vectors on the orthogonal complement ($\langle \hat{e}_y \rangle$) is simply projected out.

- For the amplitudes we can build a scalar product by using the physical spin and polarisation sums:

→ gluons: $\sum_{\lambda} \varepsilon^{\mu}(p, \lambda) \varepsilon^{\nu*}(p, \lambda) = -g^{\mu\nu} + \frac{p^{\mu} n^{\nu} + n^{\mu} p^{\nu}}{p \cdot n}$ with $n^2 = 0$

→ quarks: $\sum_{\lambda} u(p, \lambda) \bar{u}(p, \lambda) = \not{p} + m$

→ antiquarks: $\sum_{\lambda} v(p, \lambda) \bar{v}(p, \lambda) = \not{p} - m$

- This automatically ensures transversality for gluons and constraints from the Dirac equation for quarks and antiquarks.

- To build the full scalar product, we have to combine this for all Lorentz and spin indices, which we can write very compactly for the contracted tensor structures ($T = T^{\mu_1 \mu_2 \dots \mu_n}_{\nu_1 \nu_2 \dots \nu_n} \varepsilon_{\mu_1} \varepsilon_{\mu_2} \dots \bar{u}_{\nu_1} v_{\nu_2} \dots$)

$$\langle P, T \rangle = \sum_{\text{pt}} P^{\dagger} T$$

where $\sum_{\text{pt}}$ stands for the sum over all spins and polarisations using the expressions above.

- Once we have a basis of tensors $\{T_i\}$, we can compute all scalar products to construct the Gram matrix

$$G_{ij} = \langle T_i, T_j \rangle = \sum_{\text{pol}} T_i^\dagger T_j$$

- Then we can decompose the amplitude in this basis as discussed in the aside above:

$$\mathcal{M} = \sum_i c_i T_i$$

$$\Rightarrow c_i = \sum_j (G^{-1})_{ij} \langle T_j, \mathcal{M} \rangle \quad \left(\overset{\text{check}}{=} \sum_j (G^{-1})_{ij} \sum_k c_k \underbrace{\langle T_j, T_k \rangle}_{=G_{jk}} = \sum_k c_k \underbrace{\sum_j (G^{-1})_{ij} G_{jk}}_{=\delta_{ik}} = c_i \right)$$

$\Rightarrow$ We've built projectors $P_i = \sum_j (G^{-1})_{ij}^* T_j$ which project out exactly one form factor c_i from the amplitude via:

$$c_i = \langle P_i, \mathcal{M} \rangle$$

- Return to our example above (for $gg \rightarrow H$):

$\rightarrow$ The tensor structures are: $T_i^{\mu\nu} \in \{-g^{\mu\nu}(p_1 \cdot p_2) + p_1^\mu p_2^\nu, p_1^\mu p_2^\nu, p_1^\nu p_2^\mu, p_1^\mu p_2^\nu, p_1^\nu p_2^\mu\}$

$\rightarrow$ Calculate the Gram matrix via $G_{ij} = T_i^{\mu\nu\dagger} \left(-g_{\mu\nu} + \frac{p_{1\mu} n_{2\nu} + n_{1\mu} p_{2\nu}}{p_1 \cdot n_2} \right) \left(-g_{\nu\rho} + \frac{p_{2\nu} n_{1\rho} + n_{2\nu} p_{1\rho}}{p_2 \cdot n_1} \right) T_j^{\alpha\beta}$

$$G = \begin{pmatrix} (p_1 \cdot p_2)(d-2) & * & 0 & 0 & 0 \\ (p_1 \cdot p_2)^2 \left[1 + \frac{-(n_1 \cdot n_2)(p_1 \cdot p_2) + (p_1 \cdot n_2)(p_2 \cdot n_1)}{(p_1 \cdot n_1)(p_2 \cdot n_2)} \right] & 4(p_1 \cdot p_2)^2 \frac{(p_1 \cdot n_1)(p_2 \cdot n_2)}{(p_1 \cdot n_2)(p_2 \cdot n_1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rightarrow$ Make use of the fact that we can choose the auxiliary vectors n_1, n_2 independently & in a convenient way:

$$n_1 = p_2, \quad n_2 = p_1$$

$\rightarrow$ With this choice the matrix becomes:

$$G = \begin{pmatrix} (p_1 \cdot p_2)^2 (d-2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rightarrow$ Taking into account the first tensor structure is enough.

$\rightarrow$ We can extract the (only) form factor from this amplitude $\mathcal{M} = \varepsilon_{1\mu} \varepsilon_{2\nu} C (-g^{\mu\nu}(p_1 \cdot p_2) + p_1^\mu p_2^\nu)$ via

$$C = \sum_{\text{pol}} \varepsilon_{1\alpha}^* \varepsilon_{2\beta}^* \frac{(-g^{\alpha\beta} p_1^\mu p_2^\nu + p_1^\alpha p_2^\beta)}{(p_1 \cdot p_2)^2 (d-2)} \mathcal{M}$$

Let's consider a more complicated case, with quarks: $u\bar{u} \rightarrow d\bar{d}$

→ The amplitude has the form

$$\mathcal{M} = \bar{v}(p_2) \Gamma_1 u(p_1) \bar{u}(q_1) \Gamma_2 v(q_2)$$

→ The structures we can write down include

$$T_1 = \bar{v}(p_2) \gamma^\mu u(p_1) \bar{u}(q_1) \gamma_\mu v(q_2)$$

$$T_2 = \bar{v}(p_2) \not{p}_3 u(p_1) \bar{u}(q_1) \not{p}_4 v(q_2)$$

$$T_3 = \bar{v}(p_2) \not{p}_3 \gamma^\mu \gamma^\nu u(p_1) \bar{u}(q_1) \not{p}_4 \gamma_\mu \gamma_\nu v(q_2)$$

$$T_4 = \bar{v}(p_2) \gamma^\mu \gamma^\nu \gamma^5 u(p_1) \bar{u}(q_1) \gamma_\mu \gamma_\nu \gamma^5 v(q_2) \quad \text{etc.}$$

→ Certain structures cannot occur due to properties of the theory:

- $\not{p}_i u(p_i) = 0$ (Dirac eq. for massless spinors)
- Γ_i ~ even # of γ matrices cannot occur (for massless quarks due to helicity conservation)

→ But: when can we stop adding γ matrices on the chains?

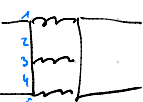
→ Problem: in dimensional regularisation ($d=4-2\epsilon$), the Dirac algebra is not closed

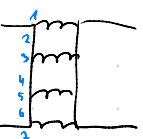
→ In $d=4$ any product of γ matrices can be expressed in terms of the 16 basis elements: $\{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu \gamma^5, \gamma^5\}$

→ If we have at least two fermion lines, we can keep adding more γ matrices, contracting their Lorentz indices between the different chains.

→ At a fixed loop order, there is obviously an upper bound on how many γ matrices can at most be generated by the Feynman rules. But if we increase the loop order, we have to also enlarge the basis for the tensor decomposition.

→ One loop:  ⇒ up to 3 γ matrices/line

Two loop:  ⇒ up to 5 γ matrices/line

Three loop:  ⇒ up to 7 γ matrices/line

→ In principle, we can simply accept this fact and work with a specific basis at a given loop order

→ In our example, we would have to calculate

$$\begin{aligned} G_{ij} &= \sum_{\lambda_a, \lambda_b} T_i^\dagger T_j = \sum_{\lambda_a, \lambda_b} \bar{u}(p_1, \lambda_a) \Gamma_{1,i}^\dagger v(p_2, \lambda_b) \bar{v}(q_2, \lambda_b) \Gamma_{2,i}^\dagger u(q_1, \lambda_a) \bar{v}(p_2, \lambda_b) \Gamma_{1,j} u(p_1, \lambda_a) \bar{u}(q_1, \lambda_a) \Gamma_{2,j} v(q_2, \lambda_b) \\ &= \text{Tr}[\not{p}_1 \Gamma_{1,i}^\dagger \not{p}_2 \Gamma_{1,j}] \text{Tr}[\not{q}_2 \Gamma_{2,i}^\dagger \not{q}_1 \Gamma_{2,j}] \end{aligned}$$

→ The expressions that came out are a bit unwieldy, but otherwise fine:

$$G = \begin{pmatrix} 4[(d-2)s^2 + 4st + 4t^2] & -4t(s+t)(s+2t) & -4t(s+t)[3d-4)s + 2(6d-2)t] & \dots \\ * & 4t^2(s+t)^2 & 4(2+d)t^2(s+t)^2 & \dots \\ * & * & 4(16-6d+3d^2)t^2(s+t)^2 & \dots \\ * & * & * & \dots \end{pmatrix}$$

→ This matrix has full rank ($\text{rank } G = 4$) and can be inverted.

→ The expressions became even more cumbersome when we calculate the inverse, and much more complicated at higher loop orders.

→ However, when we calculate the rank of this matrix at $d=4$, we find: $\text{rank } G|_{d=4} = 2$.

→ In $d=4$ there are only 2 independent tensor structures.

⇒ Can we exploit this? → Yes, by working in the 't Hooft-Veltman (tHV) scheme

- In the tHV scheme, we treat internal momenta etc. as d -dim, but the momenta, spinors, polarisation vectors etc. of external particles as 4-dim.
- We can choose $Q = \text{rank } G|_{d=4}$ tensor structures (here T_1 & T_2) as the independent ones. (We assume that they are the first Q elements of the basis - otherwise we reorder.)
- The $Q \times Q$ submatrix has full rank in $d=4$ and can be inverted also there.
- The the Q independent ($\text{ind}=4$) tensor structures remain exactly the same. In the tHV scheme they are all we need.

$$\Rightarrow \bar{T}_i = T_i \quad i=1, \dots, Q \quad P_i^{Q \times Q} = \sum_{j=1}^Q (G^{Q \times Q})_{ij}^{-1} \bar{T}_j \quad i=1, \dots, Q$$

- If we insist on working in conventional dimensional regularisation (CDR), we should ensure that the remaining basis elements are orthogonal to the first Q ones.
- We can achieve this via:

$$\bar{T}_i = T_i - \sum_{j=1}^Q \langle P_j^{Q \times Q}, T_i \rangle \bar{T}_j \quad i=Q+1, \dots, N$$

- For this new basis, the Gram matrix will be block diagonal

$$\bar{G} = \begin{pmatrix} G^{Q \times Q} & 0 \\ 0 & Y \end{pmatrix} \Rightarrow \bar{G}^{-1} = \begin{pmatrix} (G^{Q \times Q})^{-1} & 0 \\ 0 & Y^{-1} \end{pmatrix}$$

- We can again construct projectors via $\bar{P}_i = \sum_j (\bar{G}^{-1})_{ij}^* \bar{T}_j$
- We'll see how this can be applied in the exercises.

Integration-by-parts reduction

- After squaring the amplitudes or tensor decomposition, we're left with only "scalar" integrals where loop momenta appearing in the numerator only appear contracted with each other or with external momenta.
- Usually there is a huge number of these integrals, which even grows more quickly at higher loop orders.
- Fortunately, it turns out that these integrals are not all independent of each other and that we can find linear relations relating the integrals to a much smaller number of actually independent integrals, called "master integrals" that have to be calculated.
- Before we can discuss these relations, we first must discuss how to systematically organise and label the integrals.

The Feynman integrals we deal with have the form:

$$I = \int \frac{d^4 k_1}{(2\pi)^4} \dots \int \frac{d^4 k_l}{(2\pi)^4} \frac{\mathcal{N}}{D_1^{a_1} D_2^{a_2} \dots D_n^{a_n}}$$

where the $D_i = q_i^2 - m_i^2$ are the propagator denominators (for some linear combination of loop and external momenta q_i).

The numerator $\mathcal{N}$ is some monomial of scalar products $(k_i \cdot k_j)$ and $(k_i \cdot p_j)$. (Anything that's independent of the loop momenta can be pulled out of the integral.)

- We can cancel terms between the numerator and denominator, e.g.:

$$\cdot \frac{k_1^2}{k_1^2} = 1$$

$$\cdot \frac{k_1^2}{k_1^2 - m_1^2} = \frac{k_1^2 - m_1^2 + m_1^2}{k_1^2 - m_1^2} = 1 + \frac{m_1^2}{k_1^2 - m_1^2}$$

$$\cdot \frac{k_1 \cdot k_2}{k_1^2 k_2^2 (k_1 + k_2)^2} = \frac{\overbrace{2 k_1 \cdot k_2 + k_1^2 + k_2^2 - k_1^2 - k_2^2}^{=(k_1 + k_2)^2}}{2 k_1^2 k_2^2 (k_1 + k_2)^2} = \frac{1}{2} \left[\frac{1}{k_1^2 k_2^2} - \frac{1}{k_2^2 (k_1 + k_2)^2} - \frac{1}{k_1^2 (k_1 + k_2)^2} \right]$$

- In order to avoid ambiguities from how we label such integrals, we need to systematically cancel terms as far as possible (with the added benefit that integrals with fewer propagator denominators are usually easier to calculate)
- Try to express scalar products in the numerator in terms of "inverse propagators" as far as possible.
- To this end we introduce the concept of "integral families".
- An l -loop integral family is characterised by a minimal and complete ordered set of propagator denominators.
 - $\rightarrow$ Minimal and complete means that any scalar product $(k_i \cdot k_j)$ or $(k_i \cdot p_j)$ can be uniquely expressed as a linear combination of propagator denominators (plus potentially a term depending only on external kinematics and masses).
- At l loops and with e independent external momenta there are $\frac{l(l+1)}{2}$ scalar products $(k_i \cdot k_j)$ and $l \cdot e$ scalar products $(k_i \cdot p_j)$.
 - $\Rightarrow$ In total the integral family must contain $n = \frac{l(l+1)}{2} + l \cdot e$ propagator denominators

• Example: 1-loop box $\rightarrow l=1, e=3 \Rightarrow n = \frac{1 \cdot 2}{2} + 1 \cdot 3 = 4$ propagators necessary

$$D_1 = k^2 \quad D_2 = (k-p_1)^2 \quad D_3 = (k+p_2)^2 \quad D_4 = (k+p_1+q_1)^2$$

Let's see how we can express all scalar products in terms of D_i :

$$k \cdot k = D_1$$

$$k \cdot p_1 = \frac{1}{2} \underbrace{(2k \cdot p_1 + k^2 + p_1^2)}_{=(k-p_1)^2 = D_2} - \underbrace{\left(\underbrace{k^2}_{=D_1} - \underbrace{p_1^2}_{=0} \right)}_{=D_1} = -\frac{1}{2} D_2 + \frac{1}{2} D_1$$

$$k \cdot p_2 = \frac{1}{2} \underbrace{(2k \cdot p_2 + k^2 + p_2^2)}_{=D_3} - \underbrace{\left(\underbrace{k^2}_{=D_1} - \underbrace{p_2^2}_{=0} \right)}_{=D_1} = \frac{1}{2} D_3 - \frac{1}{2} D_1$$

$$k \cdot q_1 = \frac{1}{2} \underbrace{(2k \cdot q_1 + 2k \cdot p_1 + k^2 + p_1^2 + q_1^2)}_{=(k+p_1+q_1)^2 = D_4} - \underbrace{\left(\underbrace{2k \cdot p_1}_{=-\frac{1}{2}D_2 + \frac{1}{2}D_1} - \underbrace{k^2}_{=D_1} - \underbrace{p_1^2}_{=0} - \underbrace{q_1^2}_{=0} \right)}_{=-\frac{1}{2}D_2 + \frac{1}{2}D_1} = \frac{1}{2} [D_4 + D_2 - D_1 - D_1] = \frac{1}{2} D_4 + \frac{1}{2} D_2 - D_1$$

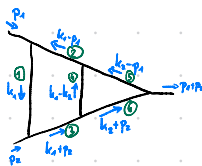
• One loop is a bit peculiar, because for $l=1$ we always have $n = \frac{1 \cdot 2}{2} + 1 \cdot e = e+1$ which is exactly the number of propagators in a one-loop diagram with e independent external momenta:

diag:						...
#legs	1	2	3	4	5	...
e	0	1	2	3	4	...
#prop	1	2	3	4	5	...

$\Rightarrow$ We can always express all scalar products in terms of propagators present in the diagram

• Beyond 1-loop we encounter cases where that's no longer the case

$\rightarrow$ Example: massless 2-loop triangle



$$\begin{aligned} D_1 &= k_1^2 \\ D_2 &= (k_1-p_1)^2 \\ D_3 &= (k_1+p_2)^2 \\ D_4 &= (k_1-k_2)^2 \\ D_5 &= (k_2-p_1)^2 \\ D_6 &= (k_2+p_2)^2 \end{aligned}$$

• With $l=2$ and $e=2$, we have $n = \frac{2 \cdot 3}{2} + 2 \cdot 2 = 7$ scalar products $\Rightarrow$ propagators of the diagram are not sufficient

• We have to add another propagator denominator to complete the integral family.

$\rightarrow$ Here we could, e.g., add $D_7 = k_2^2$ or $D_7 = (k_2+p_1+p_2)^2$ or $D_7 = k_1 \cdot k_2$ or ...

$\rightarrow$ How to complete it isn't unique, and the additional propagators don't correspond to any propagator present in the diagram

$\rightarrow$ As you see it's also possible to just directly add scalar products

$\rightarrow$ Sometimes these additional elements you have to add to complete an integral family are called "irreducible scalar products" (ISP)

• By construction once we have an integral family, we can express any scalar product in the numerator in terms of D_i 's. Then cancelling numerators against denominators is trivial

- Within an integral family, we can label any integral just by the exponents of the D_i :

$$I(a_1, a_2, \dots, a_n) = \int \frac{d^d k_1}{(2\pi)^d} \dots \int \frac{d^d k_n}{(2\pi)^d} \frac{1}{D_1^{a_1} D_2^{a_2} \dots D_n^{a_n}}$$

- Negative exponents $a_i < 0$ yield numerators.

- Diagrammatically, integrals with exponent $a_i = 0$ correspond to a diagram where the corresponding line with propagator D_i is "pinched", i.e. contracted to a point.

$$I(1, 1, 1, 1, 1, 0) \rightarrow \text{triangle diagram with one internal line pinched to a point}$$

$$I(1, 1, 1, \textcircled{0}, 1, 1, 0) \rightarrow \text{triangle diagram with one internal line contracted to a point}$$

- Commonly used terminology: the set of all integrals that have the same propagators in the denominators, i.e., the same set of positive exponents $\{a_i > 0\}_{i \in S}$ is called a "sector". (Regardless of possible negative exponents [numerators].)
- If a sector S_1 whose positive exponents $\{a_i > 0\}_{i \in S_1}$ are a strict subset of the positive exponents of another sector S_2 ($S_1 \subset S_2$) we call S_1 a "subsector" of S_2 .
- Conversely, S_2 is a "supersector" of S_1 .

- This allows us to sort sectors into a directed graph according to their subsector relations.

- With this terminology in place, we're ready to discuss how to find linear relations between different integrals.

- The relations we discuss are called "integration-by-parts" (IBP) relations. [Chetyrkin, Tkachov '81]

- They are a generalisation of the integration-by-parts method for one-dimensional integrals:

$$\int_{-\infty}^{\infty} dx \frac{\partial}{\partial x} f(x) = f(x) \Big|_{-\infty}^{\infty} = 0 \quad \text{for} \quad \lim_{x \rightarrow \pm\infty} f(x) = 0$$

- To be precise it is an application of Gauss' theorem (divergence theorem) to dimensionally regulated Feynman integrals

- One way to derive it is to use that dim. reg. Feynman integrals are translation invariant: for v^μ a linear combination of external and loop momenta (except for k_i): $\int \frac{d^d k_i}{(2\pi)^d} f(k_i) = \int \frac{d^d k_i}{(2\pi)^d} f(k_i + \alpha v)$

$\Rightarrow$ R.h.s must be independent of α , in particular if we expand in α , the $O(\alpha)$ term must be zero:

$$f(k_i + \alpha v) = f(k_i) + \frac{\partial k_i^\mu \alpha v^\mu}{\partial \alpha} \frac{\partial}{\partial k_i^\mu} f(k_i) + O(\alpha^2) = f(k_i) + \frac{\partial}{\partial k_i^\mu} v^\mu f(k_i) + O(\alpha^2)$$

$$\Rightarrow 0 = \int \frac{d^d k_i}{(2\pi)^d} \frac{\partial}{\partial k_i^\mu} v^\mu f(k_i) \quad (*)$$

- We can also choose $v = k_i$. Then we use that Feynman integrals in dim. reg. fulfil a scaling relation: $\int \frac{d^d k_i}{(2\pi)^d} f(\alpha k_i) = \alpha^{-d} \int \frac{d^d k_i}{(2\pi)^d} f(k_i)$
Here, we need it in the form

$$\int \frac{d^d k_i}{(2\pi)^d} f(k_i) = (1+\alpha)^d \int \frac{d^d k_i}{(2\pi)^d} f(k_i + \alpha k_i) \quad (**)$$

$$\text{Expand } (1+\alpha)^d = 1 + d\alpha + O(\alpha^2) \quad \text{and} \quad f(k_i + \alpha k_i) = f(k_i) + \alpha k_i^\mu \frac{\partial}{\partial k_i^\mu} f(k_i) + O(\alpha^2)$$

Rewrite: $k_i^\mu \frac{\partial}{\partial k_i^\mu} f(k_i) = \frac{\partial}{\partial k_i^\mu} [k_i^\mu f(k_i)] - f(k_i)$

Inserting this into (xx) yields (*) again, for $v = k_i$.

• We can repeat this for any loop momentum k_i and for v we can use any loop or external momentum.

• Consider how the derivative acts on a single propagator:

$$\frac{\partial}{\partial k_i^\mu} \frac{1}{[(k_i + q_j)^2 - m_j^2]^{a_j}} = \frac{-q_j^\mu}{[(k_i + q_j)^2 - m_j^2]^{a_j+1}} \frac{\partial (k_i + q_j)^2 - m_j^2}{\partial k_i^\mu} = \frac{-2q_j^\mu (k_i + q_j)_\mu}{[(k_i + q_j)^2 - m_j^2]^{a_j+1}} \quad (\text{for } q_j \neq 0)$$

• If we apply $\frac{\partial}{\partial k_i^\mu} v^\mu$ on a Feynman integral under the integral sign, the product rule yields several terms (one per propagator that contains k_i and if $v^\mu = k_i^\mu$ another term with $\frac{\partial k_i^\mu}{\partial k_i^\mu} = g^\mu_\mu = d$). Each term has the same structure as the initial term, just with new terms in the numerator which will get contracted with v^μ and shifted exponents a_i .

• All scalar products in the numerator can be expressed in terms of linear combinations of D_i , which potentially cancel denominators, thereby further shifting the exponents a_i .

• As we saw above, applying $\frac{\partial}{\partial k_i^\mu} v^\mu$ leads to a linear combination of integrals with shifted exponents, but within the same family.

• Example: one-loop tadpole



$$I(a_1) = \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{D_1^{a_1}} \quad \text{with } D_1 = k_1^2 - m^2$$

There is only one operator we can insert: $\frac{\partial}{\partial k_1^\mu} k_1^\mu$

$$\int \frac{d^d k_1}{(2\pi)^d} \frac{\partial}{\partial k_1^\mu} k_1^\mu \frac{1}{[k_1^2 - m^2]^{a_1}} = \int \frac{d^d k_1}{(2\pi)^d} \left[\frac{d}{[k_1^2 - m^2]^{a_1}} + \frac{-2a_1 k_1^2}{[k_1^2 - m^2]^{a_1+1}} \right] = (d-2a_1) I(a_1) - 2a_1 m^2 I(a_1+1) \stackrel{!}{=} 0$$

$$\Rightarrow I(a_1+1) = \frac{d-2a_1}{2a_1 m^2} I(a_1) \quad (\text{for } a_1 \neq 0)$$

$\leadsto$ We can reduce integrals with exponent $a_1 > 1$:

$$I(2) = \frac{d-2}{2m^2} I(1)$$

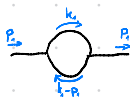
$$I(3) = \frac{d-4}{4m^2} I(2) = \frac{(d-4)(d-2)}{8m^2} I(1)$$

⋮

Any integral with $a_1 > 1$ can be expressed in terms of $I(1)$ (the tadpole with a single propagator).

We call $I(1)$ a "master integral".

• Example: equal mass one-loop bubble integral



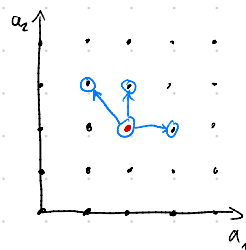
$$I(a_1, a_2) = \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{D_1^{a_1} D_2^{a_2}}$$

$$\text{with } D_1 = k_1^2 - m^2 \\ D_2 = (k_1 + p_1)^2 - m^2$$

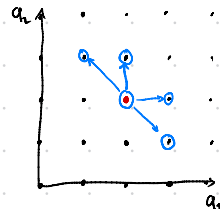
$$\Rightarrow k_1^2 = D_1 + m^2 \\ k_1 \cdot p_1 = \frac{1}{2}(D_1 - D_2 + p_1^2)$$

→ Here we can choose two different momenta for v : $v \in \{k_1, p_1\}$

$$V = k_1 : \int \frac{d^d k_1}{(2\pi)^d} \frac{\partial}{\partial k_1} k_1 \frac{1}{D_1^{a_1} D_2^{a_2}} = \int \frac{d^d k_1}{(2\pi)^d} \left[\frac{d}{D_1^{a_1} D_2^{a_2}} + \frac{-2a_1 k_1^2}{D_1^{a_1+1} D_2^{a_2}} + \frac{-2a_2 \overbrace{(k_1^2 - p_1 k_1)}^{\substack{= \frac{D_1^2}{2} + \frac{D_2^2}{2} \\ = \frac{D_1^2 + D_2^2}{2} + m^2}}}{D_1^{a_1} D_2^{a_2+1}} \right]$$



$$V^{\mu} = p_{\lambda}^{\mu} : \int \frac{d^d k_{\lambda}}{(2\pi)^d} \frac{\partial}{\partial k_{\lambda}} p_{\lambda}^{\mu} \frac{1}{D_1^{a_1} D_2^{a_2}} = \int \frac{d^d k_{\lambda}}{(2\pi)^d} \left[\frac{-2a_1 k_{\lambda} p_{\lambda}^{\mu}}{D_1^{a_1+1} D_2^{a_2}} + \frac{-2a_2 \overbrace{(k_{\lambda} \cdot p_1 - p_1^2)}^{\substack{= \frac{p_1^2}{2} - \frac{D_1^2}{2} + \frac{p_2^2}{2}}} }{D_1^{a_1} D_2^{a_2+1}} \right]$$



- We can use these relations for fixed values of $(a_1, a_2) \in \{(1,1), (1,2), (2,1), \dots\}$ to generate a (sparse) linear system for the integrals.
- If we order the integrals in some way ("simple" $\rightarrow$ "hard"), we can solve the linear system using Gaussian elimination. $\Rightarrow$ Laporta algorithm [Laporta '00]
- Taking into account the symmetry $I(a_1, a_2) = I(a_2, a_1)$, it turns out that there are two master integrals:

$I(1,0) \rightarrow$ the tadpole integral we discussed before

$I(1,1) \sim$ the bubble integral with simple propagators

- In general, there are computer programs (e.g., Reduce, Kiva, FIRE, LiteRed, ...) that implement constructing the IBP relations, and solving the Linear system.