# III) Canonical Differential Equations & Iterated Integrals

*) So far mostly $\varepsilon = 0$

*) And we have not really thought about a "good" basis of master integrals we should compute. Definitely, the simpler the $\varepsilon$ dependence in $\mathbb{D}(\underline{z};\varepsilon)$ the better, but what is the best possible?

*) We usually do not want/need a full/exact solution in $\varepsilon$. We only want a LAURENT expansion of our MIs up to some finite order in $\varepsilon$:

$$I_K(\underline{z};\varepsilon) = \underbrace{\frac{1}{\varepsilon^r} I_K^{(-r)}(\underline{z}) + \frac{1}{\varepsilon^{r-1}} I_K^{(-r+1)}(\underline{z}) + \dots +}_{\text{poles if integral is not finite in } d_0 \text{ dimensions}} \underbrace{I_K^{(0)}(\underline{z})}_{\text{finite piece}} + \underbrace{\varepsilon I_K^{(1)}(\underline{z}) + \dots + \varepsilon^s I_K^{(s)}(\underline{z})}_{\text{higher necessary/wanted orders}} + \mathcal{O}(\varepsilon^{s+1})$$

⇝ What is the best/most efficient way of deriving these coefficients in the $\varepsilon$-expansion?

Let's make the following:

Basis change: $\boxed{\mathcal{J} = T(z;\varepsilon)\,\mathcal{I}}$

we rotate where the entries can be not just rational in $z$ & $\varepsilon$

↳ depending on the problem $T$ will contain interesting transcendental functions ⇝ related solutions of GM/PF eq. @ $\varepsilon = 0$

$$d\mathcal{I} = B(z;\varepsilon)\,\mathcal{I} \quad \rightsquigarrow \quad T\left(B(z;\varepsilon)\,T^{-1} - d\,T^{-1}\right) =: \varepsilon\, A(z)$$

we want to choose $T$ such that $\varepsilon$-factorizes from kinematics

$$\Rightarrow \quad \boxed{d\mathcal{J} = \varepsilon\, A(z)\, \mathcal{J}} \quad (*)$$

$\varepsilon$-factorized diff. eq.

⇝ Nice since $\varepsilon$-dependence is simple but why is this exactly good?

finding/constructing rotation $T$ is as complicated as solving the diff. eq's @ $\varepsilon = 0$ !

⇝ this form is not for free!

(*) is nice because:

We have a formal solution given by the path-ordered exponential

$$\boxed{\mathcal{J} = \mathbb{P}\exp\left\{\varepsilon \int_\gamma A(x)\right\} \mathcal{J}_0(\varepsilon)}$$

path-ordered exponential

path connecting boundary pt. $z_0$ to pt. $z$

entries of GM/connection form $A(x)$ determines the functions we have to integrate over

boundary values of our MIs @ $z_0$ has to be known (it is ok to have it as an expansion in $\varepsilon$)

expand the path-ordered exponential to systematically get the required $\varepsilon$-orders

What is exactly the path-ordered exponential and how do we expand it? (skip?)

⇝ we want to know the $\mathbb{P}$-ord. exp. at $z$

$$\gamma : [0,t]^n \longrightarrow \mathbb{R}^{\overset{n \leftarrow \#\text{ param.}}{}} \ (\text{or } \mathbb{C}^n) \quad \text{such that } \gamma(0) = z_0 \text{ and } \gamma(t) = z$$

≙ decompose paths — a bit sloppy, better/simpler one-param.

$$\mathbb{P}\exp\left\{\varepsilon \int_\gamma A(x)\right\} = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \int_0^t \cdots \int_0^t \underbrace{\mathbb{P}\{A(t_1)\cdots A(t_k)\}}_{\substack{\text{matrices are ordered by path} \\ \text{(compare time ordering in QFT)}}} d^nt_1 \cdots d^nt_k$$

We choose a particular ordering of integration → removes $k!$ redundancy

$$= \sum_{k=0}^{\infty} \varepsilon^k \underbrace{\int_0^t d^nt_k \int_0^{t_k} d^nt_{k-1} \cdots \int_0^{t_3} d^nt_2 \int_0^{t_2} d^nt_1}_{\text{decompose path: first } t_1, t_2, \dots, t_k} A(t_k) \cdots A(t_1)$$

We see that the Taylor expansion of FIs naturally gives rise to iterated integrals over the entries of $A(z)$. At order $\varepsilon^k$ we get at most an iterated integral of length (≙ number of integrations) $k$.

if the boundary values are properly normalized in $\varepsilon$ (no poles, start const. in $\varepsilon$!)

To understand this better, let's look at the following simple example:

1-param.: $\gamma : [0,t] \to \mathbb{R}$

$$A(z) = \left(\begin{array}{c|c} \overset{1\times 1}{0} & 0 \\ \hline \frac{1}{z} & \frac{1}{1-z} \end{array}\right) \qquad \longrightarrow \text{ we see immediately from here that first MI is const.}$$

subsector ($\frac{1}{z}$), $1\times1$ ($\frac{1}{1-z}$)

$$\mathbb{P}\exp\left\{\varepsilon\int_\gamma A(x)\right\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \int_0^t dt_1 \begin{pmatrix} 0 & 0 \\ 1/t_1 & 1/1-t_1 \end{pmatrix} \overset{\frac{1}{2!}\text{ goes away from } \mathbb{P}\text{-ordering}}{+\,\varepsilon^2} \int_0^t dt_2 \int_0^{t_2} dt_1 \begin{pmatrix} 0 & 0 \\ 1/t_2 & 1/1-t_2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1/t_1 & 1/1-t_1 \end{pmatrix}$$

We regulate s.t. $\int_0^t dt_1 \frac{1}{t_1} = \log(t)$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 \\ \log t & -\log(1-t) \end{pmatrix} + \varepsilon^2 \int_0^t dt_2 \begin{pmatrix} 0 & 0 \\ \frac{1}{t_2} & \frac{1}{1-t_2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \log(t_2) & -\log(1-t_2) \end{pmatrix} + \mathcal{O}(\varepsilon^3)$$

$$= \begin{pmatrix} 0 & 0 \\ \log(t_2)/1-t_2 & -\log(1-t_2)/1-t_2 \end{pmatrix} + \mathcal{O}(\varepsilon^3)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 \\ \log t & -\log(1-t) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 & 0 \\ \mathrm{Li}_2(1-t_2) & \frac{1}{2}\log^2(1-t_2) \end{pmatrix} + \mathcal{O}(\varepsilon^3) \qquad (*)$$

if not known just leave it as an integral

Exercise: Check that indeed $(*)$ fulfills $\varepsilon$-factorized eq. $d\underline{I} = \varepsilon A(z)\underline{I}$ up to $\mathcal{O}(\varepsilon^3)$.

In the special case when the matrix $A$ can be written as

$$A(\underline{z}) = G_1 \omega_1 + \dots + G_m \omega_m \quad , \quad G_i \in M_{N\times N}(\mathbb{C}) \text{ (const. matrices)}$$

and all $\omega_i = d \log f_i(\underline{z})$ "dlog-forms" $, i=1,\dots,m$

algebraic function in $\underline{z}$

then we call the system in CANONICAL form (not just $\varepsilon$-factorized)

Comments:

*) canonical integrals only have logarithmic singularities

as expected from gauge theory amplitudes (good)

$$\int_\gamma \omega_1 \dots \omega_n \sim \underbrace{\sum_{k=0}^{n} \underset{\mathbb{C}}{\overset{}{c_k}}}_{} \log^k(f(\underline{z})) + \dots \quad : \text{at most } \log^n$$

no function in front of logarithm ($\rightsquigarrow$ Frobenius basis similar but with $\sum a_n z^n$ be in front of log!)

$\longrightarrow$ MIs will be const. linear combination of these functions ("generalized logs")
("No algebraic or rational prefactors")

*) functions made of iterat. integrals whose it. int. have no rat/alg prefactor are called PURE FUNCTIONS ($\rightsquigarrow$ take a derivatives removes one integration)

*) Sometimes people define canonical integrals if diff. eq. is $\varepsilon$-factorized, dlog-form and they are of uniform transcendental weight, i.e. @ $\varepsilon^k$ we have it. integrals of length $k-r$ time boundary cond. of weight $r$

Examples:

- $\underbrace{x}_{\text{prefactor}} \log(x) \underbrace{+\,5}_{\text{diff. weights}}$ ⤳ neither pure or uniform weight

better show this as last example → – $1 + \varepsilon\,(\log(x) + \pi) + \varepsilon^2\,(\log^2(x) + \pi^2)$ ⤳ uniform & pure

weight -1, weight 1, -2, weight 2

- $\log(x) + \log^2(x)$ ⤳ only pure

- $x \log(x) + \pi$ ⤳ uniform not pure

*) for a diff. eq. in (canonical) form integrability is simply:
$\varepsilon$-fact.

$$0 = dB - B \wedge B \quad ⤳ \quad \varepsilon\, dA - \varepsilon^2 A \wedge A = 0$$

independent constraints

$$\begin{cases} dA = 0 & \to \text{closed forms (trivial for dlog-forms, } d^2=0) \\ A \wedge A = 0 \end{cases}$$

⤳ in general, integrability tells us that P-ordered exp. is independent of the choice of path, i.e. two homotopy invariant path gives same P-ord. exp. (the paths do not encircle singularities)

*) A very important and well understood class of special functions are so-called MULTIPLE POLYLOGARITHMS (MPLs) (iterated integrals)

$$\omega_i = d\log(g_i) \quad \text{with} \quad g_i : \text{rational functions}$$

⤳ idea is that they are the functions one obtains ~~th~~ by consecutive integration of rational functions on the sphere
(naturally live on the sphere)

more and more integrations

$$\int z^n dz = \frac{1}{n+1} z^{n+1}, \quad \int \frac{dz}{z} = \log(z), \quad \int \frac{dz \log(z)}{z} = \frac{1}{2}\log^2(z),$$

works for $n \neq -1$ — rational — new, transcendental

$$\int \frac{dz \log(z)}{1-z} = \mathrm{Li}_2(z), \ldots$$

next new transc. func.

Proper definition of MPLs:

$$\boxed{G(a_1,\dots,a_n;z)=\int_0^z \frac{dt}{t-a_1}\, G(a_2,\dots,a_n;t)}$$

pay attention to order

and we define
(@ 0 we have to regularize, tangential base pt. regularization)

$$G(\underbrace{0,\dots,0}_{n\text{-times}},x) := \frac{1}{n!}\log^n(z)$$

↳ that's why we also took before $\int_0^t \frac{dt_1}{t_1} = G(0;t) = \log(t)$

*) $\{a_1,\dots,a_n\}$ are called indices of polylog

*) $n \hat{=}$ number of integration = length = transcendental weight of MPL

very special to polylogs!

→ $\log^n(z) \sim G(\underbrace{0,\dots,0}_{=n};z)$ : weight n function

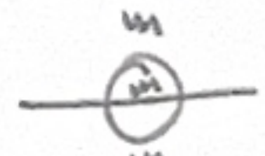
naively what we would call trans. weight

equals polylog weight

*) $a_i \in \{-1,0,+1\}$ → harmonic polylogarithms

→ Many useful properties/structure of MPLs (relations, shuffle product, coaction, ...)

↳ beyond this course

⇝ Most important is general theory of iterated integrals since many examples are known where MPLs are not enough, maybe simplest  2L sunset

→ So far we have seen how nice & useful canonical bases are, but still how can we find them?

*) Very important: Don't start with random basis to construct rotation T for!

even T always exists if canonical basis exists

↳ I will give criteria to find good initial basis

⇝ We have seen that canonical integrals are given as iterated integrations over dlog-kernels.

⇝ Can we find / look for / construct integrals that they are explicitly in such a $\overset{\text{dlog}}{\text{form}}$ without going through diff. eq.'s? At least for $\varepsilon = 0$ or on the max cut?

idea: higher orders in $\varepsilon$ naturally give rise to logs, so $\varepsilon^0$ form is most important
→ Symanzik representation

⟶ would at least give a hint that these integrals are "good"

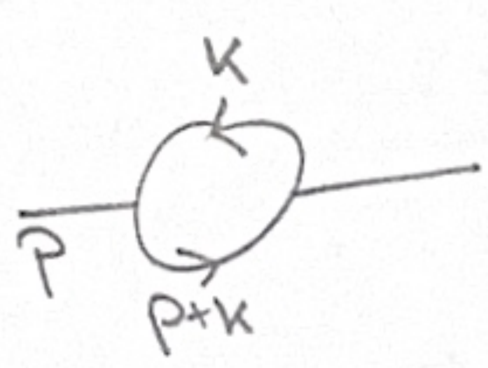
⇝ Integrals with only log singularities are also favored from physics: amplitude have only log sing. in gauge theories (causality, locality, unitarity)

"Good building blocks for scattering amplitudes"

⟶ have $\overset{\text{also}}{\text{nice}}$ diff. eq.s (cool)

So let's look for FIs such that there integral representation naturally for $\varepsilon = 0$ is in dlog form.

EXAMPLE: equal-mass bubble




$D_1 = k^2 - m^2$
$D_2 = (p+k)^2 - m^2$

$\boxed{D=2}$

same parametrization in 2D as previously: $d^2k \longrightarrow da\,db$ (up to numerical factors)

$$D_1 = 4ab - m^2$$
$$D_2 = 4(a+\alpha)(b+\beta) - m^2$$

$$\int d^2k \frac{1}{D_1 D_2} \sim \int da\,db \underbrace{\frac{1}{4ab - m^2} \; \frac{1}{4(a+\alpha)(b+\beta) - m^2}}$$

to bring the integrand in dlog form we rewrite it as

$$= -\frac{1}{16}\,\underbrace{\frac{1}{\alpha\beta\sqrt{1-\frac{m^2}{\alpha\beta}}}}\int db\,\underbrace{\left(\frac{1}{b+\beta\,\frac{1+\sqrt{\cdots}}{2}} - \frac{1}{b+\beta\,\frac{1-\sqrt{\cdots}}{2}}\right)}\int da\,\underbrace{\left(\frac{1}{a-\frac{m^2}{4b}} - \frac{1}{a+\alpha-\frac{m^2}{4(b+\beta)}}\right)}$$

We see that our integral has an algebraic prefactor
↓
called **LEADING SINGULARITY**
↓
can be also algebraic
(or even more complicated)
(before we have seen rational)

$$= \partial_b\left(\log\left(b+\beta\,\frac{1+\sqrt{\cdots}}{2}\right) - \log\left(b+\beta\,\frac{1-\sqrt{\cdots}}{2}\right)\right)$$
↓
only depends on b
↓
$d\log(\cdots) - d\log(\cdots)$

$$= \partial_a\left(\log\left(a-\frac{m^2}{4b}\right) - \log\left(a+\alpha-\frac{m^2}{4(b+\beta)}\right)\right)$$
$d\log(\cdots) - d\log(\cdots)$
↓
only da survives because of "1"

we get db so from first dlog can only take da part

This part is indeed in dlog-form ✓

→ If we multiply $I_{Bub} \cdot \underbrace{\alpha\beta\sqrt{1-\frac{m^2}{\alpha\beta}}}_{LS}$ we have a pure dlog integral in 2D.

$$\overset{4\alpha\beta = s}{=} \frac{s}{4}\cdot\sqrt{1-\frac{4m^2}{s}} \sim \sqrt{s(s-4m^2)}$$

If we look indeed at differential equation we find a canonical form:

$$\frac{\partial}{\partial s}\begin{pmatrix} I_{Tad} \\ I_{Bub} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{2\varepsilon}{s(s-4m^2)} & -\frac{s-2m^2}{s(s-4m^2)} - \varepsilon\,\frac{1}{s-4m^2} \end{pmatrix}\begin{pmatrix} I_{Tad} \\ I_{Bub} \end{pmatrix}$$

look only at s here

Exercise to check these steps explicitly

We make the rotation $T = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{s(s-4m^2)} \end{pmatrix}$

$$\Rightarrow \quad A(s) = \begin{pmatrix} 0 & 0 \\ \frac{2}{\sqrt{s(s-4m^2)}} & -\frac{1}{s-4m^2} \end{pmatrix}$$

obviously a dlog

$$= 2\,\frac{\partial}{\partial s}\left(\log\left(\frac{\sqrt{s}+\sqrt{s-4m^2}}{\sqrt{s}-\sqrt{s-4m^2}}\right)\right)$$

↳ also dlog if one also considers the $m^2$-derivatives

~> We see that we can start with Integrals having @ $\varepsilon=0$ an Integrand in dlog-form and the strip of its LS to obtain a canonical diff. eq.!

~> LS are related to max cuts / generalized residues

@ $\varepsilon=0$ homogeneous diff. eq. for bubbles is

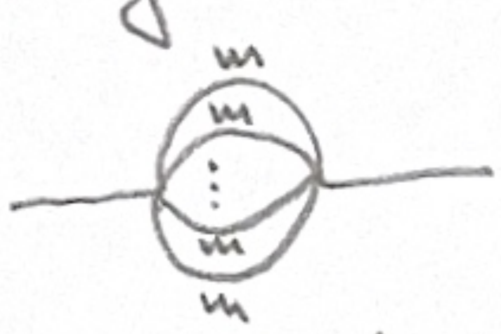
Exercise: Show this.

$$\left[\frac{\partial}{\partial s} - \left(-\frac{s-2m^2}{s(s-4m^2)}\right)\right] \frac{1}{\sqrt{s(s-4m^2)}} = 0$$

$\hat{=}$ LS = Hom. sol = Max Cut = generalized Residue

~> So we see an analysis of generalized residues / max cut together with integrand / dlog analysis is key to derive canonical forms!

~> It is conjectured that there exists always a canonical form also $\varepsilon$-factorized when forms $\omega$ are not just dlog-forms.

↳ e.g. general equal-mass bananas  are related to so-called Calabi-Yau varieties which have more general canonical forms than dlog-forms

→ analysis of corresponding PF eq. with Frobenius basis allows to find canonical form

→ still very active field of research!

Exercise: Show from the Baikov representation of the massless box that in $D=4$ the box with unit propagator powers (corner integral) is canonical, i.e. has an integrand in dlog-form.

Notice that we can do the integrand analysis in $D_0 \in 2\mathbb{N}$ arbitrary since we can use dimensional shifts to translate our integrals to $D_0 = 4$.

"If an integral is canonical in $D_0$ dimensions then it is so also in $D_0 \pm 2$"

$$d\mathcal{I}(D = D_0 - 2\varepsilon) = \underset{\frac{D_0 - D}{2}}{\underset{\parallel}{\varepsilon}}\, A\, \mathcal{I}(D)$$

$$\Big\downarrow \text{ dim. shift } D \to \tilde{D} = D \pm 2$$

$$d\mathcal{I}(\tilde{D} = D_0 \pm 2 - 2\varepsilon) = \underset{\varepsilon}{\underset{\parallel}{\frac{D_0 - (\tilde{D} \mp 2)}{2}}}\, A\, \mathcal{I}(\tilde{D})$$

→ indeed in $\tilde{D} \pm 2$ they satisfy an $\varepsilon$-fac. eq.

Before we end this lecture let us have a different view on solving the eq.

$$d\mathcal{I} = \varepsilon A \mathcal{I}$$

↙ Laurent expansion: $\mathbb{P}$-ordered exponential

↘ $\varepsilon$-resummed, exact in $\varepsilon$: general solution has the following form

$$\boxed{\mathcal{I} = \sum_{i,j,k} a_{ijk}^{(\varepsilon)} (z - z_0)^{i + j\varepsilon} \log^k (z - z_0)}$$

we can simply expand to recover Laurent expansion

but much more expensive than $\mathbb{P}$-ordered exponential

depend rationally on $\varepsilon$ parameter

as we know from Frobenius method: $j\varepsilon$ determined from indicial, $k$ depends on how many equal local exponents one finds

ranges of $j$ & $k$ are fixed from differential equation, $i_{max}$ is truncation order

→ there are different ways to compute the resummed solution

*) purely through a ansatz (hard to guess all values of $j$ & $k$)

*) Frobenius method and indicial equation (has to go through higher-order eq.)

*) use the matrix exponential to get leading terms and therefore $j\varepsilon$ scalings

Let's consider again the massless box in $D = 4-2\varepsilon$:

(for simplicity we set $t=1$ and only consider the $s$ dependence (⇝ through dimensional analysis we can recover $t$ dependence))

⇝ We have seen that in $D=2$ the bubble is canonical up to $LS = \frac{1}{s}$

↳ the dim. shift to $D=4$ only produces a dot

the box in $D=4$ is canonical up to $LS = \frac{1}{s \cdot t}$

$$\underline{\mathcal{I}} = \left( I_{\text{bubble}}, I_{\text{bubble}}, I_{\text{box}} \right)$$

$$\Rightarrow \quad d\underline{\mathcal{I}} = \frac{\partial}{\partial s} \underline{\mathcal{I}} = \varepsilon A(s) \underline{\mathcal{I}} \quad \text{with}$$

$$A(s) = \begin{pmatrix} -1/s & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2}{s(s+1)} & \frac{2}{s+1} & -\frac{1}{s(s+1)} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix} \frac{1}{s} + \mathcal{O}(s^0)$$

because of canonical form at most single poles ⇝ easier!

We compute the matrix exponential:

$$M = \exp\left( \varepsilon \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix} \log(s) \right) = \exp\left( \varepsilon \begin{pmatrix} 0 & -1/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} -1 & 1 \\ 0 & -1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix} \begin{pmatrix} 0 & -1/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \log(s) \right)$$

2x2 Jordan block

Jordan decomposition

eigenvalues $0, -1$ ⇝ $s^{-\varepsilon}, s^0$

$s^{-\varepsilon} \begin{pmatrix} 1 & \varepsilon \log(s) \\ 0 & 1 \end{pmatrix}$

explicit log due to Jordan block

this clearly solves

$$\left( \frac{\partial}{\partial s} - \varepsilon \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix} \frac{1}{s} \right) M = 0$$

in general we have $J = \begin{pmatrix} a & 1 & & 0 \\ & a & 1 & \\ & & a & 1 \\ 0 & & & \ddots \end{pmatrix}$ $r \times r$ Jordan block

$$\exp(\varepsilon J \log(s)) = s^a \begin{pmatrix} 1 & \log(s) & \frac{1}{2}\log^2(s) & \frac{1}{3!}\log^3(s) & \cdots \\ & 1 & \log(s) & \frac{1}{2}\log^2(s) & \cdots \\ & & 1 & & \ddots \\ & & & \ddots & \end{pmatrix}$$

similar as logs in Frob. basis (recall $\mu \hat{=}$ max Jordan block monodromy matrix)

Exercise: check all steps

bubble has a cut @ $s=0$

$$= \begin{pmatrix} s^{-\varepsilon} & 0 & 0 \\ 0 & 0 & 0 \\ -2\varepsilon s^{-\varepsilon}\log(s) & 0 & s^{-\varepsilon} \end{pmatrix}$$

box as well

get explicit log

will learn more about this in next lecture

## Short Recap Main Points:

*) FI satisfy 1st-order systems $d\underline{I} = \mathbb{B}(z;\varepsilon)\,\underline{I}$.

*) Find rotation to canonical form $\underline{J} = T\,\underline{I} \rightsquigarrow d\underline{J} = \varepsilon\, A(z)\,\underline{J}$.

dlog-forms

↳ solving is then trivial

(↳ expand $A(z)$ in $z$ allows to series expand $\mathbb{P}$-ordered exponential as well → easy series expansion of $\underline{J}$, only compute $\int_0^z z'^n \log^m(z')\, dz'$)

*) Key is finding $T$ (equally hard as solving diff. eq. for $\varepsilon = 0$).

*) Max Cut / residue / integrand analysis tell us a lot about structure of diff. eq. and how to find canonical integrals.

*) Frobenius method gives us a basis of solutions, particularly for $\varepsilon = 0$.

↳ analytic continuation for global solutions