14.02.'23

Advanced Methods for Collider Physics

I) Differential equations for Feynman integrals: (Part 2)

FI satisfy a first-order system of diff. eq.'s in the kinematical invariants $\underbrace{(s_{ij}, m_i^2)}_{=:\underline{z}} = (z_1, \dots, z_n)$

$$\boxed{d\underline{I} = B(\underline{z};\varepsilon)\,\underline{I}} \quad , \underline{I} = (I_1(z;\varepsilon), \dots, I_N(z;\varepsilon))$$

+ boundary info/const.

total (exterior) derivative

$$d\underline{I} = \sum_{i=1}^{n} \frac{\partial \underline{I}}{\partial z_i}\, dz_i$$

$(N\times N)$-matrix of one-form $B = \sum_{i=1}^{n} B_n\, dz_i$

"$B_i(\underline{z};\varepsilon)$, $i=1,\dots,n$ one system of diff. eq. for each variable"

$$\rightsquigarrow \quad (d - B)\,\underline{I} = 0$$

connection one-form

covariant derivative

→ is also called in geom. setting flat Gauss-Manin system

→ flatness equals integrability condition $\boxed{0 = d^2 \underline{I}}$ ⇔ $\boxed{dB - B\wedge B = 0}$

integrability is $d^2\underline{I} = 0$ ($\underline{I}$ is given in terms of proper functions)

also known as curvature two form of connection B

$$0 = d^2 \underline{I} = d\left(\frac{\partial I}{\partial z_i}\, dz_i\right) = \frac{\partial^2 I}{\partial z_j\, \partial z_i}\, dz_j \wedge dz_i = \frac{1}{2}\left(\frac{\partial^2}{\partial z_j \partial z_i} - \frac{\partial^2}{\partial z_i \partial z_j}\right) I\, dz_i \wedge dz_j$$

antisym.

can change order of double derivatives as expected/wanted

because switching a one-form

$$= d(B\underline{I}) = dB\,\underline{I} - B\wedge d\underline{I} = (dB - B\wedge B)\,\underline{I} = 0$$

⇝ So "How can we now solve these differential equations?"

↳ in realistic problems: $N \sim \mathcal{O}(100)$, n often 2 or more

↳ so far everything was rational (IBPs only give rational functions in $\underline{z}$ & ε)

↳ still complicated! stupid choice of master can make the diff. eq.'s really horrible (ε-dependent denom. mixing with $\underline{z}$)

→ so clearly one has to think which master integrals one want to compute!
if masters are computed we use IBPs to relate them to wanted integrals

Let's first organize the differential equations a bit better and give some general strategies to solve them.

1) We can sort the master integrals by the number of propagators. Integrals can only couple (talk to) integrals with the same or smaller number of propagators.

$$\frac{\partial}{\partial p^\mu}\,\frac{1}{(k-p)^2-m^2} = -\frac{1}{\left((k-p)^2-m^2\right)^2}\cdot\left[-2(k-p)^\mu\right]$$

same denom. just other exponent

can be combined to a propagator, can at most cancel a propagator

→ we can not generate new propagators/denominators by differentiation!

→ Sort integrals starting with ones with the smallest number of propagators:

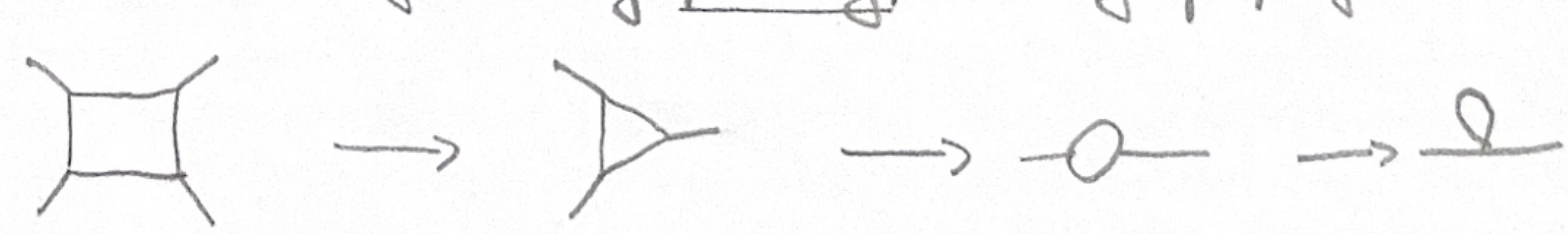
usually tadpoles

$$\underline{I} = (I_1, \dots, I_N)$$

increasing # props.

→ We can visualize this by pinching/removing propagators

(also mentioned by Lorenzo & in exercises)



Box | Triangle(s) | Bubble(s) | Tadpole(s)

Topsector → Subsectors

⇝ attention: May not be all independent under IBPs!

$$\underline{I} = (I_{Tad}, I_{Bub}, I_{Tri}, I_{Box})$$

in this order we get a block-triangular form of diff. eq.'s

$$B(\underline{z};\varepsilon) = \begin{pmatrix} \boxed{*} & & & 0 \\ & \boxed{**} & & \\ & & \boxed{***} & \\ 2 & & & \ddots \\ & & & \boxed{*} \end{pmatrix}$$

exact in ε

→ @ 1L every sector can only contain 1 Master Integral (⇝ see [illegible]'s lecture, no new ISPs)

@ higher loops sectors can become quite big (10x10 or bigger often appear)

| this gives particularly a way to solve FIs @ $\varepsilon = 0$

2) We can zoom in our diff. eq. by taking generalized cuts or more generally residues of FIs. (For definitness fix $D \in 2\mathbb{N}, \varepsilon = 0$)

Cut: $\dfrac{1}{D_i} \longrightarrow \delta(D_i)$ or a residue $\dfrac{1}{2\pi i} \oint\limits_{D_i = 0} d^D k \, \dfrac{1}{D_i}$

"replace propagators by δ-functions"

in general this requires a complex integration contour $((\mathbb{R}^D)^L \to (\mathbb{C}^D)^L)$

→ this gives simpler integrals

Max. Cut := cut all propagators in a given sector

*) taking cuts and taking derivatives "commute"
(easy to convince ourselves)

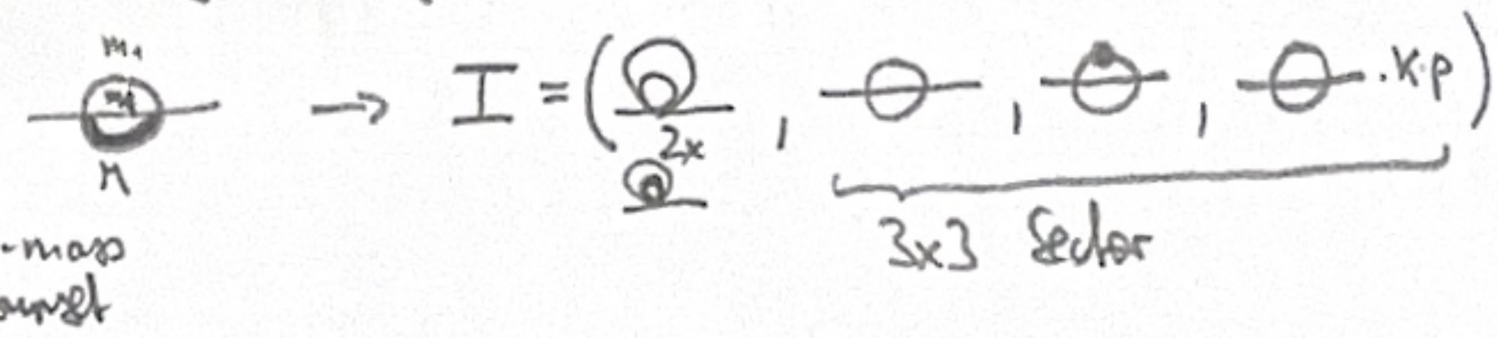
*) by taking cuts we set subsectors to zero

⟹ Considering Max. Cuts enables one to pick out the diagonal blocks of a diff. eq.,

i.e. Max. Cuts satisfy the homogeneous diff. eq.'s!

@ higher loops after having cutted all props. one can often take further residues, that's why residues are even more general

⇝ in this way one gets a refined block structure @ $\varepsilon = 0$

e.g. 

$\rightsquigarrow \begin{pmatrix} 1 & 0 \\ 2 & \begin{array}{c|c} 2\times2 & 0 \\ \hline ? & 1 \end{array} \end{pmatrix} + \mathcal{O}(\varepsilon)$ (2x, 3x3)

3) In principle, we can now solve our diff. eq. starting from the diag. blocks/max cut to have the homogeneous solutions and then integrate (Variation of constant) to obtain also inhomogeneous solutions. → "DONE!" (particularly for $\varepsilon = 0$)

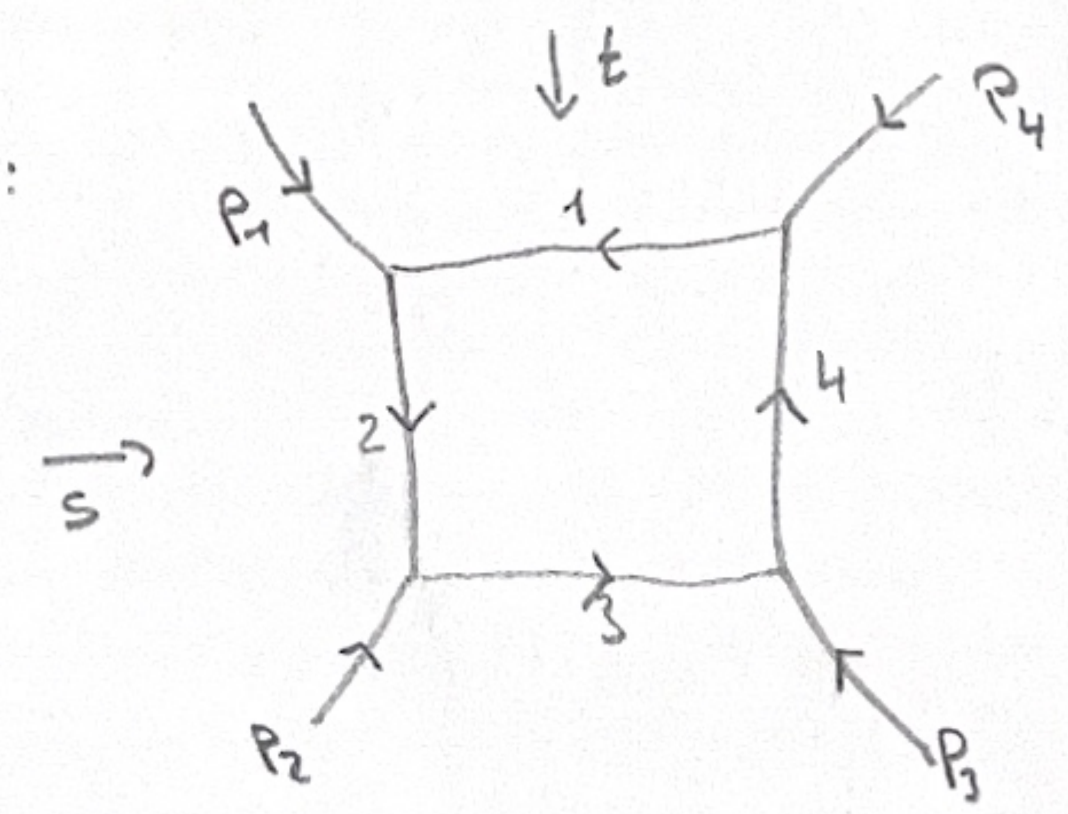
↳ kind of functions that come from max. cut define necessary function space of the whole problem "up to integration"

"if max. cuts are difficult the whole problem is, too"

"Core of a typical computation is to understand the max. cut"

EXAMPES: (How to solve FIs for $\varepsilon = 0$) (⇝ see later why this is even important if FIs are not finite)

Again 1L Box: (massless)



$D_1 = k^2$

$D_2 = (k+p_1)^2$

$D_3 = (k+p_1+p_2)^2$

$D_4 = (k+p_1+p_2+p_3)^2$

$p_i^2 = 0$

$s = (p_1+p_2)^2 = 2p_1 \cdot p_2$

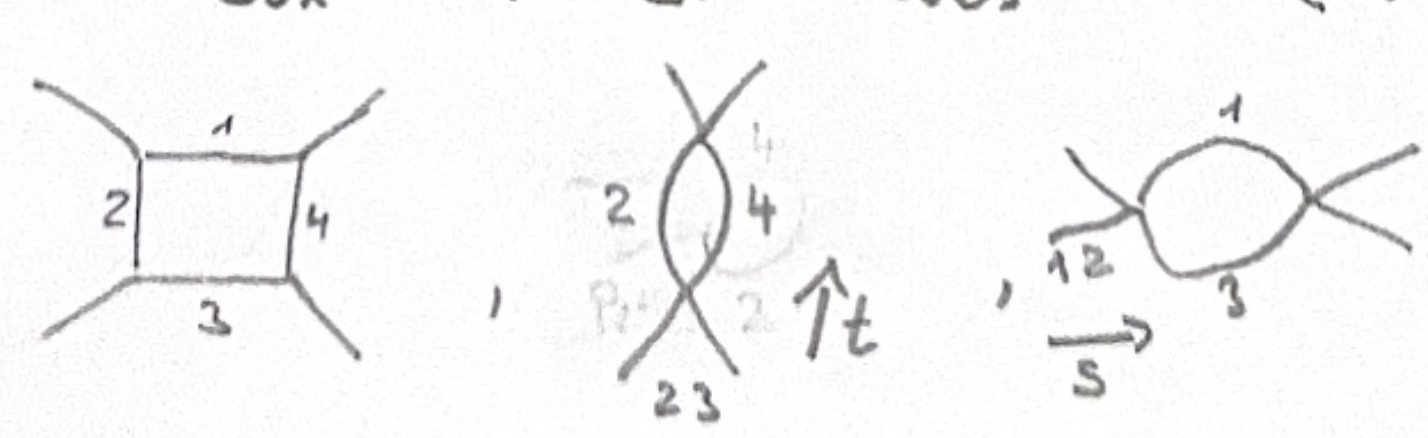
$t = (p_1+p_4)^2 = 2p_1 \cdot p_4$

$u = (p_1+p_3)^2 = 2p_1 \cdot p_3$

$s + t + u = 0$

only 3 master integrals:

1x Box + 2x Bubbles (No triangles & tadpoles for fully massless configuration)



⇝ $I = (I_{Bs}, I_{Bt}, I_{Box})$

⇝ variables s & t :

(if you want compute these diff. eq's from IBP relations)

$$B_s(s,t;d) = \begin{pmatrix} \frac{d-4}{2s} & 0 & 0 \\ 0 & 0 & 0 \\ +\frac{2(d-3)}{s^2(s+t)} & -\frac{2(d-3)}{st(s+t)} & \frac{dt-2s-6t}{2s(s+t)} \end{pmatrix}$$

more or less $s \leftrightarrow t$

$$B_t(s,t;d) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{d-4}{2t} & 0 \\ -\frac{2(d-3)}{st(s+t)} & \frac{2(d-3)}{t^2(s+t)} & \frac{ds-6s-2t}{2t(s+t)} \end{pmatrix}$$

*) easy to check integrability condition : $\partial_t B_s + \partial_s B_t + B_t B_s - B_s B_t = 0$ ✓ → Exercise

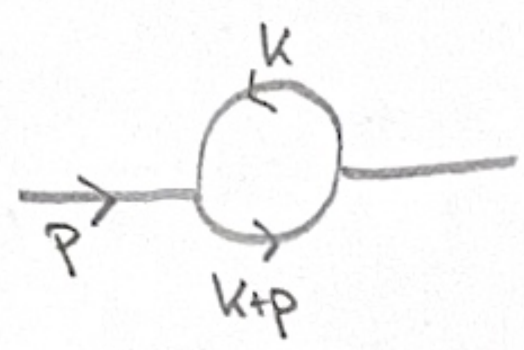
*) scaling relation $\quad s B_s + t B_s = \mathrm{diag}\left(\frac{d-4}{2}, \frac{d-4}{2}, \frac{d-8}{2}\right) = \frac{1}{2}\,\mathrm{diag}\left(\alpha_{B_s}, \alpha_{B_t}, \alpha_{Box}\right)$

$\alpha = LD + 2s - 2r$: scaling dimension

*) $3\times$ (1×1) blocks, only Box couples to bubbles (⇝ bubbles only hom. eq.)

Let's compute some maximal cuts and see that they satisfy the homogeneous differential eq's. (⇝ since we consider hom. solutions we do not care about normalizations/prefactors/numbers)

1) Bubble in 2D :



(replace appropriate for concrete momenta in box example)

$D_1 = k^2$

$D_2 = (k+p)^2$

" max cut $\hat{=}$ taking 2 residues around $D_1 = D_2 = 0$ "

⇝ we have two integration $d^2k = dk^1 \wedge dk^2$ ⇝ in principle it could work

Since we are in 2D we take the following convenient parametrization:

take $q_1, q_2 \in \mathbb{R}^2$ s.t. $q_1^2 = q_2^2 = 0$ and $q_1 \not\parallel q_2$ ⇝ q_1, q_2 span $\mathbb{R}^2$

e.g. $|q_1|\binom{1}{1}$, $|q_2|\binom{1}{-1}$

then we have: $\quad p = \alpha q_1 + \beta q_2 \quad ⇝ \quad p^2 = 2\alpha\beta\, q_1 q_2 =: s$

$k = a q_1 + b q_2 \quad ⇝ \quad k^2 = 2ab\, q_1 \cdot q_2 = \frac{ab}{\alpha\beta} s$

$(k+p)^2 = 2(a+\alpha)(b+\beta)\, q_1 q_2 = (a+\alpha)(b+\beta)\, s$

and the measure changes: $\quad d^2k = \det\left(\frac{\partial k^i}{\partial(a,b)}\right) da\, db = \det\begin{pmatrix} q_1^1 & q_1^2 \\ q_2^1 & q_2^2 \end{pmatrix} da\, db$

we take $q_1 = \begin{pmatrix}1\\1\end{pmatrix},\ q_2 = \begin{pmatrix}1\\-1\end{pmatrix}$, $q_1 \not\parallel q_2 \leadsto$ span $\mathbb{R}^2$ and $q_i^2 = 0$

$q_1 \cdot q_2 = 2$

$\leadsto$ so any 2D vector can be decomposed through q_1, q_2, particularly

$$p = \alpha q_1 + \beta q_2 \quad \leadsto p^2 = 2\alpha\beta\, q_1 \cdot q_2 = 4\alpha\beta =: s$$

$$k = a q_1 + b q_2 = \begin{pmatrix}a+b\\a-b\end{pmatrix} \leadsto k^2 = 2ab\, q_1 \cdot q_2 = 4ab$$

$$(k+p)^2 = 2(a+\alpha)(b+\beta) q_1 q_2 = 4(a+\alpha)(b+\beta)$$

and for the measure:

$$d^2k = \left| \det\left(\frac{\partial(k^1,k^2)}{\partial(a,b)} \right) \right| da\,db = \left| \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right| da\,db = \cancel{2}\, da\,db \sim da\,db$$

$$\cancel{\frac{1}{2\pi}} \int d^2k \frac{1}{D_1 D_2} \sim \int da\,db \frac{1}{4ab} \frac{1}{4(a+\alpha)(b+\beta)} \sim \int da\,db \frac{1}{ab} \cdot \frac{1}{(a+\alpha)(b+\beta)}$$

$$\xrightarrow[\text{Cut}]{\text{Max}} \oint da\,db \frac{1}{ab} \frac{1}{(a+\alpha)(b+\beta)} \begin{cases} \xrightarrow{a=0} \oint db \frac{1}{b} \frac{1}{\alpha(b+\beta)} \xrightarrow[\text{or } b=-\beta]{b=0} \pm \frac{1}{\alpha\beta} \sim \frac{1}{s} \\ \xrightarrow{b=0} \oint da \frac{1}{a} \frac{1}{(a+\alpha)\beta} \xrightarrow[\text{or } a=-\alpha]{a=0} \pm \frac{1}{\alpha\beta} \sim \frac{1}{s} \end{cases}$$

$$\Rightarrow \quad \text{Max Cut}\left(\int d^2k \frac{1}{D_1 D_2} \right) \sim \frac{1}{s}$$

and indeed @ $d=2$ we get for the s-bubble:

$$\left\{ \frac{\partial}{\partial s} - \left(-\frac{1}{s}\right) \right\} \frac{1}{s} = -\frac{1}{s^2} + \frac{1}{s^2} = 0 \quad ✓$$

$\leadsto$ satisfies homogeneous diff. eq.

2) <u>Box in 4D:</u>

($\leadsto$ naively: Box in 2D has 4 propagators and two integration variables
$\rightarrow$ can not satisfy 4 δ-constraints
$\rightarrow$ better take residues to get max cut!)

Here we want to change from $d^4k = dk^1 \wedge \dots \wedge dk^4$ to $d^4x = dx^1 \wedge \dots \wedge dx^4$ such that

$$\begin{aligned} x_1 &= D_1 = k^2 \\ x_2 &= D_2 = (k+p_1)^2 \\ &\vdots \end{aligned} \quad \leadsto \underline{x} = \begin{pmatrix} 1&0&0&0\\1&2&0&0\\1&2&2&0\\1&2&2&2 \end{pmatrix} \begin{pmatrix} k^2 \\ k\cdot p_1 \\ k\cdot p_2 \\ k \cdot p_3 \end{pmatrix} + \begin{pmatrix} 0 \\ p_1^2=0 \\ s \\ p_4^2 = 0 \end{pmatrix}$$

$$=: G\, \underline{k} + \begin{pmatrix}0\\0\\s\\0\end{pmatrix}$$

We do the change of variables in 2 steps:

(i) $k^i \longrightarrow v^i$: We have $\dfrac{\partial v}{\partial k^i} = \left(2k^i, p_1^i, p_2^i, p_3^i\right)^T$

We need the Jacobian $d^4k = |\det J| \, d^4 v$

$$J^{-1}_{ij} = \frac{\partial v^i}{\partial k^j}$$

Moreover, we have:

do not matter for us

$$J^{-1} \cdot \left(J^{-1}\right)^T = \begin{pmatrix} 2k^i \\ p_1^i \\ p_2^i \\ p_3^i \end{pmatrix} \cdot \left(2k^i, p_1^i, p_2^i, p_3^i\right) = \begin{pmatrix} 4k^2 & 2kp_1 & 2kp_2 & 2kp_3 \\ 2kp_1 & p_1^2 & p_1p_2 & p_1p_3 \\ 2kp_2 & p_1p_2 & p_2^2 & p_2p_3 \\ 2kp_3 & p_1p_3 & p_2p_3 & p_3^2 \end{pmatrix} =: \mathrm{Gram}\left(k^2, p_1, p_2, p_3\right)$$

$$\Rightarrow \det\left(J^{-1}\left(J^{-1}\right)^T\right) = \det(J)^{-2} = \mathrm{Gram}\left(k^2, p_1, p_2, p_3\right)$$

$$\Rightarrow \det J = \frac{1}{\sqrt{\mathrm{Gr}}}$$

(ii) $v^i \rightarrow x^i$: we get $(\det G)^{-1}$ for the Jacobian but these are only numbers $\rightarrow$ we don't care

$$\Rightarrow \int d^4k \, \frac{1}{D_1 D_2 D_3 D_4} \sim \int d^4x \, \frac{1}{x_1 x_2 x_3 x_4} \, \frac{1}{\sqrt{\mathrm{Gr}}}$$

Exercise: Show: $\mathrm{Gr} \sim s^2t^2 + \mathcal{O}(x_i)$ (maybe with computer)

$\Big(\mathrm{Gr} = s^2t^2 - 2st^2x_1 + t^2x_1^2 - 2s^2tx_2 + 2st\,x_1x_2 + s^2x_2^2 - 2st^2x_3 - 4st\,x_1x_3 - 2t^2x_1x_3 + 2st\,x_2x_3 + t^2x_3^2 - 2s^2tx_4 + 2st\,x_1x_4 - 2s^2x_2x_4 - 4st\,x_2x_4 + 2st\,x_3x_4 + s^2x_4 \Big)$

Max Cut is now very easy: Residues around $x_i = 0$ just mean that we set $x_i = 0$ in $\frac{1}{\sqrt{\mathrm{Gr}}}$

$$\Rightarrow \mathrm{Max\ Cut}\left(I_{\mathrm{Box}}\right) \sim \frac{1}{st}$$

and indeed:

$$\left\{\frac{\partial}{\partial s} - \left(-\frac{1}{s}\right)\right\} \frac{1}{st} = -\frac{1}{s^2t} + \frac{1}{s^2t} = 0 \ \checkmark$$

$$\left\{\frac{\partial}{\partial t} - \left(-\frac{1}{t}\right)\right\} \frac{1}{st} = 0 \ \checkmark$$

→ the idea that we use the propagators (or ISPs) directly as integration variables is rigorously worked out in the so called BAIKOV change of variables or BAIKOV Representation

⇝ also in dim. reg. ($D = d_0 - 2\varepsilon$) we can get a nice integral representation, which is very good for max. cut computations (not for whole integral since integration range is quite complicated)

⇝ full inhom. solution of Box in 4D can be computed ~~here~~ with Variation of constants
→ will not do this here, see later a better way of getting full solution!

So far it was very easy to compute max cuts. Either directly through integration or through their hom. diff. eq's. But particularly at higher loops it is better to solve homog. diff. eq's. instead of direct integration.

→ But how can we solve generally these homog. eqs., especially when the systems are bigger than 1x1?