10. The Lippmann-Schwinger Equation

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AQM

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Let us go back to general situation

$H = H_0 + V$ $\quad$ H_0 known "FREE"

V potential which generates scattering

Let $|\phi\rangle$ be an eigenstate of $H_0 \Rightarrow H_0|\phi\rangle = E|\phi\rangle$

Now notice that, if FULL HAMILTONIAN H has a continuum of states ($E>0$, no BOUND STATE) then I can always imagine to find $|\psi\rangle$ s.t.

$$H|\psi\rangle = (H_0 + V)|\psi\rangle = E|\psi\rangle$$

same energy as $|\phi\rangle$!

What is the relationship between $|\phi\rangle$ & $|\psi\rangle$?

$\Rightarrow$ given by Lippmann - Schwinger Equation!

Consider Q, hermitian operator. We know it has a complete set of eigenstates

$$Q|n\rangle = q_n |n\rangle$$

$$\Rightarrow Q = \sum\!\!\!\!\!\!\int_n q_n |n\rangle\langle n|$$

continuous + discrete in general

Now, if all $q_n \neq 0$, the INVERSE is

$$Q^{-1} = \sum\!\!\!\!\!\!\int_n q_n^{-1} |n\rangle\langle n|$$

$$\Rightarrow Q^{-1} \cdot Q = \sum\!\!\!\!\!\!\int_n \sum\!\!\!\!\!\!\int_m q_n^{-1} q_m |n\rangle \underbrace{\langle n|m\rangle}_{\delta_{n,m}} \langle m|$$

$$= \mathbb{1} \quad \text{FORMALLY !}$$

(For continuum $\delta_{n,m} \to \delta(n-m)$)

What if some eigenvalues are zero?

Let us consider $\varepsilon > 0$; $\varepsilon \ll 1$ and

$$[Q \pm i\varepsilon]^{-1} = \oint_n (q_n \pm i\varepsilon)^{-1} |n\rangle\langle n|$$

Using this, let us define 2 states $|\psi_\pm\rangle$ as

$$|\psi_\pm\rangle = |\phi\rangle + (E - H_0 \pm i\varepsilon)^{-1} V |\psi_\pm\rangle$$

$|\phi\rangle$: eigenstate of H_0, $H_0|\phi\rangle = E|\phi\rangle$ (Asymptotic solution in SCATTERING!)

$(E - H_0 \pm i\varepsilon)^{-1}$: We need $\pm i\varepsilon$ to define inverse because $E>0$, H_0 has continuous spectrum!

V: POTENTIAL

and we have

$$(H_0 - E)|\psi_\pm\rangle = (H_0 - E)(E - H_0 \pm i\varepsilon)^{-1} V |\psi_\pm\rangle$$

$$\to -V|\psi_\pm\rangle \quad \text{sending } \varepsilon \to 0$$

if all manipulations are allowed, we have then

$$(H_0 + V - E)|\psi_\pm\rangle = 0$$

$\Rightarrow$ we found 2 states $|\psi_\pm\rangle$ which have same energy under H, as $|\phi\rangle$ had under H_0

the equation that defines $|\psi_\pm\rangle$ is called

LIPPMANN - SCHWINGER EQUATION

$$|\psi_\pm\rangle = |\phi\rangle + (E - H_0 \pm i\varepsilon)^{-1} V |\psi_\pm\rangle$$

Let's contract by $\langle\vec{x}|$ from LEFT & $\int d^3\vec{x}'\, |\vec{x}'\rangle\langle\vec{x}'|$

$$\langle\vec{x}|\psi_\pm\rangle = \langle\vec{x}|\phi\rangle$$

$$+ \int d^3\vec{x}'\, \langle\vec{x}|(E - H_0 \pm i\varepsilon)^{-1}|\vec{x}'\rangle\langle\vec{x}'| V |\psi_\pm\rangle$$

$\int d^3\vec{x}''\, |\vec{x}''\rangle\langle\vec{x}''|$

use $\langle \vec{x}' | V | \vec{x}'' \rangle = V(\vec{x}')\, \delta^{(3)}(\vec{x}' - \vec{x}'')$

DEFINITION OF A LOCAL POTENTIAL

$\Rightarrow$ Assumption, we don't want action at a distance, non-local!

we get

$$\langle \vec{x} | \psi_\pm \rangle = \langle \vec{x} | \phi \rangle + \int d^3\vec{x}' \langle \vec{x} | (E - H_0 \pm i\varepsilon)^{-1} | \vec{x}' \rangle V(\vec{x}') \langle \vec{x}' | \psi_\pm \rangle$$

or with wave function notation ($\vec{x}' \to \vec{y}$ RENAME!)

$$\psi_\pm(\vec{x}) = \phi(\vec{x}) + \int d^3\vec{y}\ \underbrace{\langle \vec{x} | (E - H_0 \pm i\varepsilon)^{-1} | \vec{y} \rangle}\ V(\vec{y})\, \psi_\pm(\vec{y})$$

SAME AS

$$\boxed{\psi(\vec{x}) = \psi_0(\vec{x}) + \frac{2M}{\hbar^2} \int d^3\vec{y}\ G(\vec{x} - \vec{y})\, V(\vec{y})\, \psi(\vec{y})}$$

$\Rightarrow$ by comparison we found an OPERATOR EXPRESSION for the Green's function(s)

TWO POSSIBILITIES $\pm$

$$G_\pm(\vec{x}-\vec{y}) = \frac{\hbar^2}{2\mu} \langle \vec{x} | (E - H_0 \pm i\varepsilon)^{-1} | \vec{y} \rangle$$

We will prove that $\begin{cases} G_+ \rightarrow \text{OUTGOING WAVE} \\ G_- \rightarrow \text{INCOMING WAVE} \end{cases}$!

Before that, consider now the operator T defined as

$$T|\phi\rangle = V|\psi_\pm\rangle$$

asymptotic state solution of H_0

Eigenstate of full Hamiltonian $H = H_0 + V$

then we can write (using Lippmann - Schwinger)

$$T|\phi\rangle = V|\phi\rangle + V(E-H_0 \pm i\varepsilon)^{-1} V |\psi_\pm\rangle$$

$$= V|\phi\rangle + V(E-H_0 \pm i\varepsilon)^{-1} T |\phi\rangle$$

this should hold $\forall\ |\phi\rangle$ (arbitrary !)

$$\Rightarrow \quad T = V + V(E-H_0 \pm i\varepsilon)^{-1} T$$

this equation can be iterated to reproduce the

BORN EXPANSION at operator level

$$T = V + V(E-H_0 \pm i\varepsilon)^{-1} V + O(V^3)$$

Take matrix element between some initial $|i\rangle$

and some final state $|f\rangle$

$$\langle f|T|i\rangle = \langle f|\int d^3\vec{x}\, |\vec{x}\rangle\langle\vec{x}|V|i\rangle \cancel{+\mathcal{O}(V^2)}$$

$$= \int d^3\vec{x}\, d^3\vec{y}\, \langle f|\vec{x}\rangle\langle\vec{x}|V|\vec{y}\rangle\langle\vec{y}|i\rangle$$

$$= \int d^3\vec{x}\, \langle f|\vec{x}\rangle\langle\vec{x}|i\rangle V(\vec{x})$$

now if $|i\rangle = |\vec{p}_i\rangle$ momenta ~ plane

$|f\rangle = |\vec{p}_f\rangle$ eigenstates waves

$$\langle\vec{x}|\vec{p}\rangle = e^{i\vec{p}\cdot\vec{x}/\hbar}\frac{1}{(2\pi\hbar)^{3/2}}$$

$$\langle\vec{p}_f|T|\vec{p}_i\rangle = \frac{1}{(2\pi\hbar)^3}\int \underbrace{d^3\vec{x}\, e^{i(\vec{p}_i-\vec{p}_f)\cdot\vec{x}/\hbar}\, V(\vec{x})}_{-\frac{2\pi\hbar^2}{\mu}f_k(\theta,\varphi)}$$

($\vec{p} = \hbar\vec{k}$ as always)

$$\langle \vec{p}_f | T | \vec{p}_i \rangle = -\frac{1}{4\pi^2\hbar}\frac{1}{\mu}\underbrace{f(\vec{p}_f, \vec{p}_i)}$$

alternative way of writing $f_k(\vartheta, \varphi)$!

From which we get $|f_k(\vartheta,\varphi)|^2 = [4\pi^2\hbar\mu]^2 |\langle p_f | T | p_i \rangle|^2$

Now for the scattering RATE we have

$$\delta n = j \frac{d\sigma(\vartheta,\varphi)}{d\Omega} d\Omega$$

$$= j\, |f(\vartheta,\varphi)|^2 d\Omega$$

$$= \left[\frac{p_i}{\mu}\frac{1}{(2\pi\hbar)^3}\right](4\pi^2\hbar\mu)^2 |\langle \vec{p}_f | T | \vec{p}_i \rangle|^2 d\Omega$$

flux $\uparrow$ for $\dfrac{e^{i\vec{p}\cdot\vec{x}/\hbar}}{(2\pi\hbar)^{3/2}}$ [in fact a we had $e^{ik\cdot z}$ only !]

$$= \frac{2\pi}{\hbar} |\langle \vec{p}_f | T | \vec{p}_i \rangle|^2 \left(p_i \mu \, d\Omega \right)$$

$$= \frac{2\pi}{\hbar} |\langle \vec{p}_f | T | \vec{p}_i \rangle|^2 \left(\underbrace{p_i^2 \frac{dp_i}{dE} d\Omega}_{\rho(E)\,dE \text{ density of states}} \right)$$

$$\overset{\text{BORN}}{\underset{=}{\sim}} \frac{2\pi}{\hbar} |\langle \vec{p}_f | V | \vec{p}_i \rangle|^2 \rho(E)\, dE$$

(I used $T = V + O(V^2)$)

$\Rightarrow$ We get FERMI GOLDEN RULE $\equiv$ BORN APPROXIMATION

COMPUTATION OF THE GREEN'S FUNCTION

Let's now compute

$$G_\pm(\vec{x}-\vec{y}) = \frac{\hbar^2}{2\mu} \langle \vec{x} | (E - H_0 \pm i\varepsilon)^{-1} | \vec{y} \rangle$$

with $H_0 = \frac{\vec{p}^2}{2\mu}$ FREE HAMILTONIAN since $\mathcal{H}_0$ is diagonal in momentum space, its convenient to perform the calculation there:

$$G_\pm(\vec{x}-\vec{y}) = \frac{\hbar^2}{2\mu} \int d^3\vec{p} \, \langle \vec{x}|\vec{p}\rangle\langle\vec{p}|(E-\mathcal{H}_0 \pm i\varepsilon)^{-1}|\vec{p}\rangle\langle\vec{p}|\vec{y}\rangle$$

$$= \frac{\hbar^2}{2\mu} \frac{1}{(2\pi\hbar)^3} \int d^3 \, e^{i\vec{p}\cdot\frac{(\vec{x}-\vec{y})}{\hbar}} \frac{1}{\left(E - \frac{\vec{p}^2}{2\mu} \pm i\varepsilon\right)}$$

introduce $E = \frac{\hbar^2 \vec{k}^2}{2\mu}$ $\quad \vec{p} = \hbar \vec{k}'$

$$G_\pm(\vec{x}-\vec{y}) = \int \frac{d^3\vec{k}'}{(2\pi)^3} \frac{e^{i\vec{k}'\cdot(\vec{x}-\vec{y})}}{\vec{k}^2 - \vec{k}'^2 \pm i\varepsilon}$$

$\Rightarrow$ spherical coord $\quad d^3\vec{k}' = dk' \, k'^2 \, d(\overbrace{\cos\theta}^{z}) \, d\varphi$

$$\vec{k}'\cdot(\vec{x}-\vec{y}) = k'|\vec{x}-\vec{y}|\underbrace{\cos\theta}_{z}$$

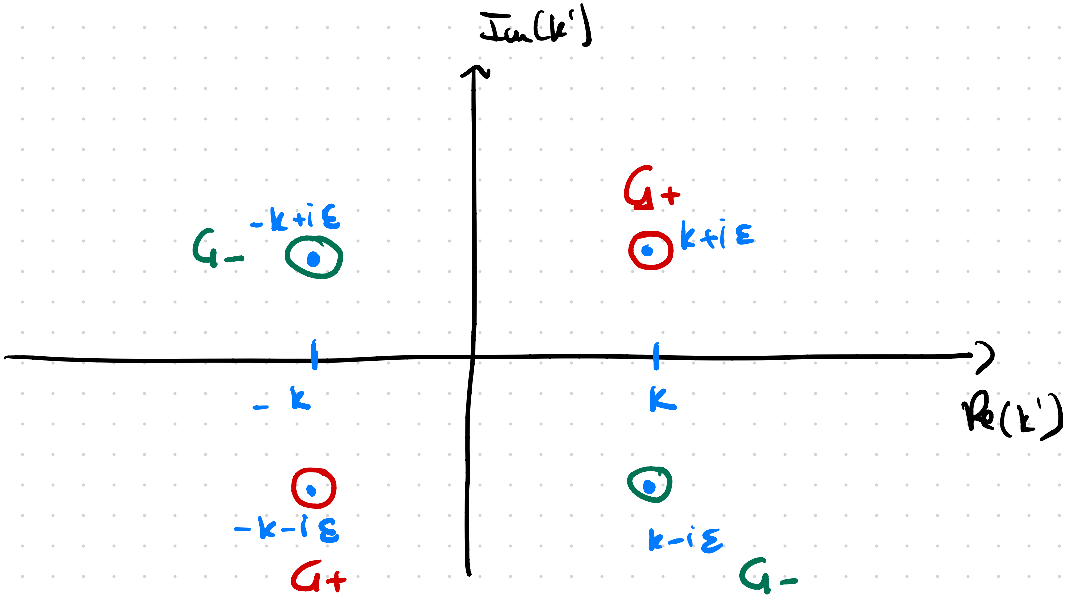
$$= \frac{1}{(2\pi)^2} \int_0^\infty k'^2 dk' \frac{1}{k^2 - k'^2 \pm i\varepsilon} \int_{-1}^{1} dz \, e^{ik'|\vec{x}-\vec{y}|z}$$

$$= -\frac{i}{(2\pi)^2}\,\frac{1}{|\vec{x}-\vec{y}|}\int_0^{\infty} dk'\,\frac{k'}{k^2-k'^2\pm i\varepsilon}\left(e^{+ik'|\vec{x}-\vec{y}|}-e^{-ik'|\vec{x}-\vec{y}|}\right)$$

$$= +\frac{i}{(2\pi)^2}\,\frac{1}{2|\vec{x}-\vec{y}|}\int_{-\infty}^{+\infty} dk'\,\frac{k'}{k'^2-k^2\mp i\varepsilon}\left[e^{ik'|\vec{x}-\vec{y}|}-e^{-ik'|\vec{x}-\vec{y}|}\right]$$

(integrand EVEN $k'\to -k'$, $\int_0^{\infty} = \frac{1}{2}\int_{-\infty}^{+\infty}$!)

Depending on $G_\pm \Rightarrow k' = \pm\sqrt{k^2 \pm i\varepsilon} = \pm(k \pm i\varepsilon)$

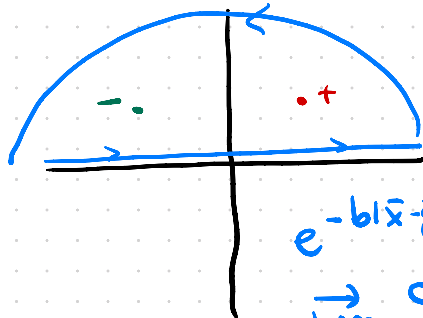


We can perform integrals with CAUCHY THEOREM

$\Rightarrow$ split into two pieces

$$I_\pm = \int_{-\infty}^{+\infty} \frac{dk'\,k'}{k'^2 - k^2 \mp i\varepsilon}\, e^{ik'|\vec{x}-\vec{y}|} \rightarrow$$

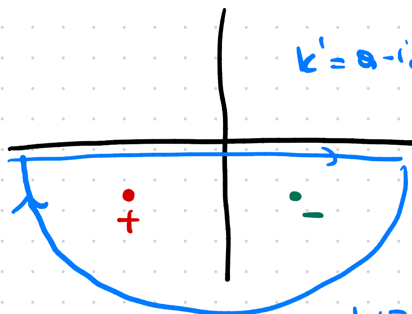
$k' = a + ib$



$e^{-b|\vec{x}-\vec{y}|} \xrightarrow[b\to\infty]{} 0$

$$II_\pm = \int_{-\infty}^{+\infty} \frac{dk'\,k'}{k'^2 - k^2 \mp i\varepsilon}\, e^{-ik'|\vec{x}-\vec{y}|} \rightarrow$$

$k' = a - ib$



$e^{-b|\vec{x}-\vec{y}|} \xrightarrow[b\to\infty]{} 0$

Counterclock wise

$$I_\pm = +2\pi i \,\frac{e^{\pm ik|\vec{x}-\vec{y}|}}{\pm 2k} \cdot (\pm k)$$

clock wise

$$II_\pm = -2\pi i \,\frac{e^{\pm ik|\vec{x}-\vec{y}|}}{\mp 2k} \cdot (\mp k)$$

$$I_\pm - II_\pm = 2\pi i \, e^{\pm ik|\vec{x}-\vec{y}|}\left[\frac{1}{2}+\frac{1}{2}\right]$$

$$= 2\pi i \, e^{\pm ik|\vec{x}-\vec{y}|}$$

and putting everything together

$$G_\pm(\vec{x}-\vec{y}) = \frac{i}{(2\pi)^2}\frac{1}{2|\vec{x}-\vec{y}|}\cdot \cancel{2\pi} i \, e^{\pm ik|\vec{x}-\vec{y}|}$$

$$G_\pm(\vec{x}-\vec{y}) = -\frac{e^{\pm ik|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|}$$

which proves the formula for G_+ that we used in Lecture 9! Outgoing plane wave as it implies

$$\psi = e^{ikz} + \frac{e^{ikr}}{r} f_k(\vartheta,\varphi)\left[1+O\left(\frac{1}{r}\right)\right] \; !$$

(see Lecture 9!)

UNITARITY & THE OPTICAL THEOREM

We will now use the Lippmann-Schwinger Eq to derive a very general result that relates the SCATTERING CROSS SECTION to the so-called FORWARD SCATTERING AMPLITUDE

Start from relation between scattering amplitude $f(\vartheta,\varphi) \equiv f(\vec{p}_f, \vec{p}_i)$ & matrix T

$$\langle \vec{p}_f | T | \vec{p}_i \rangle = -\frac{1}{4\pi^2 \hbar} \frac{1}{\mu} f(\vec{p}_f, \vec{p}_i)$$

remember

$$T|\phi\rangle = V|\psi_\pm\rangle$$

asymptotic state $\equiv |p_i\rangle$! (plane wave here)

Eigenstate of full Hamiltonian

now consider limit FORWARD SCATTERING $\vartheta = 0$

$\vec{P}_f = \vec{P}_i = \vec{P}$ then

$$f(\theta=0) = f(\vec{p},\vec{p}) = -4\pi^2\hbar\,\mu\,\langle\vec{p}|T|\vec{p}\rangle$$

now let us compute $\mathrm{Im}\,\langle\vec{p}|T|\vec{p}\rangle$

use $T|\vec{p}\rangle = V|\psi_+\rangle$ & Lippmann Schwinger

$$|\psi_\pm\rangle = \underbrace{|\phi\rangle}_{|\vec{p}\rangle} + (E - H_0 \pm i\varepsilon)^{-1} V |\psi_\pm\rangle$$

$V^\dagger = V$

$$\mathrm{Im}\langle\vec{p}|V|\psi_+\rangle = \mathrm{Im}\left[\left(\langle\psi_+| - \langle\psi_+|V\frac{1}{E-H_0-i\varepsilon}\right)V|\psi_+\rangle\right]$$

Adjoint changes sign

– because we wrote $|\vec{p}\rangle \to |\psi_+\rangle$

$$= \operatorname{Im} \langle \psi_+ | V | \psi_+ \rangle - \operatorname{Im} \langle \psi_+ | V \frac{1}{E - H_0 - i\varepsilon} V | \psi_+ \rangle$$

now use IMPORTANT FORMULA (see exercises!)

$$\frac{1}{x \pm i\varepsilon} = PV\left(\frac{1}{x}\right) \mp i\pi \delta(x) \quad \left(x = E - H_0\right)$$

$$= \operatorname{Im} \langle \psi_+ | V | \psi_+ \rangle - \operatorname{Im} \langle \psi_+ | V \, PV\left(\frac{1}{E - H_0}\right) V | \psi_+ \rangle$$

$$- \operatorname{Im} \langle \psi_+ | V \, i\pi \delta(E - H_0) V | \psi_+ \rangle$$

now $\operatorname{Im} \langle \psi_+ | V | \psi_+ \rangle = 0$

$\operatorname{Im} \langle \psi_+ | V \, PV\left(\frac{1}{E - H_0}\right) V | \psi_+ \rangle = 0$

because V is Hermitian $V^+ = V$, real potential!

only third term survives, in particular:

$$\mathrm{Im}\,\langle\vec{p}|T|\vec{p}\rangle = -\pi\,\langle\psi_+|\,V\,\delta(E-H_0)\,V\,|\psi_+\rangle$$

now $\quad V|\psi_+\rangle = T|\phi\rangle = T|\vec{p}\rangle \quad$ so

$$= -\pi\,\langle\vec{p}|\,T^\dagger\,\delta(E-H_0)\,T\,|\vec{p}\rangle$$

$$= -\pi\int d^3\vec{q}\;\langle\vec{p}|T^\dagger|\vec{q}\rangle\langle\vec{q}|T|\vec{p}\rangle\,\delta\!\left(E-\frac{\vec{q}^{\,2}}{2\mu}\right)$$

(used $\quad H_0|\vec{q}\rangle = \dfrac{\vec{q}^{\,2}}{2\mu}|\vec{q}\rangle \quad !$)

write $\quad d^3\vec{q} = q^2\,dq\,d\Omega_q = q^2\underbrace{\left(\frac{dq}{dE_q}\right)}\,dE_q\,d\Omega_q$

$\left(\dfrac{dE_q}{dq} = \dfrac{2q}{2\mu} = \dfrac{q}{\mu}\right.$ due to δ-function constraint $\Big)$

$$= -\pi\int d\Omega_q\,\cancel{dE}\;q^2\left(\frac{\mu}{q}\right)\left|\langle\vec{p}|T|\vec{q}\rangle\right|^2\delta(E-E_q)$$

this is E_p !

$$= -\pi\int d\Omega_q\,(m\,q)\left|\langle\vec{p}|T|\vec{q}\rangle\right|^2$$

Fixing $E = E_q$ means also $p = q$ since

$$E = \frac{p^2}{2\mu} \; !$$

$$\Rightarrow \; \mathrm{Im}\langle \vec{p}\,|T|\,\vec{p}\rangle = -\mu\pi p \int d\Omega_q \, |\langle \vec{p}\,|T|\,\vec{q}\rangle|^2$$

going back to original formula

$$f(\theta=0) = f(\vec{p},\vec{p}) = -4\pi^2\hbar\mu\,\langle \vec{p}\,|T|\,\vec{p}\rangle$$

$$\mathrm{Im}\left[f(\vec{p},\vec{p})\right] = -4\pi^2\hbar\mu\left[-\mu\pi p \int d\Omega_q |\langle \vec{p}\,|T|\,\vec{q}\rangle|^2\right]$$

$$= 4\pi^3\hbar\mu^2 p \int d\Omega_q \, |\langle \vec{p}\,|T|\,\vec{q}\rangle|^2$$

remember now relation with f

$$\langle \vec{p}_j|T|\vec{p}_i\rangle = -\frac{1}{4\pi^2\hbar\mu} f(\vec{k}_j,\vec{k}_i)$$

$$\mathrm{Im}\left(f(\vec{p},\vec{p})\right) = \frac{4\pi^3\hbar\mu^2 p}{(4\pi^2\hbar\mu)^2}\underbrace{\int d\Omega_q\,|f(\vec{p},\vec{q})|^2}_{\underline{\sigma_{TOT}}}$$

$$\boxed{\mathrm{Im}\left(f(\vec{p},\vec{p})\right) = \frac{p}{4\pi\hbar}\,\sigma_{TOT} = \frac{k}{4\pi}\,\sigma_{TOT}}$$

OPTICAL THEOREM

Says that <u>TOTAL CROSS-SECTION</u> is proportional to the <u>FORWARD SCATTERING AMPLITUDE</u>