Advanced Methods for Collider Physics

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Day 3: Asymptotic expansions

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1 1-loop bubble in massless limit

Confirm the expansion of the 1-loop bubble integral discussed in the lecture in momentum space.

2 1-dimensional integral

Expand around $\lambda = 0$ and determine the leading as well as the subleading asymptotic behavior and expand to order $\mathcal{O}(\epsilon)$:

$$I(\lambda) = \int_0^\infty \mathrm{d}x \, \left(\lambda + x + x^2 + \lambda x^3\right)^{-1+\varepsilon} \tag{1}$$

3 2-dimensional integral

Expand around $\lambda = 0$ and determine the leading asymptotic behavior:

$$I(\lambda) = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \, (1 + \lambda x + xy + \lambda y)^{-1+\varepsilon} \tag{2}$$

Convince yourself that an additional regulator is required. There is actually a solution for the unexpanded integral:

$$I(\lambda) = \frac{\pi}{\varepsilon}(-1+t^2)^\varepsilon \csc(\varepsilon\pi) - \frac{1}{\varepsilon(1+\varepsilon)\lambda^2} \, {}_2F_1\left[\begin{matrix}1 \;\; 1\\ 2+\varepsilon\end{matrix}\middle| \frac{1}{\lambda^2}\right] \tag{3}$$

4 Revealing relevant regions with Mathematica and Normaliz

We want to implement the geometric region-finding algorithm for the expansion of a scalar Feynman integral in a small parameter λ. To this end, we employ Mathematica together with the computational geometry package Normaliz, which can be obtained from here as a pre-compiled binary.

Download the most recent version (currently normaliz-3.10.4) compatible with your operating system and extract the complete directory "normaliz-3.10.4" to your favorite path. Make sure that the normaliz executable can be found, for example, by creating a symlink to the normaliz executable in one of your paths in the $PATH variable. To avoid any problems (in particular on Windows), you can alternatively create a Mathematica notebook in the same directory next to the normaliz executable and run as a first command in the notebook:

```
In[1]:= SetDirectory[NotebookDirectory[]];
```

Let us test the functionality of Normaliz. We need to prepare a file "normaliz.in" containing a list of points for which the convex hull is determined. The first line of the file sets the dimension of the ambient space given by $(N+1)$ for points embedded in N-dimensional space. The second line indicates after the keyword `polytope` the length of the list of points. Here is an example for a triangle with vertices $\{(0,0),(1,0),(0,1)\}$:

```
In[2]:= strm = OpenWrite["normaliz.in"];
        WriteString[strm, "amb_space 3", "\n"];
        WriteString[strm, "polytope 3", "\n"];
        WriteString[strm, "0 0", "\n"];
```

```
WriteString[strm, "1 0", "\n"];
WriteString[strm, "0 1", "\n"];
Close[strm];
```

Now, invoke Normaliz with Mathematica:

```
In[3]:= Run["normaliz --ext --cst -s --verbose normaliz.in"];
```

If successfully executed, Normalize has created the files "normaliz.cst" (facet representation) and "normaliz.ext" (vertex representation). In both files (starting from line 3) you find a matrix of integer values (ignore all output in "normaliz.cst" below the keyword `inequalities`). For the vertex representation, ignore the last column. The vertices from the input file can be found here. In general, the minimal set of required vertices is returned. For the facet representation, every row in the matrix contains a facet normal vector $\mathbf{n}_f$ and the value of a_f in the last column. The first and the second line in the files indicate the number of rows and columns, respectively. The output files can be inspected with Mathematica as follows:

```
In[4]:= ReadList["normaliz.ext", String]

Out[4]= {"3", "3", "0 0 1 ", "0 1 1 ", "1 0 1 "}

In[5]:= ReadList["normaliz.cst", String]

Out[5]= {"3", "3", "-1 -1 1 ", "0 1 0 ", "1 0 0 ", "inequalities", "0", "3", "equations", "0", "
```

Recall that in the facet representation the convex hull is given by the intersection of half-spaces:

$$\bigcap_{f\in F}\left\{\mathbf{m}\in\mathbb{R}^N \mid \langle\mathbf{m},\mathbf{n}_f\rangle + a_f \geqslant 0\right\} \tag{4}$$

Note that the facet normal vectors $\mathbf{n}_f$ point into the convex hull.
The following function has been copied from here and returns the Symanzik polynomials (and the number of loops) for a given set of propagators.

```
In[6]:= UF[xx_, yy_, z_] :=
        Module[{degree, coeff, i, t2, t1, t0, zz},
          zz = Map[Rationalize[##, 0] &, z, {0, Infinity}];
          degree = -Sum[yy[[i]]*x[i], {i, 1, Length[yy]}];
          coeff = 1;
          For[i = 1, i <= Length[xx], i++,
           t2 = Coefficient[degree, xx[[i]], 2];
           t1 = Coefficient[degree, xx[[i]], 1];
           t0 = Coefficient[degree, xx[[i]], 0];
           coeff = coeff*t2;
           degree = Together[t0 - ((t1^2)/(4 t2))];];
          degree = Together[-coeff*degree] //. zz;
          coeff = Together[coeff] //. zz;
          {coeff, Expand[degree], Length[xx]}]
```

The first argument is the list of loop momenta, the second is a list of propagators following the convention $(-p^2+m^2)$, the third is a list of replacements for the kinematics. The Mathematica function `CoefficientRules` may be useful to extract the exponents of the polynomials.

5 Analytic regulators

We would like to extend our Mathematica implementation such that it returns internal facets (as induced by the expansion in the individual regions) with $a_f = 0$. For every region, the corresponding vertices in the facet can be identified. It is sufficient for the analysis to project the vertices to the plane λ^0. For every set of vertices identified as outlined before, Normaliz should be called to return the facet representation. Select those facets, which are internal and intersect with the origin ($a_f = 0$).

6 1-loop massless box in the Regge limit (requires the Mathematica implementation)

Consider the 1-loop massless box:

$$I(\nu_1, \nu_2, \nu_3, \nu_4) = \int \frac{\mathrm{d}^d k}{i\pi^{d/2}} \frac{\mathrm{e}^{\epsilon\gamma_E}}{[k^2]^{\nu_1}[(k+p_1)^2]^{\nu_2}[(k+p_1+p_2)^2]^{\nu_3}[(k+p_1+p_2+p_3)^2]^{\nu_4}}. \tag{5}$$

with $s = -2\,p_1 \cdot p_2$ and $t = -2\,p_2 \cdot p_3$ (euclidean region to guarantee positive indices in the second Symanzik polynomial) in $d = 6 - 2\epsilon$. We are interested in the leading asymptotic behavior in the limit $t \to 0$. Run your Mathematica code to determine the relevant regions. Is the introduction of an analytic regulator required? If yes, make a reasonable choice for the shift of the propagator powers ν and perform the integration of the region integrals in Mathematica. Do not expand in the regulator(s) before integration. When using Mathematica for the integration, it might be useful to globally declare `$Assumptions` on the parameters involved.