Asymptotic Expansions

The complexity of state-of-the-art multi-loop calculations often prohibits a direct analytic evaluation of the required scattering amplitudes.

↳ for problems involving multiple scales it is sometimes sufficient for phenomenological purposes to consider approximate results being valid only in a particular region of the phase-space

↳ in particular, if one scale is much smaller compared to all other scales the hierarchy can be exploited to factorize the small scale from the remainder, which is then parametrized in terms of the other scales

- Examples: $gg \to H$ in QCD with $m_t \gg m_H$ $(m_{\{b,c,s,...\}} = 0)$

  H+jet with $m_b \ll s,t,u$

  $gg \to HH$, $gg \to ZZ$, $gg \to ZH$ at high energy

  ↳ multiple (subsequently performed) expansions are also possible, e.g. $gg \to H$ with $m_t \gg m_H \gg m_b$

↳ provided the knowledge of the differential equations for the master integrals, deep asymptotic expansions can be obtained systematically

↳ expansions of integrals can serve as a boundary condition for the solution of differential equations

↳ A systematic approach for the expansion of Feynman integrals, characterized by a small parameter, is given by the method of regions

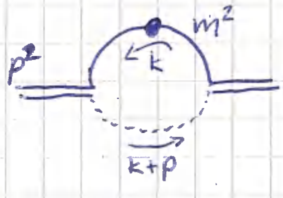
↳ we want to focus on the geometric formulation in Feynman parameter space

↳ Obviously, we want to expand before integration to reduce the complexity of the integrals

Example in momentum space

In order to explain the basic principle (and to show that a naive expansion of the integrand leads to incomplete results) we consider the bubble integral

$$I(p^2, m^2) = \int_k \mathcal{J}(k,p,m) = \int_k \frac{1}{(k+p)^2(k^2-m^2)^2}$$



for $|p^2| \gg m^2$

- Note that the integral is finite in $d=4$ dimensions
- We introduce a smallness parameter ρ (set $\rho=1$ later)

Hard region: $m^2 \ll |k^2| \sim |p^2|$

$\hookrightarrow$ interprete as Wick-rotated: "small k" $\Leftrightarrow$ small components

$\hookrightarrow m^2 \to \rho m^2$

- expand in $\rho = 0$ and then set $\rho = 1$:

$$\mathcal{J}^{(h)} = \frac{1}{(k+p)^2\,(k^2)^2}\left(1 + \boxed{\frac{2m^2}{k^2}} + \mathcal{O}\left(\frac{m^4}{(k^2)^2}\right)\right)$$

$\hookrightarrow$ expansion valid in the integration domain $|k^2| > m^2$

- since integration domain also comprises points where $|k^2| \sim m^2$, we need to take into account this region as well

Soft region: $m^2 \sim |k^2| \ll |p^2|$

$\hookrightarrow k^2 \to \rho k^2, \quad k\cdot p \to \sqrt{\rho}\, k\cdot p, \quad m^2 \to \rho m^2$

$$\mathcal{J}^{(s)} = \frac{1}{p^2\,(k^2-m^2)^2}\left(1 - \underbrace{\frac{2k\cdot p}{p^2}}_{} - \frac{k^2}{p^2} + \dots\right)$$

$\llcorner$ odd powers in $(k\cdot p)$ lead to zero after integration

$\hookrightarrow$ expansion is valid for $k^2 + 2k\cdot p < p^2$

- We introduce a cut-off Λ to split the integration domain into two parts
  - $D_h = \{k \in \mathbb{R}^d \mid |k^2| \geq \Lambda^2\}$
  - $D_s = \{k \in \mathbb{R}^d \mid |k^2| < \Lambda^2\}$

  $m^2 \ll \Lambda^2 \ll |p^2|$

$$\Rightarrow I = \int_{D_h} \mathcal{I}^{(h)} + \int_{D_s} \mathcal{I}^{(s)}$$

$\hookrightarrow$ now the expanded integrands are convergent within the respective domains

- we extend the domains to the whole integration domain and compensate with subtraction terms:

$$I = \left[\int_k \mathcal{I}^{(h)} - \int_{D_s} \mathcal{I}^{(sh)}\right] + \left[\int_k \mathcal{I}^{(s)} - \int_{D_h} \mathcal{I}^{(hs)}\right]$$

subtraction terms are also expanded in the parameter which is small in the respective integration domain

- the expansions commute (cannot be taken for granted in general [Jantzen 2011])

$$\Rightarrow I = \int_k \mathcal{I}^{(h)} + \int_k \mathcal{I}^{(s)} - \underbrace{\int_k \mathcal{I}^{(hs)}}_{\text{scaleless}}$$

$\to 0$ (in dim. reg.)

$\hookrightarrow$ expand $\mathcal{I}^{(h)}$ in soft region:

$$\mathcal{I}^{(hs)} = \frac{1}{p^2 (k^2)^2}\left(1 - \frac{2k\cdot p}{p^2} - \frac{k^2}{p^2} + \dots\right)\left(1 + \frac{2m^2}{k^2} + \dots\right) = \mathcal{I}^{(sh)}$$

$$\Rightarrow \int_k \mathcal{I}^{(hs)} = 0$$

- integrate the leading contribution:

$$I^{(h)} = \int \mu^{2\varepsilon} e^{\varepsilon\gamma_E} \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k+p)^2 (k^2)^2} + \mathcal{O}\left(\frac{m^2}{p^2}\right)$$

$$= \frac{1}{p^2}\left(-\underset{\text{IR}}{\frac{1}{\varepsilon}} + \log\left(\frac{-p^2}{\mu^2}\right)\right) + \mathcal{O}\left(\varepsilon, \frac{m^2}{p^2}\right)$$

$$I^{(s)} = \int \mu^{2\varepsilon} e^{\varepsilon \gamma_E} \frac{d^d k}{i\pi^{d/2}} \frac{1}{p^2 (k^2 - m^2)^2} + \mathcal{O}\left(\frac{m^2}{p^2}\right)$$

$$= \frac{1}{p^2}\left(\underbrace{\frac{1}{\varepsilon}}_{UV} - \log\left(\frac{m^2}{\mu^2}\right)\right) + \mathcal{O}\left(\varepsilon, \frac{m^2}{p^2}\right)$$

$$\Rightarrow I = I^{(h)} + I^{(s)} = \frac{1}{p^2} \log\left(\frac{-p^2}{m^2}\right) + \mathcal{O}\left(\varepsilon, \frac{m^2}{p^2}\right)$$

↳ note that the two expansions are separately divergent, however, the original integral is finite

↳ serves as a cross-check of the result

↳ in some cases, the individual region integrals are not regulated by dim. reg. requiring additional regulators

due to extension of domains from D_h (D_s) to whole integration domain

↳ formally, combinations $\left[I^{(h)} - I^{(sh)}\right]$ and $\left[I^{(s)} - I^{(hs)}\right]$ are separately IR and UV finite

Parametrization of Scalar Integrals

The geometric formulation of the method of regions is based on the representation of integrals in parametric space

- Consider a scalar Feynman integral (without numerators) with L loops in d dimensions:

$$I(\vec{\nu}) = \int_{k_1} \cdots \int_{k_L} \prod_{j=1}^{N} \frac{1}{D_j^{\nu_j}(\{k\},\{p\},m_j)}$$

$$\int_k = \int \frac{d^d k}{i\pi^{d/2}}$$

with denominators (propagators)

$$D_j = Q_j^2 - m_j^2 + i\varepsilon$$

Linear combination of loop momenta and external momenta with coefficients $\in \{\pm 1, 0\}$

↳ every propagator is turned into an integral over a Feynman / Schwinger parameter using

$$\frac{1}{D^{\nu}} = \frac{(-i)^{\nu}}{\Gamma(\nu)} \int_0^{\infty} dx \, x^{\nu-1} e^{i\nu D}$$

$$\hookrightarrow I(\vec{\nu}) = \int_{k_1} \cdots \int_{k_L} \frac{(-i)^{N_\nu}}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^{\infty} \left(\prod_{j=1}^{N} \frac{dx_j}{x_j} x_j^{\nu_j} \right) \exp\left(i \sum_{j=1}^{N} x_j D_j \right)$$

with $N_\nu = \sum_{j=1}^{N} \nu_j$

- the argument of the exponential function can be rearranged:

$$\sum_{j=1}^{N} x_j D_j = \sum_{i,j=1}^{L} \underbrace{M_{ij}\, k_i \cdot k_j}_{\text{loop momenta}} - 2 \underbrace{\sum_{i=1}^{L} q_i \cdot k_i}_{\text{linear combination of external momenta multiplying loop momenta}} - \underbrace{J(s_{ij}, m_i^2)}_{\text{for proper integral family depends on invariants and masses}}$$

M_{ij} admits interpretation in the context of circuit theory: product of incidence matrices describing the flow of current in a network

- the Gauß-type integral can be solved integrating out the loop momenta:

$$\int_{k_1} \cdots \int_{k_L} \exp\left(i \sum_{j=1}^{N} x_j D_j\right)$$

$$= (-i)^{\frac{d}{2}L} \mathcal{N}_L \det(\mathcal{M})^{-\frac{d}{2}} \exp\left(-i\left(\sum_{ij=1}^{L} \mathcal{M}_{ij}^{-1} q_i \cdot q_j + J\right)\right)$$

$$= (-i)^{\frac{d}{2}L} \mathcal{N}_L \, \mathcal{U}^{-\frac{d}{2}} \exp\left(-i \frac{\mathcal{F}}{\mathcal{U}}\right)$$

$\uparrow$ loop normalization, e.g. $\exp(L \varepsilon \gamma_E)$

↳ Symanzik polynomials:

$$\mathcal{U} = \det(\mathcal{M}) \qquad \mathcal{F} = \det(\mathcal{M}) \left[\sum_{ij=1}^{L} \mathcal{M}_{ij}^{-1} q_i \cdot q_j + J\right]$$

↳ can be determined from the graph:

T^1: spanning trees $\qquad$ T^2: spanning 2-trees

$$\mathcal{U} = \sum_{T \in T^1} \prod_{e \notin T} x_e \qquad \mathcal{F} = \mathcal{U} \sum_{j=1}^{N} m_j^2 x_j - \mathcal{V}$$

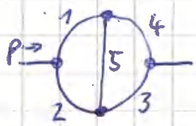
$$\mathcal{V} = \sum_{T \in T^2} \prod_{e \notin T} x_e \, S_T$$

Note: $\deg(\mathcal{U}) = L \quad \deg(\mathcal{F}) = L+1$

- T^1: connected graph containing all vertices without closed loops

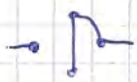
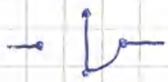
  T^2: T^1 with one edge deleted

  S_T: square of sum of momenta flowing from one of the two disconnected components into the other

Example: 

$\mathcal{U} =$  $+$  $+$  $+ \ldots$

$\sim x_1 x_3 \qquad \sim x_1 x_4 \qquad \sim x_1 x_5$

$\mathcal{V} =$  $+$  $+ \ldots$

$\sim -p^2 x_1 x_2 x_3 \qquad \sim -p^2 x_1 x_2 x_4$

- Schwinger representation:

$$I(\vec{\nu}) = \frac{(-i)^{N_\nu + \frac{d}{2}L}}{\prod_{j=1}^{N} \Gamma(\nu_j)} \mathcal{N}_L \int_0^\infty \left(\prod_{j=1}^{N} \frac{dx_j}{x_j} x_j^{\nu_j} \right) \mathcal{U}^{-\frac{d}{2}} \exp\left(-i \frac{\mathcal{F}}{\mathcal{U}}\right)$$

↳ to derive the Feynman representation insert $1 = \int_0^\infty d\eta \, \delta\left(\sum_{j=1}^{N} x_j - \eta\right)$

↳ apply rescaling $x_j = \eta x_j'$, $dx_j = \eta \, dx_j'$, and recall that $\mathcal{U}$ and $\mathcal{F}$ are homogeneous in the x_i

$$\Rightarrow I(\vec{\nu}) = \frac{(-i)^{N_\nu + \frac{d}{2}L}}{\prod_{j=1}^{N} \Gamma(\nu_j)} \mathcal{N}_L \int_0^\infty \left(\prod_{j=1}^{N} \frac{dx_j'}{x_j'} (x_j')^{\nu_j} \right) \mathcal{U}^{-\frac{d}{2}} \int_0^\infty d\eta \, \delta\left(\sum_{j=1}^{N} x_j' - 1\right) \eta^{N_\nu - \frac{d}{2}L - 1} e^{-i \frac{\mathcal{F}}{\mathcal{U}} \eta}$$

$$\Rightarrow \boxed{I(\vec{\nu}) = (-1)^{N_\nu} \frac{\Gamma(N_\nu - \frac{d}{2}L)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \mathcal{N}_L \int_0^\infty \left(\prod_{j=1}^{N} dx_j \, x_j^{\nu_j - 1} \right) \delta\left(\sum_{j=1}^{N} x_j - 1\right) \frac{\mathcal{U}^{N_\nu - \frac{d}{2}(L+1)}}{\mathcal{F}^{N_\nu - \frac{d}{2}L}}}$$

↳ the N-fold integration in Schwinger representation has been reduced to a surface integral over one facet in the first orthant of the N-dimensional cross polytope

↳ Cheng-Wu theorem: in the argument of the δ we can sum over any non-empty subset of parameters

↳ obviously, we could have inserted a smaller subset of x_j in the identity-integral above

- Lee-Pomeransky representation:

$$I(\vec{\nu}) = (-1)^{N_\nu} \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2}(L+1) - N_\nu) \prod_{j=1}^{N} \Gamma(\nu_j)} \mathcal{N}_L \int_0^\infty \left(\prod_{j=1}^{N} dx_j \, x_j^{\nu_j - 1} \right) (\mathcal{U} + \mathcal{F})^{-\frac{d}{2}}$$

↳ crucial point relevant for method of regions: Lee-Pomeransky representation contains sum of Symanzik polynomials (as opposed to the ratio in Feynman parametrization)

↳ can be turned back to Feynman parametrization in analogy to derivation above

Geometric determination of regions

A first geometric approach to expansion by regions was introduced in [Pak, Smirnov 2010] and relies on the Feynman parameter representation. It was later shown that the regions can also be inferred from the Lee-Pomeransky representation [Semenova, Smirov, Smirnov 2018].

↳ the argumentation applies to even more general polynomials than just $\mathcal{U}+\mathcal{F}$!

- We want to disuss the expansion of regulated parametric integrals over polynomials of the form:

$$P(\vec{x},\lambda) = \sum_{i=1}^{m} c_i \prod_{j=1}^{N} x_j^{p_{i,j}} \lambda^{p_{i,N+1}}$$

↳ for Feynman integral in Lee-Pomeransky representation:

$$I = \int_0^\infty \frac{dx}{x} x^{\vec{\nu}} \lambda^{\nu_{N+1}} \left(\sum_{i=1}^{m} c_i x^{\vec{p}_i} \lambda^{p_{i,N+1}} \right)^b$$

↳ $b = -\frac{d}{2}$

↳ multi-index notation: $x^{\vec{\nu}} = \prod_{j=1}^{N} x_j^{\nu_j}$

↳ require $c_i \geqslant 0$ for the monomial coefficients

↳ organize exponents in terms of $(N+1)$-dimensional vectors $\vec{p}_i{}' = (\vec{p}_i, p_{i,N+1})$ and $\vec{\nu}' = (\vec{\nu}, \nu_{N+1})$ where $\vec{p}_i{}' \in \mathbb{Z}^{N+1}$ (for negative p_{ij}': factor out the corresponding terms and absorb in $\vec{\nu}'$)

↳ smallness parameter λ in which we expand the integral

- The prescription for expansion by regions dictates to rescale the integration variables in every region:

$$\lambda \to z\lambda \qquad x_j \to z^{r_j} x_j$$

↳ expand in auxiliary parameter $z=0$ and set $z=1$ afterwards

↳ we refer to $\vec{r} = (r_1, \dots, r_N, 1)$ as the region vector

↳ the region vector induces a scaling hierarchy among the monomials in $P(\vec{x},\lambda)$ such that some terms survive the expansion in z whereas others are power-suppressed

↳ this reflects the fact that care must be taken when the limit $\lambda \to 0$ is taken

↳ since $\vec{x}$ is integrated over $\mathbb{R}_+^N$ different hierarchies / scalings between x_j and λ can be encountered, e.g. $\lambda \ll x \sim 1$ in one region and $\lambda \sim x \ll 1$ in another region

Example: $P(x,\lambda) = \lambda + x + x^2$

↳ for $\vec{r}_1 = (0,1)$: $\lambda \ll x$

$$P(x,\lambda) = (x+x^2)\left(1+\frac{\lambda}{x+x^2}\right)$$

↳ remainder not small for $\lambda \sim x \ll 1$

$\Rightarrow$ other region $\vec{r}_2 = (1,1)$:

$$P(x,\lambda) = (\lambda + x)\left(1+\frac{x^2}{\lambda+x}\right)$$

↳ $\vec{r}_3 = (2,1)$: $x \ll \lambda \ll 1$

$$P(x,\lambda) = \lambda\left(1+\frac{x+x^2}{\lambda}\right)$$

- for the three region vectors we factored out the largest terms of the polynomial according to the scaling implied by $\vec{r}$

  ↳ the remainder is small in the corresponding region such that the expansion is convergent

  ↳ for our example it turns out that expanding in $\vec{r}_1$ and $\vec{r}_2$ is sufficient to cover the whole integration domain

- we note that $\lambda \to z\lambda$ and $x_j \to z^{r_j} x_j$ expanding in $z=0$ and setting $z=1$ afterwards is equivalent to replacing $x_j \to \lambda^{r_j} x_j$ and expanding in $\lambda = 0$ up to the desired order

- in order to discuss the geometric algorithm for finding the regions, we need to introduce the *Newton polytope* of the polynomial $P(\vec{x}, \lambda)$

  ↳ the Newton polytope can be determined from the exponent vectors:

$$\Delta' = \text{conv Hull}\,(\vec{p}_1{}', \vec{p}_2{}', \dots) = \left\{ \sum_i \alpha_i \vec{p}_i{}' \,\middle|\, \alpha_i \geq 0 \wedge \sum_i \alpha_i = 1 \right\}$$

$\boxed{\text{vertex representation}}$

like probabilities weighting the exponent vectors

sum not necessarily over all $\vec{p}_i{}'$, i.e. there is a minimal set

  - facet representation:

$$\Delta' = \bigcap_{f \in F} \{ \vec{m} \in \mathbb{R}^{N+1} \mid \underbrace{\langle \vec{m}, \vec{n}_f \rangle + a_f \geq 0}_{} \}$$

intersection of affine half-spaces

equality defines a hyperplane with normal vector $\vec{n}_f$ and distance $|a_f|$ from the origin

  ↳ F: set of polytope facets with inward-pointing normal vectors

  $a_f \in \mathbb{Z}$

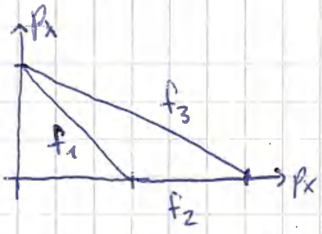
  ↳ we define the "lower" facets as those with $(\vec{n}_f)_{N+1} > 0$

$$F^+ = \{ f \in F \mid (\vec{n}_f)_{N+1} > 0 \}$$

- the tool Normaliz allows to translate between facet and vertex representation

Example: $P(x,\lambda) = \lambda + x + x^2$

↳ $\vec{p}_1{}' = (0,1)$ $\quad \vec{p}_2{}' = (1,0)$ $\quad \vec{p}_3{}' = (2,0)$



f_1: $\vec{n}_f = (1,1)$ $\quad a_f = -1$

f_2: $\vec{n}_f = (0,1)$ $\quad a_f = 0$

f_3: $\vec{n}_f = (-1,-2)$ $\quad a_f = 2$

- Convergence domain of an expansion with a region vector:

↳ Let $\vec{r}$ denote a general region vector and consider the polynomial $P(\vec{x},\lambda) = \sum_{i=1}^{m} c_i \, x^{\vec{p}_i} \lambda^{p_{i,N+1}}$

↳ define the N-dimensional vector $\vec{u}$ by $x_j = \lambda^{u_j}$

$\Rightarrow \vec{u}' = (\vec{u}, 1)$

change of variables mapping $\vec{x} \in \mathbb{R}_+^N$ to $\vec{u} \in \mathbb{R}^N$

↳ under this transformation, the polynomial becomes

$$P(\lambda^{\vec{u}},\lambda) = \sum_{i=1}^{m} c_i \, \lambda^{\langle \vec{p}_i{}', \vec{u}' \rangle}$$

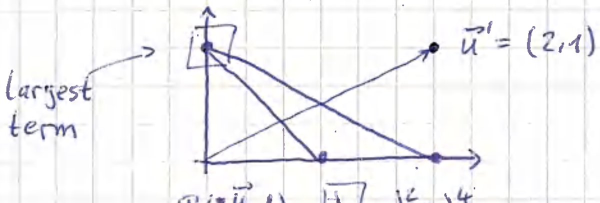
- Now, we can geometrically determine the convergence domain of an expansion around $\lambda = 0$

↳ we can assess which expansions defined by $\vec{r}$ converge at an integration point defined by $\vec{u}$

for $\lambda \ll 1$, the largest term of P is characterized by the smallest $\langle \vec{p}_i{}', \vec{u}' \rangle$ (lowest order in λ)

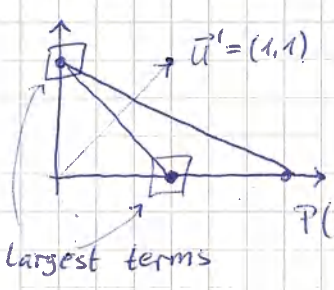
↳ there can be multiple points $\vec{p}_i{}'$ at the same "distance" from $\vec{u}'$ (same projection from $\vec{p}_i{}'$ to $\vec{u}'$)

Example:



$P(\lambda^{\vec{u}},\lambda) = \boxed{\lambda} + \lambda^2 + \lambda^4$

$(\lambda + x + x^2)$



$P(\lambda^{\vec{u}},\lambda) = \boxed{\lambda + \lambda} + \lambda^2$

$(\lambda + x + x^2)$

Let us assess which expansions (defined by $\vec{r}$) converge at $\vec{x}$ (defined by $\vec{u}'$):

↳ split the polynomial $P(\vec{x},\lambda) = \underbrace{Q(\vec{x},\lambda)}_{\text{contains all lowest order terms in } \lambda} + R(\vec{x},\lambda)$

↳ the series expansion of $P(\vec{x},\lambda)^b$ in λ is equivalent to the binomial expansion

$$P(\vec{x},\lambda)^b = Q(\vec{x},\lambda)^b \left(1 + \frac{R(\vec{x},\lambda)}{Q(\vec{x},\lambda)}\right)^b$$

$\uparrow$ converges for $\vec{x}$ (defined by $\vec{u}'$) if $R(\vec{x},\lambda)/Q(\vec{x},\lambda) < 1$

⇒ We conclude that an expansion with a region vector $\vec{r}$ converges at a point given by $\vec{u}'$ if the terms with minimum $\langle \vec{p}_i', \vec{u}' \rangle$ are contained in the terms with minimum $\langle \vec{p}_i', \vec{r} \rangle$

↳ we are free to assign points in the integration region to different expansion regions by choosing any region whose lowest order terms in λ contain all of the lowest order terms in λ of the integration point

↳ Example: $P(x,\lambda) = \lambda + x + x^2$

- for $\vec{u}' = (3,1)$ we have an infinite number of different converging expansions: $\vec{r} = (t,1)$ with $t \geq 1$
- compare with figures on previous page
- for $\vec{u}' = (1,1)$ the only convergent expansion is given by $\vec{r} = (1,1)$

- Summary of observations:

  ①: an expansion with region vector $\vec{r}$ converges at a point $\vec{u}'$ if
  $\{\text{terms with min. } \langle \vec{p}_i', \vec{u}'\rangle\} \subset \{\text{terms with min. } \langle \vec{p}_i', \vec{r}\rangle\}$

  ②: for any $\vec{u}'$, the vertices with smallest $\langle \vec{p}_i', \vec{u}'\rangle$ must be part of some facet of the polytope

  ↳ since $u_{N+1} > 0$ by definition, the lowest order terms for any $\vec{u}'$ lie on a facet in F^+ (lower facets)

  ⇒ choosing the normal vectors of $f \in F^+$ as region vectors $\vec{r}_i$ we have a converging expansion for every $\vec{u}'$ ⇒ the whole integration domain is covered

  ↳ faces of co-dimension > 1 are scaleless ($\hat{=}\, (\mathcal{U}\mathcal{F})(c^{\vec{u}}\vec{x}) = c^k(\mathcal{U}\mathcal{F})(\vec{x})$ for $\vec{u} \neq n\mathbb{1}, n \in \mathbb{R}$)

  ↳ region vectors not normal to a facet give scaleless integrals

  ↳ "overlap regions" ($\hat{=}$ rescaling by two region vectors) are scaleless

- What about the non-commuting regions?

  ↳ [Jantzen 2012]: the term collecting all contributions from non-commuting regions vanishes if there is one region which commutes with all other regions

  ↳ expansions commute if the corresponding facets have a common vertex, otherwise they do not commute

  ↳ for typical Feynman integrals the hard region with $\vec{r} = (\vec{0}, 1)$ commutes with all other regions

Example : $I(\lambda) = \int_0^\infty dx \, (\lambda + x + x^2)^{-1+\varepsilon}$

$\hookrightarrow$ region vectors : $\vec{r}_1 = (0,1)$ $\vec{r}_2 = (1,1)$

- expand to order $\mathcal{O}(\lambda^0)$ and $\mathcal{O}(\varepsilon^0)$

$\hookrightarrow$ region 1: $\begin{cases} x \to x \\ \lambda \to \lambda \end{cases}$

$$(\lambda + x + x^2)^{-1+\varepsilon} = (x + x^2)^{-1+\varepsilon} + \mathcal{O}(\lambda)$$

$$\hookrightarrow \int_0^\infty (x + x^2)^{-1+\varepsilon} dx = \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)$$

$\hookrightarrow$ region 2: $\begin{cases} x \to \lambda x \\ \lambda \to \lambda \end{cases}$ $dx \to dx \, \lambda$

$$(\lambda + \lambda x + \lambda^2 x^2)^{-1+\varepsilon} = (\lambda + \lambda x)^{-1+\varepsilon} \left(1 + \mathcal{O}(\lambda)\right)$$

$$\hookrightarrow \int_0^\infty \lambda^\varepsilon (1+x)^{-1+\varepsilon} dx = -\frac{1}{\varepsilon} \lambda^\varepsilon$$

$$\Rightarrow I(\lambda) = \frac{1}{\varepsilon} - \frac{1}{\varepsilon} \lambda^\varepsilon + \mathcal{O}(\varepsilon, \lambda)$$

$$= -\log(\lambda) + \mathcal{O}(\varepsilon, \lambda)$$

Note the cancellation of ε-divergences

$\hookrightarrow$ can be checked numerically with pySecDec

Exercise :

Expand the integral $I(\lambda) = \int_0^\infty dx \, (\lambda + x + x^2 + \lambda x^3)^{-1+\varepsilon}$ for $\lambda \ll 1$ up to (and including) $\mathcal{O}(\lambda^1)$ and $\mathcal{O}(\varepsilon^1)$

- Expansion by regions in Feynman parameter representation:

Recall the integrand:

$$\left(\prod_{j=1}^{N} dx_j\, x_j^{\nu_j - 1}\right) \delta\left(\sum_j x_j - 1\right) \frac{\mathcal{U}^{N_\nu - \frac{d}{2}(L+1)}}{\mathcal{F}^{N_\nu - \frac{d}{2}L}}$$

we can choose any non-empty subset to be summed over, so that the argument is invariant

rescaling affects only terms $x_j^{\nu_j}$

after rescaling the Feynman parameters according to the region vector $\vec{r}$, the expansions of $\mathcal{U}$ and $\mathcal{F}$ start at orders u_{min} and f_{min}, respectively, in the smallness parameter

↳ we deduce the so-called external degree corresponding to the overall scaling of the region:

$$\sum_{j=1}^{N} r_j \nu_j + u_{min}\left(N_\nu - \frac{d}{2}(L+1)\right) + f_{min}\left(-N_\nu + \frac{d}{2}L\right)$$

↳ Since the Symanzik polynomials are homogeneous functions with $\deg(\mathcal{U}) = L$ and $\deg(\mathcal{F}) = L+1$, we can easily convince ourselves that the external degree remains invariant under constant shifts of $\vec{r}$

- Since in Lee-Pomeransky representation the polynomial in the integrand is $\mathcal{U}+\mathcal{F}$, we determine the relevant regions geometrically as discussed before

- Back to our example from the beginning:

$$\int_k \frac{1}{(k+p)^2 (k^2 - m^2)^2} \qquad \text{expand around } \frac{m^2}{|p^2|} = \lambda \ll 1$$

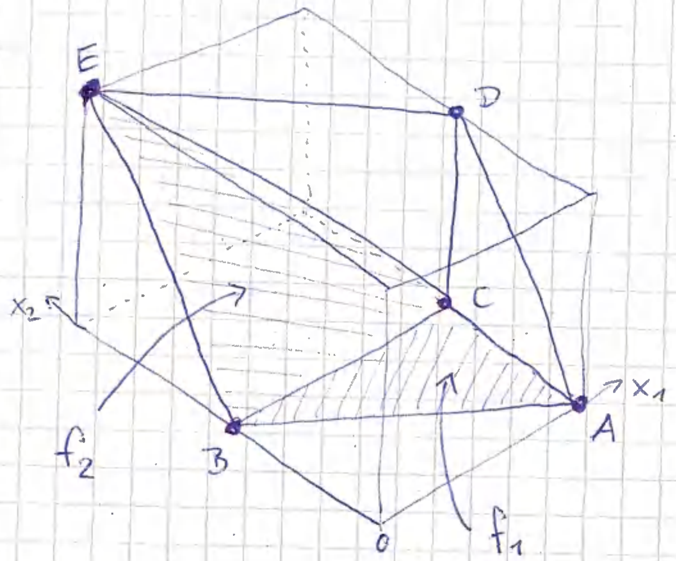
↳ euclidean: $p^2 = -s$

↳ Symanzik polynomials:

$$\mathcal{U} = x_1 + x_2$$

$$\mathcal{F} = x_2\left((s+m^2)\, x_1 + m^2 x_2\right)$$

$$\mathcal{U}+\mathcal{F}: \quad \text{vertices} \left\{ \underset{A}{\overset{(x_1\ x_2\ \lambda)}{(1,0,0)}}, \underset{B}{(0,1,0)}, \underset{C}{(1,1,0)}, \underset{D}{(1,1,1)}, \underset{E}{(0,2,1)} \right\}$$



f_1 : $\vec{n}_{f_1} = (0, 0, 1)$ $\longrightarrow$ ext. degree : 0

f_2 : $\vec{n}_{f_2} = (0, -1, 1)$ $\longrightarrow$ ext. degree : $\lambda^{-\varepsilon}$ (convenient to shift region to $(1, 0, 1)$)

$\hookrightarrow$ Exercise : confirm the results from the beginning

- Additional regulators

  ↳ we have already noticed that expansion by regions introduces spurious singularities in our regulator

  ↳ can even introduce singularities beyond those regulated with only one regulator

  ↳ need another regulator, e.g. symbolic "propagator" power

- Consider integral in Lee-Pomeransky representation with given vector $\vec{\nu}$ (and forget about λ for the moment)

  ↳ the facets of the Newton polytope of $P(\vec{x})$ encode the divergences of the integral

  ↳ using arguments from sector decomposition, we know that for each facet a singularity is present if

$$\langle \vec{n}_f, \vec{\nu} \rangle + a_f \frac{d}{2} \leq 0$$

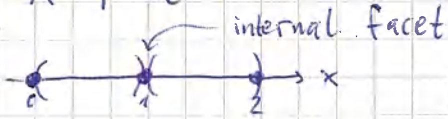
  ↳ method of regions will effectively subdivide the Newton polytope of the unexpanded integrand

  (for every region, the vertices contained in the facet will be selected)

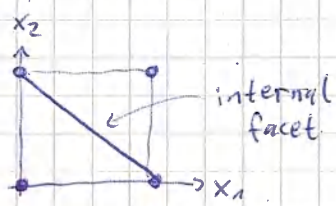
  ↳ new internal facets

- Visualize the subdivision in λ^0-plane:

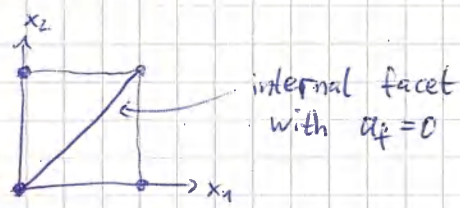
  ↳ $P(x,\lambda) = \lambda + x + x^2$



  ↳ $P(\vec{x},\lambda) = \lambda + x_1 + \lambda x_1 x_2 + x_2$



  ↳ $P(\vec{x},\lambda) = 1 + \lambda x_1 + x_1 x_2 + \lambda x_2$



- If the set of internal facets with $a_f = 0$ is non-empty => additional regulator required
  ↳ we can deform $\vec{\nu}$ such that

  $$\langle \vec{n}_f, \vec{\nu}_\delta \rangle \neq 0 \quad \text{for all internal facets with } a_f = 0$$

  where $\vec{\nu} \to \vec{\nu} + \vec{\nu}_\delta$

- Building bridges to the expansion of differential equations:
  ↳ expansion ansatz:

  $$I = \sum_j \sum_k \sum_n c_{jkn} \, x^{j+k\varepsilon} \log^n(x)$$

  logarithms beyond those generated by $\varepsilon \to 0$

  $j+k\varepsilon$ corresponds to external degree

  ↳ the appearance of $\log^n(x)$ terms can be attributed to the need for an analytic regulator (in addition to ε) in expansion by regions

  ↳ for a system in Fuchsian form in $x=0$, one can read out the $k\varepsilon$-branches from the eigenvalues of the residue
    ↳ $\log^n(x)$ terms originate from degenerate eigenvalues with algebraic multiplicity larger than the geometric one