13. Low-Energy Scattering:

more examples

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AQM

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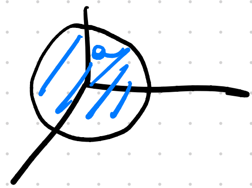
We consider here a further example of scattering by a "bounded potential", in particular the so-called POTENTIAL WELL (spherical well!)

$$V = \begin{cases} -V_0 & \text{fr } r \leq a \\ 0 & \text{fr } r > a \end{cases}$$



Let's consider solution of Schrödinger Eq inside & outside

INSIDE (using $V(r) = -V_0$ in radial Eq)

$$-\frac{\hbar^2}{2\mu}\frac{1}{r}\frac{d^2(rR)}{dr^2} + \frac{\hbar^2}{2\mu r^2}\ell(\ell+1)R = \underbrace{(V_0 + E)}_{\mathcal{E}}R$$

same as free solutions with $E \to \mathcal{E} = E + V_0$

define $K' = \frac{2\mu}{\hbar^2}(E+V_0)$ then

$r \leq a \quad R_{k'\ell}(r) = A\, j_\ell(k'r)$ (spherical Bessel smooth @ $r=0$)

outside we have standard solution

$$r>a \qquad R_{k\ell}(r) = B\, j_\ell(kr) + C\, y_\ell(kr) \quad (*)$$

$\uparrow$ ADMISSIBLE for $r \neq 0$

Remember now the asymptotic form of the solution that we found in Lecture 11

$$\psi(\vec{r}) = \sum_{\ell=0}^{\infty} (2\ell+1) \frac{P_\ell(\cos\theta)}{2ik} \left[S_\ell(k) \frac{e^{ikr}}{r} - \frac{e^{-i(kr-\pi\ell)}}{r} \right]$$

$$S_\ell(k) = 1 + 2i e^{i\delta_\ell} \sin\delta_\ell = 1 + \cancel{2i} \frac{e^{i\delta} - e^{-i\delta}}{\cancel{2i}} e^{i\delta} = e^{2i\delta_\ell}!$$

$$R_{k\ell}(r) \sim e^{2i\delta_\ell} \frac{e^{ikr}}{2ikr} - \frac{e^{-i(kr-\pi\ell)}}{2ikr}$$

$$= \frac{e^{-i(kr-\pi\ell/2)}}{2ikr} - e^{2i\delta_\ell} \frac{e^{i(kr-\pi\ell/2)}}{2ikr}$$

$$= e^{i\delta_\ell}\left[\frac{e^{-i(kr-\frac{\ell\pi}{2}+\delta_\ell)}}{2ikr} - \frac{e^{+i(kr-\frac{\ell\pi}{2}+\delta_\ell)}}{2ikr}\right]$$

$$= \frac{e^{i\delta_\ell}}{kr}\sin\left(kr-\frac{\ell\pi}{2}+\delta_\ell(k)\right)$$

phase shift

$$= \frac{e^{i\delta_\ell}}{kr}\left[\sin\left(kr-\frac{\ell\pi}{2}\right)\cos\delta_\ell + \sin\delta_\ell\cos\left(kr-\frac{\ell\pi}{2}\right)\right]$$

these are all alternative versions of same formula!

now expanding general solution ✱ @ $r\to\infty$

$$R_{k\ell}(r) \sim B\,\frac{\sin(kr-\ell\pi/2)}{kr} - C\,\frac{\cos(kr-\ell\pi/2)}{kr}$$

Comparing with behaviour above, gives [ignore overall phase!]

$$\frac{C}{B} = -\tan\delta_\ell \Rightarrow$$

so if I can get B, C, then I can obtain phase shift to find B, C I need to match solution inside and outside potential well

$$A\, j_\ell(k'a) = B\, j_\ell(ka) + C\, y_\ell(ka)$$

$$k' A \left.\frac{d\, j_\ell(\rho)}{d\rho}\right|_{\rho = k'a} = k \left[B \left.\frac{d\, j_\ell(\rho)}{d\rho}\right|_{\rho = ka} + C \left.\frac{d\, y_\ell(\rho)}{d\rho}\right|_{\rho = ka} \right]$$

ratio gives

$$k' \frac{j_\ell'(k'a)}{j_\ell(k'a)} = k \frac{j_\ell'(ka) + \frac{C}{B} y_\ell'(ka)}{j_\ell(ka) + \frac{C}{B} y_\ell(ka)}$$

$j_\ell' = \dfrac{d\, j_\ell(\rho)}{d\rho}$ etc , while $\underline{k' \neq k}$!

this formula can be inverted to give $\frac{C}{B}$, which in turn determines the tangent of the phase shift

assume $V_0 > 0$ [ATTRACTIVE POTENTIAL] there we can write

$$\left(-\frac{C}{B}=\right) \tan \delta_\ell = \frac{k\, j_\ell'(ka)\, j_\ell(k'a) - k'\, j_\ell(ka)\, j_\ell'(k'a)}{k\, y_\ell'(ka)\, j_\ell(k'a) - k'\, y_\ell(ka)\, j_\ell'(k'a)}$$

What if $k\, y_\ell'(ka)\, j_\ell(k'a) - k'\, y_\ell(ka)\, j_\ell'(k'a) = 0$?

$$\tan \delta_\ell = \infty \quad \Rightarrow \quad \delta_\ell = \frac{\pi}{2} + n\pi$$

Cross section $\sigma_k = \frac{4\pi}{k^2} \sum_\ell (2\ell+1) \sin^2 \delta_\ell(k)$

if $\delta_\ell(k) = \frac{\pi}{2}$ $\quad \sigma_k =$ MAXIMUM VALUE for that value of ℓ

$\Rightarrow$ RESONANT SCATTERING

Consider case when potential is VERY DEEP

$V_0 \gg E \Rightarrow k' \gg k$ so we can say

$$k'a \gg \ell \gg ka$$

in this further limit we can expand relation for $\tan\delta_\ell$. Assume first $ka \ll \ell$, then

$$\tan\delta_\ell \sim \frac{(2\ell+1)}{[(2\ell+1)!!]^2}(ka)^{2\ell+1}\underbrace{\left[\frac{\ell j_\ell(k'a) - k'a\, j_\ell'(k'a)}{(\ell+1) j_\ell(k'a) + k'a\, j_\ell'(k'a)}\right]}_{\text{now expand this for } k'a \gg \ell}$$

same behaviour we found for hard sphere

due to low energy $ka \ll \ell$

$$(\ell+1)\, j_\ell(k'a) + k'a\, j_\ell'(k'a) = 0$$ resonance condition

$$j_\ell(k'a) \sim \frac{1}{k'a}\cos\left(k'a - \frac{(\ell+1)}{2}\pi\right)$$

$$j_\ell'(k'a) \sim -\frac{1}{k'a}\left[\sin\left(k'a - \frac{\pi}{2}(\ell+1)\right) - \frac{1}{k'a}\cos\left(k'a - \frac{\pi}{2}(\ell+1)\right)\right]$$

so resonance condition becomes

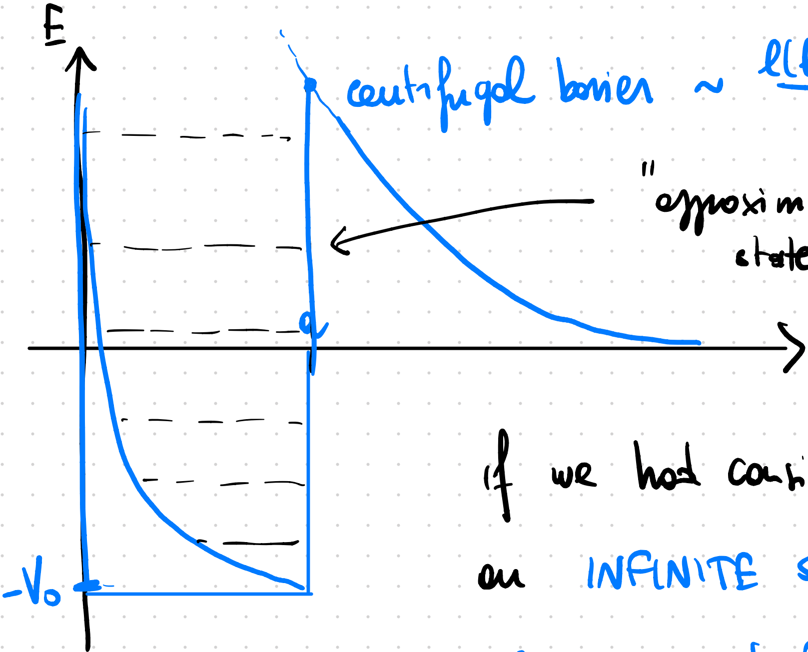
$$\frac{\ell}{k'a}\cos\left(k'a - \frac{(\ell+1)}{2}\pi\right) = \sin\left(k'a - \frac{(\ell+1)}{2}\pi\right)$$

$$\tan\Big(\underbrace{k'a - \frac{(\ell+1)}{2}\pi}_{\text{this must be small too (modulo } n\pi)}\Big) = \frac{\ell}{k'a} \sim 0$$

$$k'a - \frac{(\ell+1)}{2}\pi \sim \cancel{\frac{\ell}{k'a}} + n\pi$$

$$k'a \cong \left[\frac{\ell+1}{2} + n\right]\pi \sim \left(\frac{\ell}{2} + n\right)\pi \Rightarrow$$ energy levels in infinite WELL $\rightarrow$

IMPORTANT $\Rightarrow$ LINEAR POLE IN k' $\Rightarrow$ IN Energy !



if we had considered an INFINITE SPHERICAL WELL $V = \begin{cases} 0 & r \leq a \\ \infty & r > a \end{cases}$

$$R_{k\ell}(r) = A\, j_\ell(kr) \quad \text{for} \quad r \leq a$$

$$R_{k\ell}(r \to a) = 0 \Rightarrow j_\ell(ka) = 0 \quad \text{for } ka \gg \ell$$

$$\frac{1}{ka} \sin\left(ka - \frac{\ell\pi}{2}\right) = 0 \Rightarrow \underline{ka = \left(n + \frac{\ell}{2}\right)\pi}$$

RESONANT SCATTERING occurs when energy matches the energy levels that "would be bound states" if $\underline{V_0 \to \infty}$

"pseudo bound states" since now $E > 0$ (SCATTERING!)

going back to general behaviour of δ_ℓ for $ka \ll \ell$

$$\tan\delta_\ell \sim \frac{(2\ell+1)}{[(2\ell+1)!!]^2}(ka)^{2\ell+1}\left[\frac{\ell j_\ell(k'a) - k'a\, j_\ell'(k'a)}{(\ell+1)j_\ell(k'a) + k'a\, j_\ell'(k'a)}\right]$$

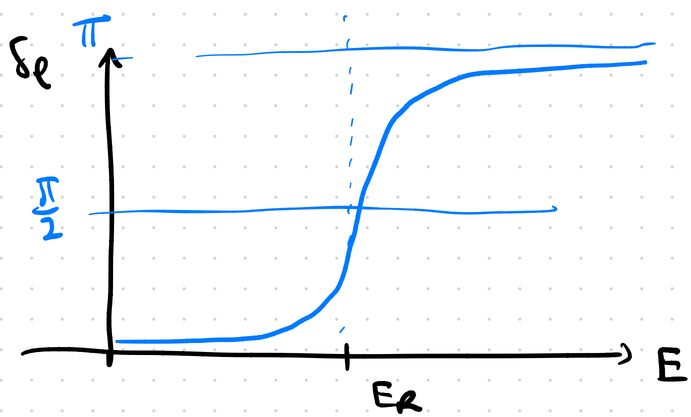
$\rightarrow 0$ if ka is small

so very small δ_ℓ

but there are resonances, as we saw, where

δ_ℓ rises rapidly from $0 \rightarrow \pi$ ($\tan\delta_\ell = \infty$)

Can we parametrize behaviour close to resonance?



In general always SINGLE POLE we can parametrize this as

$$\tan \delta_\ell \simeq \frac{\gamma (ka)^{2\ell+1}}{E - E_{res}}$$

DIVERGES TO POWER 1

resonant energy (whatever it is)

now what does this mean for cross-section?

we write $\sigma_{TOT} = \sum_\ell \sigma_\ell$

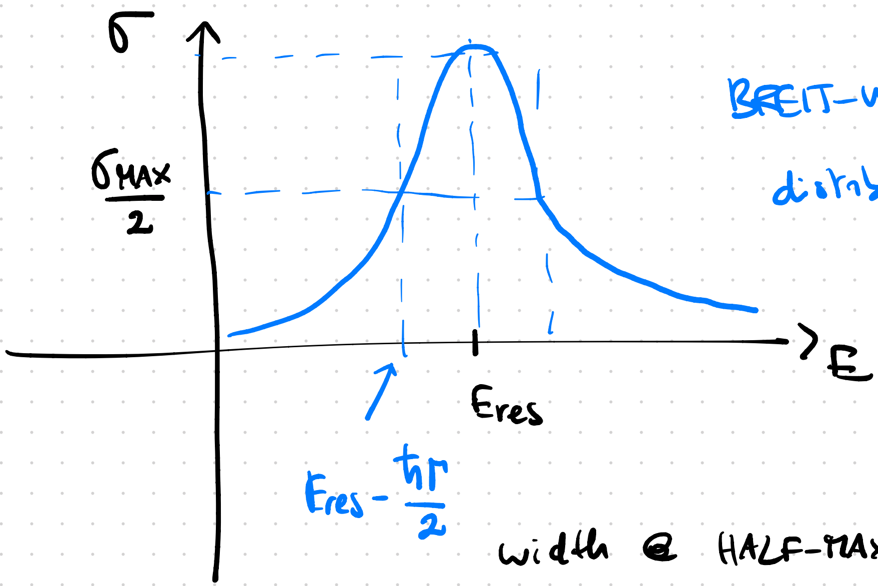
$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell = \frac{4\pi}{k^2}(2\ell+1)\frac{\tan^2\delta_\ell}{1+\tan^2\delta_\ell}$$

$$\left[\text{inverting } \tan^2\delta_\ell = \frac{\sin^2\delta_\ell}{\underbrace{1-\sin^2\delta_\ell}_{\cos^2\delta_\ell}}\right]$$

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\frac{\left[\gamma(ka)^{2\ell+1}\right]^2}{(E-E_{res})^2+\left[\gamma(ka)^{2\ell+1}\right]^2}$$

calling $\hbar^2\Gamma^2 = 4\left[\gamma(ka)^{2\ell+1}\right]^2$ (usually very small!) $ka \to 0$!

$$\sigma_\ell \sim \frac{4\pi}{k^2}(2\ell+1)\frac{\hbar^2\Gamma^2/4}{(E-E_{res})^2+\frac{\hbar^2\Gamma^2}{4}}$$



BREIT-WIGNER distribution!

width @ HALF-MAXIMUM!

remember, we found this general behaviour when studying DECAYS of UNSTABLE STATES (see Lecture 6!)

Now let's use the fact that at LOW ENERGY

$\delta \sim (ka)^{2\ell+1}$ and so $\ell = 0$ dominates

S-WAVE APPROXIMATION

let's do matching of solutions from scratch

INSIDE $\quad u(r) = r R_0(r) = C \sin k'r$

(regular at $r=0$)

OUTSIDE $\quad u(r) = \sin(kr + \delta(k))$ [up to $e^{i\delta}$]

$$C \sin(k'a) = \sin(ka + \delta)$$

$$C k' \cos(k'a) = k \cos(ka + \delta)$$

ratio

$$k' \cot(k'a) = k \cot(ka + \delta)$$

$$\tan\delta = \frac{k/k' \tan k'a - \tan ka}{1 + k/k' \tan(k'a) \tan(ka)}$$

let's define $\tan(qa) = \frac{k}{k'}\tan(k'a)$ then

$$\tan\delta = \frac{\tan qa - \tan ka}{1 + \tan(qa)\tan(ka)} = \tan(qa - ka)$$

$$\Rightarrow \quad \delta = \arctan\left[\frac{k}{k'}\tan(k'a)\right] - ka$$

remember now that

$$(k'a)^2 = (ka)^2 + \frac{2mV_0a^2}{\hbar^2}$$ from definition ! $(V_0 > 0)$

Low Energy gives then $[(ka) \ll 1 \,,\ \delta \ll 1]$

$$\delta \sim \frac{ka}{k'a}\tan k'a - ka = ka\left(\frac{\tan k'a}{k'a} - 1\right)$$

again imagine to increase V_0 slowly ; $k'a$ increases

when $k'a$ goes through $\frac{\pi}{2}$, $\tan k'a \to \infty$

here cannot use $\tan\delta \sim \delta$

$$\tan\delta = \frac{k/k' \tan k'a - \tan ka}{1 + k/k' \tan(k'a) \tan(ka)}$$

$$\sim \frac{1}{\tan ka} \to \infty \quad \text{because } \underline{\underline{ka \to 0}} \, !$$

so on resonance $\delta \to \frac{\pi}{2}$ as we saw already

$k'a = \frac{\pi}{2}$ is condition that well deep enough to give a <u>BOUND STATE</u> in well

(again bound state ~ resonance)

keep on increasing V_0, then WITH BOUND STATE $\delta \sim \pi + ka\left(\frac{\tan k'a}{k'a} - 1\right)$

this is required by continuity

$\tan\delta \sim O(ka) \Rightarrow \delta = ka\left(\frac{\tan k'a}{k'a} - 1\right) + n\pi$

so if $\delta > \frac{\pi}{2}$, it must be with $\underline{n = 1}$

as V_0 becomes larger and larger, the well becomes deeper and deeper.

$k'a$ passes through $\frac{3\pi}{2}$

$$\delta = 2\pi + ka\left[\frac{\tan k'a}{k'a} - 1\right]$$ and so on

each time that we ADD a BOUND STATE

LEVINSON'S THEOREM (For spherically symm V)

$$\delta(k\to 0) - \delta(k\to\infty) = N_B \cdot \pi$$

all bound states "seen"

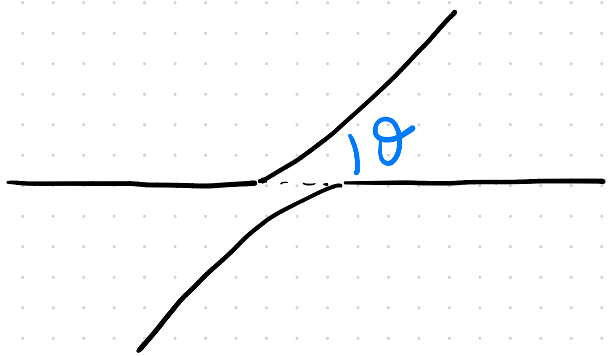
no bound states

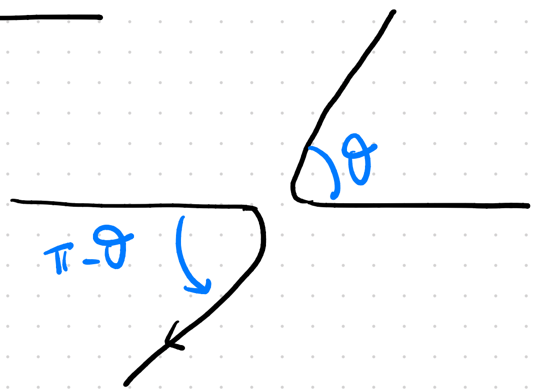
<u>number</u> of <u>Bound States</u>

SCATTERING OF IDENTICAL PARTICLES

We close with simple considerations on what happens when we scatter IDENTICAL PARTICLES

If we scatter two CLASSICAL identical particles





$\vartheta \sim \pi - \vartheta$ deflection

classically $\quad \underbrace{\sigma_{cl}(\vartheta) = \sigma(\vartheta) + \sigma(\pi - \vartheta)}_{}$

sum particles at the detector

QM $\Rightarrow$ Particles are INDISTINGUISHABLE

$\Rightarrow$ this means scattering amplitudes can INTERFERE

$$\frac{d\sigma_{QM}}{d\Omega} = |f(\theta) + f(\pi-\theta)|^2$$

$$= |f(\theta)|^2 + |f(\pi-\theta)|^2 + \underbrace{2\mathrm{Re}\left[f^*(\theta)f(\pi-\theta)\right]}_{\text{Quantum Correction to classical result}}$$

For example, @ $\theta = \frac{\pi}{2}$

$$\left(\frac{d\sigma_{QM}}{d\Omega}\right)_{\frac{\pi}{2}} = 4\left|f\left(\frac{\pi}{2}\right)\right|^2 \Rightarrow$$ Interference gives a factor of 2 extra!

things become even more interesting when SPIN involved

$\Rightarrow$ proton/proton or electron/electron scattering it depends if particles are in SINGLET / TRIPLET

Spin singlet is antisymmetric $\frac{\chi_+^1\chi_-^2-\chi_+^2\chi_-^1}{\sqrt{2}}$

then spatial wave function must be symmetric

$$\frac{d\sigma_s}{d\Omega}=|f(\theta)+f(\pi-\theta)|^2$$

symmetry exchanging two particles!

Spin triplet is symmetric $\left(\chi_+^1\chi_+^2,\ \chi_-^1\chi_-^2,\ \frac{\chi_+^1\chi_-^2+\chi_+^2\chi_-^1}{\sqrt{2}}\right)$

spatial wave function must be antisymmetric

$$\frac{d\sigma_t}{d\Omega}=|f(\theta)-f(\pi-\theta)|^2$$

antisymm exchanging particles!

if we scatter UNPOLARIZED beams of protons or electrons, ALL SPIN STATES EQUALLY LIKELY

$\Rightarrow$ probability of triplet 3 times singlet !

$$\frac{d\sigma}{d\Omega} = \frac{3}{4}\frac{d\sigma_t}{d\Omega} + \frac{1}{4}\frac{d\sigma_s}{d\Omega}$$

$$= \frac{3}{4}|f(\vartheta) - f(\pi-\vartheta)|^2 + \frac{1}{4}|f(\vartheta) + f(\pi-\vartheta)|^2$$

$$= |f(\vartheta)|^2 + |f(\pi-\vartheta)|^2 - \cancel{\frac{2}{2}}\, \mathrm{Re}[f^*(\vartheta)\, f(\pi-\vartheta)]$$

As long as the potential V is SPIN-INDEPENDENT

the − sign means we have DESTRUCTIVE INTERFERENCE

especially @ $\frac{\pi}{2}$ $\Rightarrow$ $2|f(\frac{\pi}{2})|^2 - |f(\frac{\pi}{2})|^2$

$$= |f(\tfrac{\pi}{2})|^2 \; !$$