11. Partial waves Expansion

SS 2024

AQM

[illegible]

[illegible]

Until now we considered scattering from a "weak" potential $\Rightarrow$ BORN APPROXIMATION

we also always assumed that the potential is confined in a SMALL REGION OF SPACE

Here we consider POTENTIALS WITH ROTATIONAL SYMMETRY

$$V(x,y,z) \equiv V(r) \qquad r = \sqrt{x^2+y^2+z^2}$$

We will still require $V(r) \to 0$ as $r \to \infty$ so that we can look at asymptotic solutions when $V=0$!

[we'll try to make also this statement more precise : how fast must V go to zero ?]

if $V = V(r)$ then $[\vec{L}, H] = 0$ angular momentum commutes with Hamiltonian $\Rightarrow$

we can go to spherical coordinates and look for solutions of Schrödinger Eq. in factorized form

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\psi}{\partial\theta}\right]+\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right)\right]$$

$$+V(r)\,\psi = E\,\psi$$

And we use $\psi(x,y,z) = \underbrace{R_{E,\ell}(r)}_{R_{k\ell}(r)}\, Y_{\ell,m}(\theta,\varphi)$

$R_{k\ell}(r)$ ← $E=\frac{\hbar^2k^2}{2\mu}$

$Y_{\ell,m}$ ← spherical harmonics

$$\begin{cases} L^2 Y_{\ell m} = \hbar^2\,\ell(\ell+1)\,Y_{\ell,m} \\ L_z Y_{\ell m} = \hbar\, m\, Y_{\ell m}\end{cases}$$

which gives a RADIAL Eq. for $R_{E,\ell}$

$$-\frac{\hbar^2}{2\mu}\frac{1}{r}\frac{d^2(rR)}{dr^2}+\frac{\hbar^2}{2\mu r^2}\ell(\ell+1)R+V(r)R = ER$$

as before let's put
$$\begin{cases} E = \dfrac{\hbar^2 k^2}{2\mu} \qquad \rho = kr \\ V = \dfrac{\hbar^2}{2\mu} \mathcal{U}(r) \end{cases}$$

$$\frac{d}{dr} = \frac{d\rho}{dr}\frac{d}{d\rho} = k\frac{d}{d\rho}$$

$$\frac{\hbar^2}{2\mu}\left[-\frac{\cancel{k}}{\rho} k^2 \frac{d^2}{d\rho^2}\left(\frac{\rho}{\cancel{k}} R\right) + \frac{k^2}{\rho^2} l(l+1) R + \mathcal{U}\cdot R - k^2 R \right] = 0$$

$$\Downarrow$$

$$k^2\left[-\frac{1}{\rho}\frac{d^2}{d\rho^2}(\rho R) + \left(\frac{1}{\rho^2} l(l+1) - 1\right) R(\rho) + \frac{\mathcal{U}(\rho)}{k^2} R \right] = 0$$

now consider asymptotic regime $\Rightarrow$ $\mathcal{U}(\rho) = 0$

$$-\frac{1}{\rho}\frac{d^2}{d\rho^2}\left(\rho R(\rho)\right) + \left(\frac{1}{\rho^2}\ell(\ell+1) - 1\right) R(\rho) = 0$$

SPHERICAL BESSEL EQUATION

it's a second order Eq, it has two solutions

denoted $j_\ell(\rho)$, $y_\ell(\rho)$

Close to $\rho \to 0$ solution is $R(\rho) \sim \rho^\alpha$

$$-\frac{1}{\rho}\frac{d^2}{d\rho^2}\left[\rho^{\alpha+1}\right] + \frac{1}{\rho^2}\left[\ell(\ell+1) - 1\right]\rho^\alpha = 0$$

$$-\rho^{\alpha-2}\left[\alpha(\alpha+1) - \ell(\ell+1) + \cancel{\rho^2}\right] = 0$$

$\sim \rho = 0$

$\Rightarrow$ two solutions $\begin{cases} \alpha = \ell \\ \alpha = -(1+\ell) \end{cases}$ $\quad j_\ell(\rho) \sim \rho^\ell$, $\quad y_\ell(\rho) \sim \frac{1}{\rho^{\ell+1}}$

this says that at $\rho = 0$ only $j_\ell(\rho)$ is an acceptable solution

EXPLICIT SOLUTIONS (with normalization)

$$j_0(\rho) = \frac{\sin\rho}{\rho} \qquad y_0(\rho) = -\frac{\cos\rho}{\rho}$$

$$j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho} \qquad y_1(\rho) = -\frac{\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho}$$

in general the ASYMPTOTIC BEHAVIOUR is :

$$\begin{cases} \lim_{\rho\to 0} j_\ell(\rho) \sim \dfrac{\rho^\ell}{(2\ell+1)(2\ell-1)\cdots 1} = \dfrac{\rho^\ell}{(2\ell+1)!!} \\ \lim_{\rho\to 0} y_\ell(\rho) \sim -\dfrac{(2\ell-1)\cdots 3\cdot 1}{\rho^{\ell+1}} = -\dfrac{(2\ell-1)!!}{\rho^{\ell+1}} \end{cases}$$

$$\left.\begin{cases} \lim_{\rho\to\infty} j_\ell(\rho) \sim \dfrac{1}{\rho}\cos\left(\rho - (\ell+1)\dfrac{\pi}{2}\right) = \dfrac{1}{\rho}\sin(\rho - \ell\pi/2) \\ \lim_{\rho\to\infty} y_\ell(\rho) \sim \dfrac{1}{\rho}\sin\left(\rho - (\ell+1)\dfrac{\pi}{2}\right) = -1/\rho\,\cos(\rho - \ell\pi/2) \end{cases}\right\}$$

can also be rewritten as complex exponentials

GENERAL SOLUTION of Schrödinger Eq. with $V(r)$

$$\boxed{\psi(x,y,z) = \psi(\vec{r}) = \sum_{\ell,m} A_{\ell m} R_{k\ell}(r) Y_{\ell m}(\theta,\varphi)}$$

if there was NO POTENTIAL down to $r=0$
we could then build general solution using only

$$\psi^{free}_{k\ell m}(r,\theta,\varphi) = j_\ell(\underbrace{kr}_{\rho}) Y_{\ell m}(\theta,\varphi)$$

in first lecture we used normal coordinates
and wrote for incoming wave $e^{i\vec{k}\cdot\vec{r}}$
or fixing $k=(0,0,k) \Rightarrow e^{ikz}$

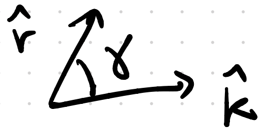
it MUST be possible to write one in terms of
the other, and in fact one can prove that

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell,m} 4\pi\, i^\ell j_\ell(kr) Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r})$$

PARTIAL WAVES EXPANSION !

where $k = |\vec{k}|$ & $Y_{\ell m}(\hat{r}) = Y_{\ell m}(\theta, \varphi)$

we can use ADDITION OF SPHERICAL HARMONICS



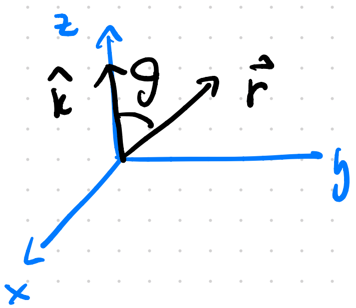
$$P_\ell(\hat{r}\cdot\hat{k}) = P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_m Y^*_{\ell m}(\hat{k})\,Y_{\ell m}(\hat{r})$$

and we can write

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, j_\ell(kr)\, P_\ell(\cos\gamma)$$

other form of <u>PARTIAL WAVES EXPANSION</u>

now if $\vec{k} = (0, 0, k)$ then $\gamma = \theta$



$$\left[\begin{aligned} e^{i\vec{k}\cdot\vec{r}} &= e^{ikz} \\ &= \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, j_\ell(kr)\, P_\ell(\cos\theta) \end{aligned}\right]$$

Now the EXACT solution of the Schrödinger eq AFTER SCATTERING with the potential will be

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \psi_{scatt}(\vec{r}) \qquad (*)$$

we have decomposed this into PARTIAL WAVES

We would like now to obtain the ASYMPTOTIC form of the solution for $r \to \infty$

We can decompose everything in spherical harmonics

$$\psi_{SCATT}(\vec{r}) = \sum_{\ell,m} S_{\ell,m}(r)\, Y_{\ell m}(\hat{r})$$

where, using ORTHOGONALITY, we have

$$S_{\ell m}(r) = \int d\Omega\, Y^{*}_{\ell m}(\theta,\varphi)\, \psi_{SCATT}(\vec{r})$$

finally, we already wrote:

$$\psi(\vec{r}) = \sum_{\ell,m} A_{\ell,m} R_{k\ell}(r) Y_{\ell m}(\hat{r})$$

Now, using all angular expansions in (*) we can write [remember $Y_{\ell m}(\hat{r})$ are BASIS!]

$$A_{\ell m} R_{k\ell}(r) = 4\pi i^{\ell} j_{\ell}(kr) Y^{*}_{\ell m}(\hat{k}) + S_{\ell m}(r)$$

we need asymptotic behaviour of

$$R_{k\ell}(r), \quad \underbrace{j_{\ell}(kr)}_{\text{known}}, \quad S_{\ell m}(r)$$

$S_{\ell,m}(r)$ is EASY if we remember that:

$$\psi_{SCATT}(\vec{r}) \sim \frac{e^{ikr}}{r} \underbrace{f_k(\theta,\varphi)}_{\text{SCATTERING AMPLITUDE}}$$

which means for $r \to \infty$ we can write

$$S_{\ell m}(r) \cong \frac{e^{ikr}}{r} \underbrace{\int d\Omega \, Y^*_{\ell m}(\vartheta,\varphi) \, f_k(\vartheta,\varphi)}_{f_{\ell m}}$$

$\Downarrow$

$$\begin{cases} f_{\ell m} = \int d\Omega' \, Y^*_{\ell m}(\vartheta',\varphi') \, f_k(\vartheta',\varphi') \\ f_k(\vartheta,\varphi) = \sum_{\ell,m} f_{\ell m} \, Y_{\ell m}(\vartheta,\varphi) \end{cases}$$

To find asymptotics of $R_{k\ell}$ of general solution we have to be careful.

We argue that if $V(r) \xrightarrow{r\to\infty} 0$ "FAST ENOUGH"

then $R_{k\ell}(r) \sim \alpha_\ell \, j_\ell(kr) + \beta_\ell \, y_\ell(kr)$

COMBINATION OF FREE SOLUTIONS

Assuming $R_{k\ell}(r)$ REAL, we can rewrite it as

$$R_{k\ell}(r) = B_\ell \underbrace{\left[j_\ell(kr) + i\, y_\ell(k\rho) \right]}_{h_\ell^{(1)}(kr)} + B_\ell^* \underbrace{\left[j_\ell(kr) - i\, y_\ell(kr) \right]}_{h_\ell^{(2)}(kr)}$$

$$h_\ell^{(2)}(kr) = h_\ell^{(1)}(kr)^* \qquad \text{HANKEL FUNCTIONS}$$

$$h_\ell^{(1)}(\rho) \xrightarrow[\rho \to \infty]{} \frac{e^{i[\rho - (\ell+1)\pi/2]}}{\rho}$$

up to a phase $\quad h_\ell^{(1)} \sim \dfrac{e^{i\rho}}{\rho}$ OUTGOING SPHERICAL WAVE

NOTICE $R_{k\ell}(kr)$ fulfils REAL DIFF EQ so we can pick $R_{k\ell}$ real too!

We can write $\quad B_\ell = |B_\ell|\, e^{i\delta_\ell}$ AND $\Rightarrow$

$$B_\ell^* = |B_\ell|\, e^{-i\delta_\ell}$$

$$R_{k\ell}(kr) \sim e^{i\delta_\ell} h_\ell^{(1)}(kr) + e^{-i\delta_\ell} h_\ell^{(2)}(kr)$$

$$\sim \frac{1}{kr}\left[e^{i(kr-(\ell+1)\pi/2+\delta_\ell)} + e^{-i(kr-(\ell+1)\pi/2+\delta_\ell)} \right]$$

$j_\ell(kr)$ in this notation $= \frac{1}{2}\left[h_\ell^{(1)}(kr) + h_\ell^{(2)}(kr)\right]$

$$\sim \frac{1}{kr}\left[e^{i(kr-(\ell+1)\pi/2)} + e^{-i(kr-(\ell+1)\pi/2)} \right]$$

$\Rightarrow$ Asymptotic solution parametrized by $\delta_\ell = \delta_\ell(k)$

$\uparrow$
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δ_ℓ called PHASE SHIFT

(with respect to incoming plane wave!)

Now we use these asymptotic behaviors in

$$A_{\ell m} R_{k\ell}(r) = 4\pi i^{\ell} j_\ell(kr) Y^*_{\ell m}(\hat{k}) + S_{\ell m}(r)$$

which gives when $r \to \infty$

$$A_{\ell m} \frac{1}{kr}\left[e^{i(kr - (\ell+1)\pi/2 + \delta_\ell)} + e^{-i(kr - (\ell+1)\pi/2 + \delta_\ell)} \right]$$

$$= 4\pi i^{\ell} \frac{1}{2kr}\left[e^{i(kr - (\ell+1)\pi/2)} + e^{-i(kr - (\ell+1)\pi/2)} \right] Y^*_{\ell m}(\hat{k})$$

$$+ \frac{e^{ikr}}{r} f_{\ell m}$$

scattered part has only OUTGOING WAVE

e^{ikr} & e^{-ikr} are independent, we can equate their coefficients (at order $\frac{1}{r}$) separately

FROM $\dfrac{e^{-ikr}}{r}$: $A_{\ell m} = 4\pi i^{\ell} \frac{1}{2} Y_{\ell m}^{*}(\hat{k})\, e^{i\delta_\ell}$

FROM $\dfrac{e^{ikr}}{r}$: $A_{\ell m} = 4\pi i^{\ell} \frac{1}{2} Y_{\ell m}^{*}(\hat{k})\, e^{-i\delta_\ell}$

$$+ e^{i(\ell+1)\frac{\pi}{2}} e^{-i\delta_\ell} k f_{\ell m}$$

insert first in second to get

$$\frac{4\pi i^{\ell}}{2} Y_{\ell m}^{*}(\hat{n})\, e^{i\delta_\ell} = \left[\frac{4\pi i^{\ell}}{2} Y_{\ell m}(\hat{n}) + e^{i(\ell+1)\frac{\pi}{2}} k f_{\ell m}\right] e^{-i\delta_\ell}$$

use : $e^{+i(\ell+1)\frac{\pi}{2}} = i^{\ell+1}$

$$\frac{4\pi \cancel{i^{\ell}}}{2} Y_{\ell m}^{*}(\hat{n})\, e^{2i\delta_\ell} = \cancel{i^{\ell}}\left[\frac{4\pi}{2} Y_{\ell m}^{*}(\hat{n}) + i k f_{\ell m}\right]$$

$$f_{\ell m} = \frac{4\pi}{k}\left[\frac{1}{2i} Y^*_{\ell m}(\hat{k})\right]\left(e^{2i\delta_\ell} - 1\right)$$

$$= \frac{4\pi}{k} e^{i\delta_\ell}\left[\frac{e^{i\delta_\ell} - e^{-i\delta_\ell}}{2i}\right] Y^*_{\ell m}(\hat{k})$$

$$= \frac{4\pi}{k} e^{i\delta_\ell} \sin\delta_\ell \, Y^*_{\ell m}(\hat{k})$$

which then finally gives expansion for scattering amplitude in terms of PHASE SHIFT

$$f_k(\theta,\varphi) = \frac{4\pi}{k}\sum_{\ell m} e^{i\delta_\ell} \sin\delta_\ell \, Y_{\ell m}(\hat{r}) Y^*_{\ell m}(\hat{k})$$

Again if $k = (0,0,k) = k\hat{z}$ we can use addition theorem and write

$$f_k(\theta,\varphi) = \frac{1}{k}\sum_{\ell=0}^{\infty} (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell \, P_\ell(\cos\theta)$$

remember $\delta_\ell = \delta_\ell(k)$ depends on k !

the last boxed equation is usually referred to as

PARTIAL-WAVES EXPANSION OF SCATTERING AMPLITUDE

What does this tell us about the wave function?

$$\psi(\vec{r}) \underset{r\to\infty}{\cong} e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f_k(\vartheta, \varphi)$$

using asymptotics for $e^{i\vec{k}\cdot\vec{r}}$ from plane waves

$$e^{i\vec{k}\cdot\vec{r}} = e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\vartheta)$$

for large $r\to\infty$

$$j_\ell(kr) = \frac{1}{2ikr}\left[e^{i(kr-\frac{\ell\pi}{2})} - e^{-i(kr-\ell\pi/2)}\right]$$

$\vec{k} = (0, 0, k)$ for definitenen;

$$\Rightarrow e^{ikz} \underset{\sim}{=} \sum_{\ell=0}^{\infty} \underbrace{e^{i\frac{\pi}{2}\ell}}_{i^\ell} (2\ell+1) \left[\frac{e^{i(kr - \frac{\ell\pi}{2})} - e^{-i(kr - \frac{\ell\pi}{2})}}{2ikr} \right] P_\ell(\cos\theta)$$

$$= \frac{1}{2ikr} \sum_{\ell=0}^{\infty} (2\ell+1) \left[e^{ikr} - e^{-i(kr - \ell\pi)} \right] P_\ell(\cos\theta)$$

so we get

$$\psi(\vec{r}) \sim \sum_{\ell=0}^{\infty} (2\ell+1) \left[\frac{e^{ikr} - e^{-i(kr-\ell\pi)}}{2ikr} \right] P_\ell(\cos\theta)$$

$$+ \frac{e^{ikr}}{r} \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell \, P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} (2\ell+1) \frac{P_\ell(\cos\theta)}{2ik} \left[\left(1 + 2i e^{i\delta_\ell} \sin\delta_\ell \right) \frac{e^{ikr}}{r} - \frac{e^{-i(kr - \pi\ell)}}{r} \right]$$

Scattering has effect of modifying outgoing wave solution as follows

$$\frac{e^{ikr}}{r} \to \underbrace{\left[1+2ie^{i\delta_\ell}\sin\delta_\ell\right]}_{S_\ell(k)} \frac{e^{ikr}}{r}$$

$$\left|1+2ie^{i\delta_\ell}\sin\delta_\ell\right|^2 = \left(1+2ie^{i\delta_\ell}\sin\delta_\ell\right)\left(1-2ie^{-i\delta_\ell}\sin\delta_\ell\right)$$

$$= 1-2ie^{-i\delta_\ell}\sin\delta_\ell+2ie^{i\delta_\ell}\sin\delta_\ell + 4\sin^2\delta_\ell$$

$$= 1+2i\sin\delta_\ell\left[2i\sin\delta_\ell\right]+4\sin^2\delta_\ell \equiv 1 \;//$$

$S_\ell(k)$ is a ~~phase~~ phase ! $\quad |S_\ell(k)|^2 = 1$

required to respect UNITARITY !

Finally, we can derive partial-wave expansion for CROSS SECTION

$$\frac{d\sigma}{d\Omega} = |f_k(\vartheta,\varphi)|^2$$

$$= \left(\frac{4\pi}{k}\right)^2 \sum_{\ell m \ell' m'} \Big[e^{i(\delta_\ell - \delta_{\ell'})} \sin\delta_\ell \sin\delta_{\ell'} \times$$

$$\times Y^*_{\ell m}(\hat{k})\, Y_{\ell' m'}(\hat{k})\, Y_{\ell m}(\hat{r})\, Y^*_{\ell' m'}(\hat{r}) \Big]$$

where $Y_{\ell m}(\hat{r}) = Y_{\ell m}(\vartheta,\varphi)$

if we integrate over $\int d\Omega = \int d\cos\vartheta \, d\varphi$

we can use ORTHONORMALITY of $Y_{\ell m}(\hat{r})$ again:

$$\sigma = \int d\Omega \, |f_k(\vartheta, \varphi)|^2$$

$$= \left(\frac{4\pi}{k}\right)^2 \sum_{\ell m} \sin^2\delta_\ell \, Y^*_{\ell m}(\hat{k}) \, Y_{\ell m}(\hat{k})$$

NOW USE ADDITION THEOREM with $\hat{r} = \hat{k}$

$$P_\ell(\hat{r} \cdot \hat{k}) = \frac{4\pi}{2\ell+1} \sum_m Y^*_{\ell m}(\hat{k}) \, Y_{\ell m}(\hat{r})$$

$$\Downarrow \quad (\hat{k} \cdot \hat{k} = 1 \; ; \; P_\ell(1) = 1)$$

$$P_\ell(1) = 1 = \underbrace{\frac{4\pi}{2\ell+1} \sum_m Y^*_{\ell m}(\hat{k}) \, Y_{\ell m}(\hat{k})}$$

$$\Rightarrow \boxed{\sigma_{TOT} = \frac{4\pi}{k^2} \sum_\ell (2\ell+1) \sin^2\delta_\ell}$$

let us conclude noticing that for FORWARD scattering

$$\theta = 0$$

$$f_k(0) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)\, e^{i\delta_l} \sin\delta_l \, \cancel{P_l(1)}$$

$$\mathrm{Im}\, f(0) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \sin\delta_l \, \mathrm{Im}(e^{i\delta_l})$$

$$\mathrm{Im}(e^{i\delta_l}) = \frac{e^{i\delta_l} - e^{-i\delta_l}}{2i} = \sin\delta_l \quad !$$

$$= \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l$$

$$= \frac{k}{4\pi}\, \sigma_{TOT} \quad !$$

WE FIND OPTICAL THEOREM AGAIN

(this proof valid only for CENTRAL POTENTIALS !)

ASYMPTOTIC BEHAVIOUR & COULOMB POTENTIAL

Now we have skipped the important issue of how FAST must $V(r) \to 0$ when $r \to \infty$ for this construction to make sense.

Consider again RADIAL SCHRÖDINGER EQ

$$-\frac{\hbar^2}{2\mu}\frac{1}{r}\frac{d^2(rR)}{dr^2} + \frac{\hbar^2}{2\mu r^2}\ell(\ell+1)R + V(r)R = ER$$

and rewrite it for $u(r) = r\,R(r)$

$$\Rightarrow \quad u''(r) + k^2 u(r) = \underbrace{\left[\frac{\ell(\ell+1)}{r^2} + \frac{2\mu}{\hbar^2}V(r)\right]}_{W(r)} u(r)$$

effective potential including centrifugal part if $V(r)$ can be neglected, $\begin{cases} u(r) \sim e^{\pm ikr} \\ R(r) \sim \dfrac{e^{\pm ikr}}{r} \end{cases}$

let us study effect of $V(r) \sim \dfrac{1}{r^p}$ on $u(r)$

for some $p \in \mathbb{N}$

PARAMETRIZE EFFECT $u(r) = e^{\pm ikr + g(r)}$

new function

plugging this in RADIAL EQUATION we get

$$g''(r) + [g'(r)]^2 \pm 2ik\,g'(r) = W(r)$$

study solution at $r \to \infty$, two cases depending $\begin{matrix} l = 0 \\ l \neq 0 \end{matrix}$

Consider $l \neq 0$, then $W \sim \dfrac{\#}{r^2} + \dfrac{\#}{r^p}$

- $p > 2$ then $\frac{1}{r^2}$ wins (~ as centrifugal)

- $p < 2$ then $\frac{1}{r^p}$ wins (goes slower to zero!)

all in all, we must study asymptotic solution to

$$g''(r) + [g'(r)]^2 \pm 2ik\, g'(r) = \frac{a}{r^p}$$

search for solution

$$\begin{cases} g(r) = \dfrac{b}{r^s} \\ g'(r) = -\dfrac{sb}{r^{s+1}} \\ g''(r) = \dfrac{s(s+1)\,b}{r^{s+2}} \end{cases}$$

$\underline{\underline{s > 0}}$!

$$\frac{s(s+1)b}{r^{s+2}} + \frac{s^2 b^2}{r^{2s+2}} \mp 2ik \frac{sb}{r^{s+1}} = \frac{a}{r^p}$$

$$\frac{sb}{r^{s+1}}\left[\mp 2ik + \cancel{\frac{s+1}{r}} + \cancel{\frac{sb}{r^{s+1}}}\right] = \frac{a}{r^p}$$

negligible for $r \to \infty$

$$\mp 2ik\,sb\,\frac{1}{r^{s+1}} = \frac{a}{r^p} \Rightarrow \boxed{s = p-1}$$

as long as $p > 1$ this has a solution!

$$g(r) = \frac{b}{r^{p-1}} \longrightarrow 0 \text{ as } r \to \infty$$

indeed $e^{\pm ikr}$ are right solutions!

if $p = 1 \Rightarrow s = 0$ ↯ no solution as power series!

if we still use $\{g'', (g')^2\} \ll g'$, we can approx

$$\pm 2ik\,g'(r) = \frac{a}{r} \Rightarrow g(r) = \mp \frac{ia}{2k} \ln(r)$$

$g(r) \sim \ln(r) \rightarrow$ explodes when $r \to \infty$

PHASE SHIFT DOES NOT GO TO ZERO !

$p = 1$ $\quad V(r) \sim \frac{1}{r}$ coulomb potential !!

$$u(r) \sim e^{\pm i\left(kr - \frac{a}{2k} \ln(r)\right)}$$

Coulomb Potential must be treated with care !