II) Some basics of Differential equations:

So far we have discussed <u>1st-order system</u>. But we can equivalently consider <u>higher-order</u> equations: (here for <u>simplicity one-parameter</u>)

$$\boxed{\mathcal{L} f(z) = 0} \quad \text{with operator} \quad \boxed{\mathcal{L} := q_n(z)\, \partial_z^n + q_{n-1}(z)\, \partial_z^{n-1} + \dots + q_0(z)}$$

(we take always this order)

*) n th-order operator, linear → has n-solutions
*) polynomial coeff. (can multiply with denom. since homog. eq.) (no common factor)
*) $q_n(z)$ also called <u>discriminant locus</u> → gives location where operator gets singular (+ possibly ∞)

⇝ How can we systematically solve these diff. eq's.?

i) An n-th-order operator is equivalent to a system of size n:

Because: $\mathcal{L} f(z) = 0 \quad \Rightarrow \quad \partial_z^n f(z) = - q_0/q_n \, f(z) - \frac{q_1}{q_n} \partial_z f(z) - \dots - \frac{q_{n-1}}{q_n} \partial_z^{n-1} f(z)$

and we have: $I = \underbrace{\left(f(z), \partial_z f(z), \dots, \partial_z^{n-1} f(z)\right)}_{\text{length } n \text{ vector}}$

$$\frac{\partial}{\partial z} I = B(z) I \quad \text{with} \quad B(z) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdot & 0 \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & & & & \\ -q_0/q_n & \cdots & & - & q_{n-1}/q_n \end{pmatrix} \leftarrow \text{diff. eq. is in last row of connection matrix}$$

and in the other direction we can decouple the system / go to a "derivative basis"

For $B(z)$ generic we have $\partial_z J = B(z) J$ with $J = T I$

rational functions inside

Exercise: Find the equivalent 3^{rd}-order operator: (use mathematica)

$$B(z) = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ -\frac{2(5-7z+3z^2)}{3(1-z)^2} & -\frac{4-3z}{3(1-z)} & 2/3 \\ +\frac{63-1467z-12932z^2+54629z^3-58624z^4+19456z^5}{3(1-z)^3 z^2 (1-16z)^2} & -\frac{9-1035z+18589z^2-38144z^3+19456z^4}{6(1-z)^2 z^2 (1-16z)^2} & -\frac{9-298z+304z^2}{3(1-z)z(1-16z)} \end{pmatrix}$$

also find T.

ii) It is also very useful to use the logarithmic derivative / Euler operator $\boxed{\theta := z \frac{\partial}{\partial z}}$

Then we have: $\mathcal{L} = \tilde{q}_n(z)\, \theta^n + \tilde{q}_{n-1}(z)\, \theta^{n-1} + \dots + \tilde{q}_0(z)$

One can prove: $\theta^n = \sum_{k=1}^{n} S_2(n,k)\, z^k \partial_z^k$ and $z^n \partial_z^n = \prod_{j=0}^{n-1} (\theta - j)$

(Stirling numbers 2nd kind $S_2(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$)

$\rightsquigarrow$ not complicated to go from ∂_z^k to θ^k form of operators

$\rightsquigarrow$ see soon why θ-form is useful

iii) We can classify the singularities of the diff. operator:

z_0 is singular if $q(z_0) = 0$ or possibly $z_0 = \infty$ (use $t = 1/z$ to check)

z_0 is a regular singular point if $(z-z_0)^{n-i}\, q_i(z)/q_n(z)$ are analytic (finite at z_0)

otherwise it is a irregular singular point $\rightsquigarrow$ essential singularity!

Fuchs : at a reg. sing. point there is at least one solution of the form

$$\boxed{f(z) = (z-z_0)^{\alpha} \sum_{n=0}^{\infty} a_n z^n} \quad , \alpha : \text{indicial / local exponent} \in \mathbb{C} \text{ (usually for us } \mathbb{Q})$$

and we can have at z_0 that a solution $f(z)$ is either

analytic, has a pole, or algebraic or logarithmic branch cut

$\rightsquigarrow$ diff. eq. which have only regular sing. points are called Fuchsian.

$\rightsquigarrow$ It is expected that FI have always Fuchsian diff. eq.s

$\rightarrow$ We call them also Picard - Fuchs equation

$\llcorner$ refers to geometric interpretation of FIs as Gauss-Manin

iv) How do we find all n solutions now? (assume for simplicity $z_0 = 0$)

First ansatz : $\varpi_0(z) = z^{\alpha} \sum_{k=0}^{\infty} a_k z^k$

("varpi")

Now we find : $\theta\, z^{k+\alpha} = (k+\alpha)\, z^{k+\alpha}$

$\llcorner$ because after deriv. we multiply again with z

$\hookrightarrow$ in θ-form it is easy / convenient to work with power series

lowest exponent in z determines α $\qquad z=0$

$$\mathcal{L}\, \varpi_0(z) = \underbrace{\left\{ \tilde{q}_n(0)\, \alpha^n + \tilde{q}_{n-1}(0)\, \alpha^{n-1} + \dots + \tilde{q}_0(0) \right\}}_{\stackrel{!}{=} 0 \text{ determines } \alpha} z^{\alpha} + z^{\alpha}\, \mathcal{O}(z^1)$$

indicial equation

*) a_k are determine from higher orders (in θ-form it is easy to find a recursion for the coeff.'s a_k or make an ansatz)

*) if indicial equation has n independent solutions we have all solutions found

*) otherwise we add logarithmic solutions of the following type

FROBENIUS METHOD

$$\boxed{\varpi_1(z) = z^{\alpha}\left(\log(z)\sum_{k=0}^{\infty} a_k z^k + \sum_{k=0}^{\infty} b_k z^k\right)}$$

indicial which appears multiple times, can be many as for which we do this

and continue with higher log solution until we have found n indep. solutions

$$\boxed{\varpi_k(z) = z^{\alpha}\left(\frac{1}{k!}\log^k(z)\sum_{i=0}^{\infty} a_i z^i + \frac{1}{(k-1)!}\log^{k-1}(z)\sum_{i=0}^{\infty} b_i z^i + \dots + \sum_{i=0}^{\infty} d_i z^i\right)}$$

→ this gives a constructive way to find all solutions

→ this gives local solutions valid/convergent at least until the next singularity

→ in principle, find basis of solutions around all sing. points and match them in common regions of convergence (analytic continuation) to obtain global solution

→ to understand the properties/singularities of a diff. oper. we write down all indicials in a matrix called RIEMANN P-symbol

$$P\left\{\begin{array}{ccc} z_1 & \cdots & z_K \\ \hline \alpha_1^{(1)} & & \alpha_1^{(K)} \\ \vdots & \cdots & \vdots \\ \alpha_n^{(1)} & & \alpha_n^{(K)} \end{array}\right\} \qquad \text{e.g. } P\left\{\begin{array}{cccc} 0 & 4 & 16 & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right\}$$

two square root branch cut

full tower of logarithms appear {hol, log, log²}

└ also called a point a maximal unipotent monodromy (MUM point)

⇝ So in principle we know how to solve n-th diff.eq. and therefore also a 1st-order system of size n!

Exercises: Find all local solutions to the diff. operator (up to a truncation order)

(use mathematica)

$$\mathcal{L}^{(3)} = (1-16z)^2\theta^3 - 48z(1-16z)\theta^2 - 32z(1-24z)\theta - 8z(1-32z)$$

operator of a very specific special double box graph

Compute a global solution numerically (voluntary).